Set-membership Identification and Fault Detection using a Bayesian Framework

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Abstract—**This paper deals with the problem of set-membership identification and fault detection using a Bayesian framework. The paper presents how the set-membership model estimation problem can be reformulated from a Bayesian viewpoint in order to determine the feasible parameter set and, in a posterior fault detection stage, to check the consistency between data and the model. The paper shows that, assuming uniform distributed measurement noise and flat model prior probability distribution, the Bayesian approach leads to the same feasible parameter set than the set-membership strips technique and, additionally, can deal with models nonlinear in the parameters. The procedure and results are illustrated by means of the application to a quadruple tank process.**

I. INTRODUCTION

IN the Control Engineering field, the so-called Robust Identification techniques deal with the problem of obtaining not only a nominal model of the plant, but also an estimate of the uncertainty associated to the nominal model. Such model of uncertainty is typically characterized as a region in the parameter space or as an uncertainty band around the frequency response of the nominal model.

Uncertainty models have been widely used in the design of robust controllers [1] and, recently, their use in model-based fault detection procedures is increasing [2]. In this later case, consistency between new measurements and the uncertainty region is checked. When an inconsistency is found, the existence of a fault is decided.

There exist two main approaches to the modeling of model uncertainty: the deterministic/worst case methods and the stochastic/probabilistic methods. For a survey, see e.g. [3]. Deterministic methods lead to hard bounds on the uncertainty region and the most representative are the set membership (SM) techniques [4] and the deterministic versions of the model error modeling (MEM) approach [5].

Stochastic methods, such as the Non Stationary Stochastic Embedding (NSSE) [6], lead to probabilistic bounds on the uncertainty region. In early years, this fact was perceived as a drawback but recent advances in robust risk adjusted controllers [7] and probabilistic fault detection [8] have given raise to stochastic methods. In particular, there is a renewed interest for the Bayesian point of view in system

identification [9]-[10]. The topic is not new since early works in system identification already considered the Bayesian parameter estimation problem [11] and model classification problem [12]. The Bayesian ideas, although appealing, have largely not been implemented due to the difficult computation of the integrals involved in the posterior distributions. Recent advances in simulation techniques such as the Markov chain Monte Carlo (MCMC) have overcome this situation [13].

In this paper, we focus on the problem of set-membership identification and fault detection using Bayesian tools. The paper presents how the set-membership model estimation problem can be reformulated such that the Bayesian tools can be used to determine the feasible parameter set and to check the consistency between data and the model. The paper shows that the Bayesian approach, assuming uniform distributed measurement noise and flat model prior probability distribution, leads to the same feasible parameter set than the set-membership approach. Additionally, the Bayesian methodology can deal with models nonlinear in the parameters such the ones derived from the use of observers. These features are exemplified using a quadruple tank process.

This paper is organized as follows: Section II establishes the model parameterization that is going to be used and formulates the parameter set estimation problem and the fault detection problem. Section III addresses both problems from a Bayesian viewpoint. In particular, we define the so-called Bayesian credible model set and particularize it in order to solve the set-membership parameter estimation problem. We also derive a test to check for faults on the basis of the resulting feasible parameter set. Section IV illustrates the application of the proposed method to a quadruple tank process and presents the results in both the linear and the nonlinear cases. Finally, Section V concludes the paper.

II. PROBLEM DEFINITION

A. Model Parameterization

Let us assume that the system can be expressed by means of the following regression model

$$y(k) = F(k, \boldsymbol{\theta}) + e(k) = \hat{y}(k, \boldsymbol{\theta}) + e(k), \quad k = 1, ..., M \quad (1)$$

where
- $F(k, \boldsymbol{\theta})$ is the regression function, or observation function, which, in a general case, is assumed to be nonlinear in the

parameters $\boldsymbol{\theta}$, and it can contain any function of inputs $u(k)$ and outputs $y(k)$.

- $\boldsymbol{\theta} \in \boldsymbol{\Theta}_o$ is the parameter vector of dimension $n_\theta \times 1$.

- $\boldsymbol{\Theta}_o$ is the set in the parameter space that contains the *a priori* bounds for the parameter values.

- $e(k)$ is an additive error term which is unknown but it is assumed to be bounded by a constant $|e(k)| \le \sigma(k)$.

