Improved actuator-fault detection and isolation strategy using interval observers and invariant sets

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Abstract: In this paper, an improved algorithm for actuator-fault detection and isolation (FDI) using a bank of interval observers is presented, where each interval observer matches one considered system mode. In this approach, interval observers and invariant sets are simultaneously used for FDI. Under a collection of improved FDI conditions, this new algorithm can detect and isolate the considered actuator faults. At the end of this paper, a circuit example is used to illustrate the effectiveness of the proposed strategy.

1. INTRODUCTION

Interval observers have been successfully used for fault detection (FD) but have only recently been extended to fault isolation (FI) [Raïssi et al., 2010, Guerra et al., 2008, Xu et al., 2013b,a].

In [Xu et al., 2013b], an FDI framework using a bank of interval observers is firstly proposed, where invariant sets and interval observers are used to establish FDI conditions and implement an FDI mechanism, respectively. Additionally, a direction different from that in [Xu et al., 2013b] is followed in [Xu et al., 2013a], where a single interval observer is used to detect and isolate a group of faults by focusing on the transient behaviors of the system.

As per the previous results, the common weakness of the proposed FDI frameworks stem from the definition of the FDI guarantees. Although the previous algorithm proposed in [Xu et al., 2013b] can effectively detect and isolate faults, there are several points that could be improved from the point of view of the FDI guarantees. Those points are analyzed as follows.

First, whenever a fault occurs, there exists an uncertain transition window between FD activation step and the FI decision. Since the transient-state behaviors are unknown, the algorithm in [Xu et al., 2013b] defines a waiting time to avoid the transient-state uncertainties and starts the FI task after the waiting time. Since the waiting time is subjectively decided by the designers, this leads to an inevitable drawback, i.e., how to define a proper waiting time as short as possible while being also accurate in terms of the FI implementation.

Second, in [Xu et al., 2013b], invariant sets are only used for establishing FDI conditions, while on-line FDI completely depends on interval observers. FD is performed by testing if the inclusion between the origin and residual intervals estimated by the interval observer matching the current mode is violated, while FI is implemented by searching the interval observer that can always contain the origin after a waiting time. Actually, there exist two potential and independent FDI mechanisms (interval observer-based and invariant set-based) in the proposed FDI framework. If one can simultaneously make use of both the FDI mechanisms with a better integration instead of only using interval observers, the FDI guarantees can be improved.

Third, in [Xu et al., 2013b], if the FDI conditions are satisfied, the faults are detectable and isolable and the FDI guarantees are obtained by allowing the residual intervals of one and only one interval observer to contain the origin after a waiting time. This implies that the FDI decisions are only given by the interval observer matching the current mode and ignore the useful process information provided by the other interval observers. Thus, if all the interval observers can be used for the implementation of FDI, the FDI approach can be enhanced.

The objectives of this paper are to address the three aforementioned problems and obtain an improved FDI strategy. First, the new algorithm avoids the specific definition of a waiting time and the designer's subjectivity by using the invariant set-based mechanism. Second, the new algorithm uses the two FDI mechanisms simultaneously and the final FDI decision is made by using both interval observers and invariant sets. Third, the system information provided by all the interval observers is used in the implementation. Consequently, the new algorithm is more sensitive to the faults with less conservative FDI conditions and higher FDI reliability.

2. PLANT AND INTERVAL OBSERVERS

2.1 Plant Models

The linear discrete time-invariant plant is given as

$$x_{k+1} = Ax_k + BF_i u_k + \omega_k, \tag{1a}$$
$$y_k = Cx_k + \eta_k, \tag{1b}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^p$ and $y_k \in \mathbb{R}^q$ are states, inputs and outputs at time instant k, respectively, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$ and $C \in \mathbb{R}^{q\times n}$ are constant matrices, and $F_i \in \mathbb{R}^{p\times p}$ ($i \in \mathbb{I} = \{0, 1, 2, \dots, N\}$ where N denotes the number of considered actuator faults) is a diagonal matrix [1] modeling the i-th mode where F_0 is the identity matrix denoting the healthy mode. The signals $\omega_k \in W$ and $\eta_k \in V$ represent bounded disturbances and noises, respectively [2] , where the sets W and V are defined as

$$W = \{\omega_k \in \mathbb{R}^n : |\omega_k - \omega^c| \le \bar{\omega}, \omega^c \in \mathbb{R}^n, \bar{\omega} \in \mathbb{R}^n\}, \tag{2a}$$
$$V = \{\eta_k \in \mathbb{R}^q : |\eta_k - \eta^c| \le \bar{\eta}, \eta^c \in \mathbb{R}^q, \bar{\eta} \in \mathbb{R}^q\}, \tag{2b}$$

where the vectors ω^c, η^c, $\bar{\omega}$ and $\bar{\eta}$ are constant, $|\cdot|$ denotes the elementwise absolute value and the inequalities are understood elementwise.

Furthermore, W and V can be rewritten as zonotopes [3] $W = \omega^c \oplus H_{\bar{\omega}}\mathbf{B}^n$ and $V = \eta^c \oplus H_{\bar{\eta}}\mathbf{B}^q$, where $H_{\bar{\omega}} \in \mathbb{R}^{n\times n}$ and $H_{\bar{\eta}} \in \mathbb{R}^{q\times q}$ are two diagonal matrices with the main diagonals being $\bar{\omega}$ and $\bar{\eta}$, respectively.

Assumption 1. The faults persist a sufficiently long time such that the FDI strategy can detect and isolate them.■

Assumption 2. The matrix A in (1) is a Schur matrix (or the pairs (A, BF_i), for all $i \in \mathbb{I}$, are stabilizable and the given control inputs u_k guarantee that the plant (1) is always stable). The pair (A, C) is detectable. ■

2.2 Interval Observers

The j-th ($j \in \mathbb{I}$) interval observer corresponding to the j-th system mode is designed as

$$\hat{X}^j_{k+1} = (A - L_jC)\hat{X}^j_k \oplus \{BF_j u_k\} \oplus \{L_j y_k\} \oplus (-L_j)V \oplus W, \tag{3a}$$
$$\hat{Y}^j_k = C\hat{X}^j_k \oplus V, \tag{3b}$$

where $\hat{X}^j_k$ and $\hat{Y}^j_k$ are the estimated state and output sets, and the interval observer gain L_j is selected to assure that $A - L_jC$ is a Schur matrix. Additionally, two properties of zonotopes are introduced in the following.

Property 1. Given zonotopes $X_1 = g_1 \oplus G_1\mathbf{B}^{r_1}$ and $X_2 = g_2 \oplus G_2\mathbf{B}^{r_2}$, $X_1 \oplus X_2 = \{g_1 + g_2\} \oplus [G_1 \quad G_2]\mathbf{B}^{r_1+r_2}$. ♦

Property 2. Given a zonotope $X = g \oplus G\mathbf{B}^r$ and a compatible matrix K, $KX = Kg \oplus KG\mathbf{B}^r$. ♦

Remark 1. From a computational point of view, the zonotopic sets are used to propagate the dynamics of interval observers in this paper. ◇

[1] All diagonal elements of F_i belong to the interval [0, 1] and a value taken from (0, 1) characterizes the performance degradation of the corresponding actuator. The limits of the interval, 0 and 1, stand for completely faulty and healthy functioning, respectively.

[2] For brevity, this paper uses ω_k and η_k to denote uncertainties under all the considered modes in (1). In general, the bounds of uncertainties in different modes may be different.

[3] Given a vector $g \in \mathbb{R}^n$ and a segment matrix $G \in \mathbb{R}^{n\times m}(n \le m)$, a zonotope X is defined as $X = g \oplus G\mathbf{B}^m$, where $\oplus$ denotes the Minkowski sum and $\mathbf{B}^m$ is a box composed of m unitary intervals.

Using Property 1 and Property 2, (3) can be transformed into the equivalent center-segment matrix form

$$\hat{x}^{j,c}_{k+1} = (A - L_jC)\hat{x}^{j,c}_k + BF_j u_k + L_j y_k - L_j\eta^c + w^c, \tag{4a}$$
$$\hat{H}^{x,j}_{k+1} = [(A - L_jC)\hat{H}^{x,j}_k \quad -L_jH_{\bar{\eta}} \quad H_{\bar{\omega}}], \tag{4b}$$
$$\hat{y}^{j,c}_k = C\hat{x}^{j,c}_k + \eta^c, \tag{4c}$$
$$\hat{H}^{y,j}_k = [C\hat{H}^{x,j}_k \quad H_{\bar{\eta}}], \tag{4d}$$

where $\hat{x}^{j,c}_{k+1}$, $\hat{y}^{j,c}_k$, $\hat{H}^{x,j}_{k+1}$ and $\hat{H}^{y,j}_k$ are the centers and segment matrices of $\hat{X}^j_{k+1}$ and $\hat{Y}^j_k$, respectively.

Assumption 3. The initial plant state x_0 is inside an initial zonotope $\hat{X}_0$ of interval observers, i.e., $x_0 \in \hat{X}_0$.