B. Parameter Estimation

According to [14], the parameter estimation problem consists in determining the region in the parameter space that contains all models consistent with the *M* input/output data. This consistency region is known as Feasible Parameter Set (FPS) and, for the parameterization (1), it is defined as follows

$$FPS = \left\{ \boldsymbol{\theta} \in \boldsymbol{\Theta}_o \mid y(k) - \sigma(k) \le F(k, \boldsymbol{\theta}) \le y(k) + \sigma(k), \, k = 1, \cdots, M \right\}$$ (2)

In the case that the regression function is expressed linearly as $F(k, \boldsymbol{\theta}) = \boldsymbol{\varphi}^T(k)\boldsymbol{\theta}$, the parameterization (1) can be formulated as

$$y(k) = \boldsymbol{\varphi}^T(k)\boldsymbol{\theta} + e(k) = \hat{y}(k) + e(k)$$ (3)

where $\boldsymbol{\varphi}^T(k)$ is the regressor vector of dimension $1 \times n_\theta$. In this case, the FPS is a polytope that can be described in the $\mathcal{H}$-polytope form [15]. Also, in the simplest case the FPS can be obtained by intersecting all *M* strips defined by the pairs of parallel lines defined by $y(k) - \boldsymbol{\varphi}^T(k)\boldsymbol{\theta} = 2\sigma(k)$.

In the case that the regression function is nonlinear in the parameters, the resulting FPS is no longer a convex polytope but a set with a complex shape. In order to avoid dealing with the exact description of this FPS, several algorithms exist that obtain inner or outer simpler regions that approximate the exact FPS. Such regions are known as *Approximated Feasible Parameter Sets* (AFPS).

Inner approximations find the parameter set of maximum volume such that all the parameters of the approximate set are inside the feasible parameter set, $AFPS_{in} \subseteq FPS$. On the other hand, outer approximation algorithms find the parameter set of minimum volume that guarantees that the feasible parameter set is inside the approximate set, $FPS \subseteq AFPS_{out}$.

When $F(k, \boldsymbol{\theta})$ is linear, boxes, parallelotopes, ellipsoids or zonotopes are used to obtain the AFPS [16]-[21]. In the nonlinear case, a minimum outer box can be determined by means of a set of optimization problems [18]. But since the parameters enter in a nonlinear way in (1), the resulting optimization problems are nonconvex and obtaining the solution is NP-hard.

As an alternative, recursive algorithms can be used as $FPS(k+1) = FPS(k) \cap S(k)$, where $S(k)$ is the set of parameters consistent with data at instant *k*.

$$S(k) = \left\{ \boldsymbol{\theta} \in \mathbb{R}^{n_\theta} \mid -\sigma(k) \le y(k) - F(k, \boldsymbol{\theta}) \le \sigma(k) \right\}$$ (4)

Recursive algorithms allow the efficient computation of inner, $AFPS_{in}(k+1) \subseteq AFPS_{in}(k) \cap S(k)$, or outer approximations, $AFPS_{out}(k) \cap S(k) \subseteq AFPS_{out}(k+1)$.

In this approach, the AFPS can be approximated by using subpavings and the SIVIA algorithm which is based on refining the initial *a priori* set $\boldsymbol{\Theta}_o$ by iteratively bisecting it [18].

C. Fault Detection

Once the FPS (or its approximation) has been estimated with nonfaulty data, the fault detection test consists in checking if new data (possibly containing faults) are inconsistent with the FPS. The inconsistency can be checked by means of the intersection of $S(k)$ with the FPS. A fault will be indicated if this intersection leads to an empty set

$$S(k) \cap FPS = \varnothing$$ (5)

In the linear case, if the identification data length is moderate, fault detection test (5) can be solved efficiently by determining the feasibility of a linear optimization problem. However, when the data length increases inner/outer approximations must be used and missed alarms (in outer approximations) and false alarms (in inner approximations) can appear [20].

III. SET-MEMBERSHIP ESTIMATION AND FAULT DETECTION IN THE BAYESIAN FRAMEWORK

A. Bayesian Credible Model Set

In the Bayesian framework, the model set that characterizes the model uncertainty can be described by means of a *Bayesian Credible Model Set* (BCMS) which is defined as follows:

$$\mathcal{B} \equiv \left\{ G \in \mathcal{G} : p(G|\mathbf{y}) \ge c(\alpha) \right\}$$ (6)

where $\mathbf{y} = \begin{pmatrix} y(1) & \ldots & y(M) \end{pmatrix}^T$.

The BCMS contains all the models *G* belonging to a model class $\mathcal{G}$ whose posterior probability distribution conditioned to measurement data, $p(G|\mathbf{y})$, is higher than a given critical value $c(\alpha)$, where 100(1- α)% is the desired credibility level.

The set $\mathcal{B}$ is inspired in the *Feasible Model Set* (FMS) of deterministic methods in the sense that it also combines *a priori* information with *a posteriori* information [22]. In the FMS, the *a priori* information is contained in the Candidate Model Set (CMS) which consists of a noise class and a model class. In the set $\mathcal{B}$, the *a priori* information is defined by means of the prior probability distributions on the error term $p_e(e)$ and on the model $p(G)$.

The measurement data **y**, i.e. the *a posteriori* information, is introduced into the credible set by means of the likelihood

function of the observations **y** conditioned to the model *G*, $p(\mathbf{y}|G)$.

The posterior distribution $p(G|\mathbf{y})$ of the model *G* conditioned to the observations **y** is obtained by applying the Bayes rule,

$$p(G\,|\,\mathbf{y}) = \frac{p(\mathbf{y}\,|\,G)\,p(G)}{p(\mathbf{y})} \qquad (7)$$

where the factor $p(\mathbf{y})$ is just a normalizing constant. In summary, we have $p(G\,|\,\mathbf{y}) \propto p(\mathbf{y}\,|\,G)\,p(G)$, where the prior distribution $p(G)$ contains the information about the plant before the data is obtained while the posterior distribution $p(G|\mathbf{y})$ contains the information about the plant updated by the measurements **y**.

B. Set-membership Estimation

Let us illustrate how the FPS region defined in (2) can be obtained by means of the application of (7). Since the region defined in (2) describes parametric-type uncertainty, the Bayesian credible model set reduces to the Bayesian credible parameter set:

$$\mathcal{B}_{\boldsymbol{\theta}} \equiv \left\{ \boldsymbol{\theta} \in \mathbb{R}^{n_\theta} : p(\boldsymbol{\theta}|\mathbf{y}) \geq c(\alpha) \right\} \qquad (8)$$

where the process model is characterized by means of the parameter vector $\boldsymbol{\theta}$. Now, we have to decide which is the model prior probability distribution, $p(\boldsymbol{\theta})$. In the Bayesian framework, this probability is a *subjective* probability [22]. For instance, here it is assumed that we have no information about which the value of the "true" parameter vector $\boldsymbol{\theta}$ will be; consequently we take a flat $p(\boldsymbol{\theta})$ over the initial set $\boldsymbol{\Theta}_o$. This way the model posterior distribution is directly proportional to the likelihood function of the observations, $p(\boldsymbol{\theta}\,|\,\mathbf{y}) \propto p(\mathbf{y}\,|\,\boldsymbol{\theta})$, in the considered initial support $\boldsymbol{\Theta}_o$.

The likelihood of the observations jointly conditioned to the model (parameter vector) and to the error bound $\boldsymbol{\sigma} = \left(\sigma(1) \;\; ... \;\; \sigma(M) \right)^T$ on the additive error coincides in form with the error term probability distribution, i.e., $p(\mathbf{y}\,|\,\boldsymbol{\theta},\boldsymbol{\sigma}) \equiv p_e\left(\mathbf{y} - \hat{\mathbf{y}}\,|\,\boldsymbol{\theta},\boldsymbol{\sigma} \right)$, where $\hat{\mathbf{y}} = \left(\hat{y}(1) \;\; ... \;\; \hat{y}(M) \right)^T$, since $\hat{y}(k) = F(k,\boldsymbol{\theta})$, $\forall k$, are deterministic quantities.

If we want to estimate a hard-bounded uncertainty/credible region, we must assume that the error term is uniform distributed, $e(k) \sim \mathcal{U}\left(-\sigma(k), \sigma(k) \right)$, $\forall k$, where $\sigma(k)$ is selected to be the additive error bound of the set-membership setup presented in Section II. In this case, the resulting likelihood function is nonzero and constant in the region where models (parameters) are consistent with the measurements and it is zero outside this region.