3. RESIDUAL ZONOTOPES

3.1 Residual Zonotopes

In this paper, the residual uncertainty is bounded using zonotopes. When the system is in the i-th mode, residual zonotopes corresponding to the j-th interval observer are defined as

$$\begin{aligned} R^{ij}_k &= \{y_k\} \oplus (-\hat{Y}^j_k) \\ &= \{Cx_k + \eta_k\} \oplus \{(-C\hat{X}^j_k) \oplus (-V)\} \\ &= C\{\{x_k\} \oplus (-\hat{X}^j_k)\} \oplus \{\eta_k\} \oplus (-V), \end{aligned} \tag{5}$$

where R^{ij}_k denotes the residual zonotopes from the j-th interval observer under the i-th mode.

In order to analyze the residual zonotopes (5), the term $\{x_k\} \oplus (-\hat{X}^j_k)$ is denoted as $\tilde{X}^{ij}_k$, which is derived as

$$\tilde{X}^{ij}_k = \{(x_k - \hat{x}^{j,c}_k)\} \oplus \hat{H}^{x,j}_k\mathbf{B}^{s^j_k}, \tag{6}$$

where s^j_k denotes the order of $\hat{X}^j_k$ at time instant k.

For brevity, the term $x_k - \hat{x}^{j,c}_k$ in (6) is denoted as $\tilde{x}^{ij,c}_k$. Furthermore, using (1) and (3), (6) can be derived as the equivalent center-segment matrix form

$$\tilde{x}^{ij,c}_{k+1} = (A - L_jC)\tilde{x}^{ij,c}_k + B(F_i - F_j)u_k - L_j(\eta_k - \eta^c) + (\omega_k - \omega^c), \tag{7a}$$
$$\tilde{H}^{ij}_{k+1} = \hat{H}^{x,j}_{k+1} = [(A - L_jC)\hat{H}^{x,j}_k \quad -L_jH_{\bar{\eta}} \quad H_{\bar{\omega}}], \tag{7b}$$

where $\tilde{x}^{ij,c}_k$ and $\tilde{H}^{ij}_k$ are the center and segment matrix of $\tilde{X}^{ij}_k$, respectively. Thus, the residual zonotopes (5) can be further rewritten as

$$R^{ij}_k = C\tilde{X}^{ij}_k \oplus \{\eta_k\} \oplus (-V). \tag{8}$$

3.2 Residual-bounding Zonotopes

Assumption 4. The input u_k is bounded by a known set

$$U = \{u_k \in \mathbb{R}^p : |u_k - u^c| \le \bar{u}, u^c \in \mathbb{R}^p, \bar{u} \in \mathbb{R}^p\},$$

where the vectors u^c and $\bar{u}$ are constant. ■

Similar with W and V in (2), U can be rewritten as a zonotope $U = u^c \oplus H_{\bar{u}}\mathbf{B}^p$, where $H_{\bar{u}} \in \mathbb{R}^{p\times p}$ is a diagonal matrix with the main diagonal being $\bar{u}$. By using V, W and U to replace η_k, ω_k and u_k in (7), respectively, one can derive a bounding zonotope denoted as $\check{X}^{ij}_{k+1}$ to confine

$\tilde{X}^{ij}_{k+1}$. By using zonotope operations, the center $\breve{x}^{ij,c}_{k+1}$ and segment matrix $\breve{H}^{ij}_{k+1}$ of $\breve{X}^{ij}_{k+1}$ can be derived as

$$\breve{x}^{ij,c}_{k+1} = (A - L_jC)\breve{x}^{ij,c}_k + B(F_i - F_j)u^c, \tag{9a}$$

$$\breve{H}^{ij}_{k+1} = [(A - L_jC)\breve{H}^{ij}_k \;\; B(F_i - F_j)H_{\bar{u}} \;\; -L_jH_{\bar{\eta}} \;\; L_jH_{\bar{\eta}} \;\; H_{\bar{\omega}} \;\; -H_{\bar{\omega}}]. \tag{9b}$$

The equivalent compact description of (9) is given as

$$\breve{X}^{ij}_{k+1} = (A - L_jC)\breve{X}^{ij}_k \oplus B(F_i - F_j)U \oplus L_j(-V) \oplus W \oplus L_jV \oplus (-W). \tag{10}$$

Remark 2. By zonotope operations, (10) can be transformed into the center-segment matrix form that is equal to (9). Thus, (10) and (9) are equivalent. ◇