Note that, in this approach, we are not really concerned on obtaining the posterior distribution for $\boldsymbol{\theta}$; instead, what we obtain is the region (within the initial support $\boldsymbol{\Theta}_o$) for which the posterior distribution for $\boldsymbol{\theta}$ is constant and nonzero. This region serves as a characterization of the FPS. Note also that, since the value of the posterior distribution

for $\boldsymbol{\theta}$ is constant over the FPS, the α value is not relevant here. All models $\boldsymbol{\theta}$ will be equally probable to occur, and the probability level will be related to the FPS size.

If we were interested in different levels of probability inside the hard-bounded FPS, we could use different prior distributions for $\boldsymbol{\theta}$, i.e. Gaussian distributions. Still, if we were interested in soft-bounded FPSs we could use, for instance, Gaussian likelihood functions instead of uniform likelihood functions. These two latter situations are out of the scope of this paper and will not be treated here.

The likelihood function can be numerically estimated by using a Monte Carlo approach (see e.g. [9]-[10]). However, in the present paper, we estimate it by means of the gridding of the candidate parameter vectors $\boldsymbol{\theta}_i$ and taking the so-called *equation-error* assumption [14]. On the contrary to, the *error-in-variables* approach, where the regression function itself presents an error term, the equation-error approach assumes that the error term is additive to data at each time sample *k*. This way, we can assume that the error samples $e(k) = y(k) - \hat{y}(k)$, where $\hat{y}(k) = F(k,\boldsymbol{\theta}_i)$, are i.i.d. (independent and identically distributed), and we can compute the likelihood function as follows

$$p_e(\mathbf{y}\,|\,\boldsymbol{\theta}_i,\boldsymbol{\sigma}) = \prod_{k=1}^{M} p_e\left(y(k) - \hat{y}(k)\,|\,\boldsymbol{\theta}_i, \sigma(k) \right) \qquad (9)$$

Therefore, the approximation for the FPS in this approach is the following:

$$\text{AFPS}_{\mathcal{B}} = \left\{ \boldsymbol{\theta}_i \in \boldsymbol{\Theta}_o \mid p_e\left(y(k) - F(k,\boldsymbol{\theta}_i)\,|\,\boldsymbol{\theta}_i, \sigma(k) \right) \neq 0, \; k = 1, \cdots, M \right\} \qquad (10)$$

A high level description of the procedure is summarized in the following algorithm:

Algorithm

(*i*) Define a grid of candidate models $\boldsymbol{\theta}_i \in \boldsymbol{\Theta}_o$.

(*ii*) For a fixed $\boldsymbol{\theta}_i$

    (*ii*.1) compute the error between the measured response $y(k)$ and the output predicted by the model $\hat{y}(k)$, $k=1,\ldots,M$.

    (*ii*.2) estimate the probability that $\boldsymbol{\theta}_i$ has generated $y(k)$ by computing $p_e\left(y(k) - \hat{y}(k)\,|\,\boldsymbol{\theta}_i, \sigma(k) \right)$, where this term corresponds to a uniform probability density function defined between $-\sigma(k)$ and $\sigma(k)$.

    (*ii*.3) perform the product of all the previous terms $\forall k$ in order to obtain the likelihood function corresponding to the *M* samples, $p_e(\mathbf{y}\,|\,\boldsymbol{\theta}_i, \boldsymbol{\sigma})$. See (9)

(*iii*) Repeat (*ii*) for all the models $\boldsymbol{\theta}_i$ in the grid in order to obtain the sampled credible region.

It is noteworthy that there is no difference in the computation of the likelihood function whether $F(k,\boldsymbol{\theta})$ is linear in the parameters or not. Also, note that the resulting region is characterized by means of a grid of points

belonging to the inner approximation of the feasible parameter set defined in (7).

C. Fault Detection

Once we have calibrated the model (i.e, once we have obtained the likelihood function $p_e(\mathbf{y}\,|\,\boldsymbol{\theta}_i,\boldsymbol{\sigma})$ for all the points $\boldsymbol{\theta}_i$ in the parameter grid and for the nonfaulty samples $k = 1,\ldots,M$), the detection of faults can be performed, for every new measurement $y(k)$, $k>M$, by computing the new likelihood function $p_e(y(k)-\hat{y}(k)\,|\,\boldsymbol{\theta}_i,\sigma(k))$ and verifying whether there is at least one parameter vector $\boldsymbol{\theta}_j$ in the grid for which both $p_e(\mathbf{y}\,|\,\boldsymbol{\theta}_j,\boldsymbol{\sigma})$ and $p_e(y(k)-\hat{y}(k)\,|\,\boldsymbol{\theta}_j,\sigma(k))$ are nonzero. If this parameter (or set of parameters) exists, we conclude that the new measurement is consistent with the feasible parameter set.

The consistency can be checked by simply multiplying both likelihood functions for each parameter $\boldsymbol{\theta}_i$ in the grid. If the product is equal to zero for all the parameters in the grid,

$$p_e(\mathbf{y}\,|\,\boldsymbol{\theta}_i,\boldsymbol{\sigma})\cdot p_e(y(k)-\hat{y}(k)\,|\,\boldsymbol{\theta}_i,\sigma(k)) = 0, \forall\boldsymbol{\theta}_i, \quad (11)$$

we conclude that the new measurement $y(k)$ is not consistent with the feasible parameter region and therefore we decide that a fault has occurred.

Of course, the ability to detect "small" faults depends on the grid density. A denser grid will be able to detect smaller deviations of the parameter vector. This implies a more computationally intense calibration stage. However, the fault detection stage is not so intensive computationally since it can consider one sample at once. This feature allows the on-line implementation of the method.

IV. EXAMPLE

A. Plant Description

A quadruple-tank process, proposed by Johansson [23], is used to illustrate the methodology presented in this paper. The process inputs are the input voltages to the pumps, v_1 and v_2, and the process outputs are the tank levels $h_i, i = 1,\cdots,4$.

For illustrative purposes, we focus only on a part of the whole system. We assume that the levels h_1, h_3 and the voltage v_1 can be directly measured. The equation that describes the dynamic behavior of this part of the system is:

$$\dot{h}_1 = -\frac{a_1}{A_1}\sqrt{2gh_1} + \frac{a_3}{A_1}\sqrt{2gh_3} + \frac{\rho_1 k_1}{A_1}v_1 \quad (12)$$

where $\dot{h}_1 = dh_1/dt$, a_1 and a_3 are the cross-sections of the outlet holes of tanks 1 and 3, $A_1 = 28\,cm^2$ is the cross-section

of tank 1. The term $k_1 v_1$ with $k_1 = 3.33\,cm^3/Vs$ is the first pump flow and the parameter $\rho_1 = 0.7$ is determined from how the first valve is set prior to the experiment. The gravity acceleration is $g = 981\,cm/s^2 g$. The parameters a_1 and a_3 are the ones to be estimated and their nominal values are assumed to be $a_1 = a_3 = 0.071\,cm^2$.

B. Discrete Models

Discrete models for the linear and non-linear regression cases will be used to illustrate that the proposed approach works well in either case:

1) Linear case. A discrete, linearized version of (12) can be obtained by means of the forward approximation of the derivative $\dot{h}_1 \approx (h(k)-h_1(k-1))/T_s$ with sampling time $T_s = 1s$. This way, (12) can be expressed in the following linear regression form

$$h_1(k) = h_1(k-1) + \boldsymbol{\varphi}^T(k)\boldsymbol{\theta} + \frac{k_1\rho_1}{A_1}v_1(k-1) + e(k) \quad (13)$$

where $\boldsymbol{\varphi}^T(k) = \left(-\dfrac{1}{A_1}\sqrt{2gh_1(k-1)} \quad \dfrac{1}{A_1}\sqrt{2gh_3(k-1)}\right)$ is the

regressor vector and $\boldsymbol{\theta} = \begin{pmatrix} a_1 & a_3 \end{pmatrix}^T$ is the model parameter vector to be estimated. The term $e(k)$ is the additive error due to the measurement noises and discretization and it is assumed to be bounded, $|e_1(k)| \le \sigma = 0.05\,cm$.

2) Non-linear case. A model nonlinear in the parameters can be obtained if an output observer is used. Observers improve the ability of detecting output faults but lead to structures nonlinear in the parameters. In our example, the resulting expression is

$$\begin{aligned} \hat{h}_1(k) &= \hat{h}_1(k-1) + \boldsymbol{\varphi}^T(k)\boldsymbol{\theta} + \frac{\rho_1 k_1}{A_1}v_1(k-1) \\ &+ e(k) + L\left(h_1(k-1) - \hat{h}_1(k-1)\right) \end{aligned} \quad (14)$$

where $\boldsymbol{\varphi}^T(k) = \left(-\dfrac{1}{A_1}\sqrt{2g\hat{h}_1(k-1)} \quad \dfrac{1}{A_1}\sqrt{2gh_3(k-1)}\right)$,

$\boldsymbol{\theta} = \begin{pmatrix} a_1 & a_3 \end{pmatrix}^T$, and $|e_1(k)| \le \sigma = 0.05\,cm$.

C. Uncertainty estimation in a fault-free scenario

To obtain the uncertainty region (FPS), i.e., to determine the uncertainty region for a_1 and a_3 in the parameter space, a set of $M=140$ measurements has been obtained in a fault-free scenario.

1) Linear case. Fig. 1a shows the FPS obtained by the strips intersection using the set-membership technique described in Section II. The red little circles indicate the final (i.e., after M intersections) polytope vertices.

Fig. 1b shows the FPS region obtained by computing the contour of the sampled likelihood function assuming that the error is uniform distributed as $\mathcal{U}(-\sigma,\sigma)$ for a grid of 60×60

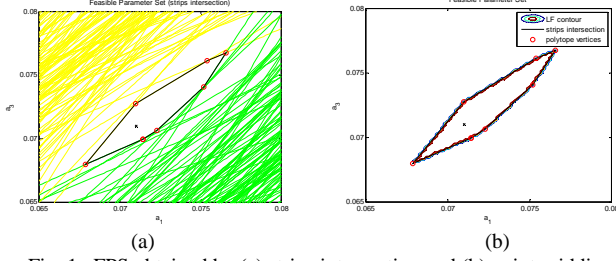parameters. As expected, this region coincides to the one obtained by the strips intersection method shown in Fig. 1a.

Fig. 1. FPS obtained by (a) strips intersection and (b) point gridding

2) Non-linear case. In the observer case, since the resulting recursive structure is nonlinear in the parameters, the strips intersection technique cannot be applied. By contrast, in the Bayesian approach, the same methodology (9)-(10) can be used for either linear or nonlinear systems.

Fig. 2a shows the FPS region obtained for the case when an observer with gain L=0.1 is used. As expected, the use of the observer leads to a tightened FPS region compared to those of Fig. 1.
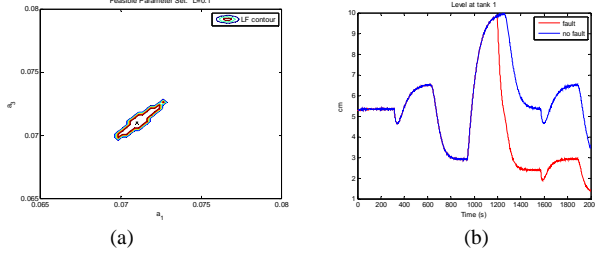

Fig. 2. (a) FPS for the nonlinear case. (b) Faulty scenario

D. Fault detection results

In order to compare the performance of the strips intersection and Bayesian fault detection tests, different fault scenarios have been created.

Here we illustrate the case when a fault consisting of an additive constant of value 0.035 acting over the parameter a_1 is introduced at k=1201. The faulty behavior is shown in Fig. 2b.

Fig. 3 illustrates the fault detection test (5) for the set-membership technique based on strip intersection in the linear case. Fig. 3a shows the FPS and the consistency parameter strip S_k corresponding to the measured data at k=1200. Since the intersection between the FPS and S_k is not empty, we conclude that the observed deviation from the nominal behavior is due to the model uncertainty and not to a fault. In other words, we say that the measurement at k=1200 is *consistent* with the model and consequently we (correctly) decide that there is no fault. On the other hand, Fig. 3b shows that at k=1201 the FPS and the strip S_k are disjoint, so their intersection is empty. This indicates that the deviation of the behavior cannot be explained by the model uncertainty and therefore we (correctly, again) decide that a fault has occurred.
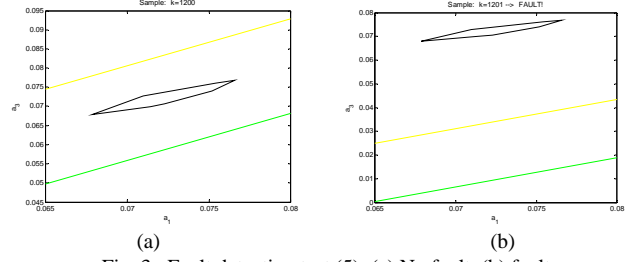

Fig. 3. Fault detection test (5): (a) No fault, (b) fault

Finally, Fig. 4 illustrates the Bayesian fault detection test in the linear case for a grid of 60×60 parameters. Fig. 4a shows the initial likelihood function corresponding to the FPS (10) and the likelihood function computed for the new measurement at k=1200. The top value in both functions has been scaled to 5 and 10 for comparison purposes. In this case, the new likelihood totally covers the FPS and so their product is nonzero over the entire FPS region. Since the product of the two likelihood functions is nonzero in at least one point of the grid, we conclude that the data are consistent with the model and therefore we (correctly) decide that there is no fault. On the other hand, Fig. 4b illustrates that at k=1201, the two likelihood functions are totally separated. This way, their product is zero for all the values over the parameter grid. The conclusion is that the observed deviation of the behavior is not due to the uncertainty because the FPS does not contain any value consistent with the observed data. In this case we (correctly, again) decide that a fault has taken place.
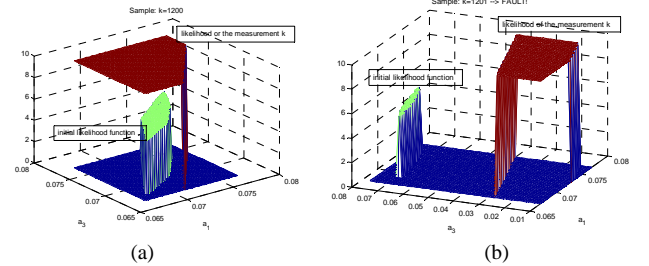

Fig. 4. Likelihood fault detection test (11): (a) No fault, (b) fault

In the example above, for the linear case, the strips intersection test and the Bayesian test have led to the same successful results since the obtained FPS regions were the same.

In the non-linear case, the comparison cannot be performed since the strips technique cannot deal with structures nonlinear in the parameters. However, for the case of plant plus observer, the Bayesian fault detection test has been applied and has successfully detected the fault at k=1200. Even more, in the case when an output observer is used, since the resulting FPS regions may be smaller, the methodology is able to detect faults of smaller magnitude. In this example, the test (11) can detect faults as small as 0.001cm^2, for an observer gain of 0.1 and a 60×60 parameter grid in the range [0.076 0.066]×[0.076 0.066].

V. Conclusion

In this paper we have presented a new set-membership estimation approach to obtain hard-bounded feasible parameter regions and to perform fault detection on the basis of them. The method is based on a Bayesian framework for system identification assuming that the error bounds are uniform distributed and that the model prior distribution is flat. In the linear case, the method presented here leads to the same FPS regions obtained by the strips intersection set-membership technique.

The Bayesian approach presents some advantages and drawbacks compared to the set-membership technique. In the identification stage, the computation of the likelihood function by means of the expression (9) may result a computationally intensive task and the more points in the grid of θ, the more computation resources are needed. On the other hand, the Bayesian approach can deal with nonlinear parameterizations of the system. This is especially interesting when nonlinear structures, such as observers, are used to improve the model estimation.

Although in the quadruple tank case study considered here we have obtained a deterministic region as a particular case of the Bayesian methodology, it has to be stressed that the Bayesian approach is a probabilistic approach, and that this stochastic nature is an advantage rather than the reverse. In a general case, the adequate selection of the model prior probability distributions may lead to probabilistic uncertainty regions that are tighter than the ones obtained by conventional system identification methods.

Regarding the fault detection stage, we have illustrated the detection of faults for the non-linear case. Since the FPS regions obtained in the calibration stage were similar for the set-membership technique and the Bayesian technique, the two fault detection procedures (5) and (11) lead to the same results. In this stage, the Bayesian method presents a computation cost similar to the set-membership strips approach and it can also be implemented on-line.

It is important to mention that the characterization of the FPS region by means of a point-wise gridding of the initial parameter set presents some shortcomings in the fault detection stage. For example, very small FPS could lay in the spaces between the points grid, thus giving zero likelihood for all the points and deciding erroneously that a fault has taken place. This drawback can be overcome by taking a denser grid, by implementing an adaptive mechanism in the points' selection stage, or even by generalizing the method in order to characterize the FPS by means model intervals instead of model points.

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