Comparing (7) with (9), it is known that, as long as $\tilde{X}^{ij}_{k^*} \subseteq \breve{X}^{ij}_{k^*}$ holds, then $\tilde{X}^{ij}_k \subseteq \breve{X}^{ij}_k$ will always hold for all $k \geq k^*$, where $\tilde{X}^{ij}_k$ and $\breve{X}^{ij}_k$ are generated by (7) and (9), respectively. Thus, according to (8), zonotopes to bound the residual zonotopes R^{ij}_k can be derived as

$$\breve{R}^{ij}_k = C\breve{X}^{ij}_k \oplus V \oplus (-V). \tag{11}$$

Note that the set-based dynamics (10) correspond to an equivalent dynamics with the form

$$\breve{x}^{ij}_{k+1} = (A - L_jC)\breve{x}^{ij}_k + B(F_i - F_j)u_k - L_j\eta_k + \omega_k + L_j\breve{\eta}_k - \breve{\omega}_k, \tag{12}$$

where $\breve{\eta}_k \in V$ and $\breve{\omega}_k \in W$ are used to describe the effect of V and W in (3), respectively. Considering that $\breve{\eta}_k$, $\breve{\omega}_k$, η_k, ω_k and u_k are bounded, a *robust positively invariant* (RPI) set of (12), denoted as $\mathring{X}^{ij}$, can be constructed (see [Kofman et al., 2007, Olaru et al., 2010] for the notion and computation of invariant sets). Besides, the *minimal robust positively invariant* (mRPI) set for the dynamics (12) is denoted as $\breve{X}^{ij}$.

Furthermore, as per the results given in [Olaru et al., 2010], the set sequence generated by (10) converges to the mRPI set $\breve{X}^{ij}$. Because the mRPI set is contained inside any RPI set of the dynamics, i.e., $\breve{X}^{ij} \subseteq \mathring{X}^{ij}$, the set sequence generated by (10) converges into the RPI set $\mathring{X}^{ij}$. Additionally, since $\breve{X}^{ij}_k$ bounds $\tilde{X}^{ij}_k$, as k increases, $\tilde{X}^{ij}_k$ will finally enter into $\mathring{X}^{ij}$ and stay inside.

4. IMPROVED FDI STRATEGY

4.1 Interval Hull

Definition 1. The interval hull of $X = g \oplus G\mathbf{B}^r$ is the smallest interval box that contains X, which is computed as $\square X = \{x : |x_i - g_i| \leq \| G_i \|_1\}$, where G_i is the i-th row of G, x_i and g_i are the i-th components of x and g, and $\| \cdot \|_1$ is the 1-norm of vectors, respectively. ▽

Based on Definition 1, one further gives a definition for the width of the interval hull of a zonotope.

Definition 2. The interval hull width of $X = g \oplus G\mathbf{B}^r$ is defined as a vector

$$width(X) = (2\|G_1\|_1,\ 2\|G_2\|_1,\ \ldots,\ 2\|G_n\|_1),$$

where n denotes the dimension of X and $\|G_i\|_1$ denotes the width of the i-th interval component of $\square X$. ▽

Remark 3. As observed in (7b), the segment matrix of $\tilde{X}^{ij}_k$ is not affected by the mode i, i.e., for the j-th interval observer, the mode switching does not affect the evolution of the interval hull width of $\tilde{X}^{ij}_k$. Furthermore, according to (8), the evolution of the interval hull width for the residual zonotopes corresponding to a certain observer is free from the effect of the mode switching. ◇

Remark 4. Since $A - L_jC$ is a Schur matrix, as k tends to infinity, the interval hull width of $\tilde{X}^{ij}_k$ corresponding to the j-th interval observer converges to a fixed vector independent of the effect of mode switching. The same result holds for R^{ij}_k. ◇

4.2 The FDI Algorithm in [Xu et al., 2013b]

In the previous work [Xu et al., 2013b], the FD and FI are implemented only using the interval observer matching the current mode. For example, if the plant is in the i-th mode, the FD principle consists in real-time testing if

$$\mathbf{0} \in R^{ii}_k, \tag{13}$$

where R^{ii}_k is generated by the i-th interval observer matching the current i-th mode.

Furthermore, it is assumed that whenever a fault is detected, then the FI is based on searching an interval observer in real time, whose residual zonotopes satisfy

$$\mathbf{0} \in R^{ff}_k \tag{14}$$

after a waiting time, where f denotes the index of the fault that is unknown before FI.

4.3 Enhanced FDI Conditions

When the system is functioning in a certain mode, all residual zonotopes estimated by a bank of interval observers can convey the system-operating information in that mode. If it can be guaranteed that, using the system-operating information provided by the interval observers, all the modes can be distinguished from each other, then all the faults can be detected and isolated.

At time instant k, the interval hull width of the residual zonotope predicted by the j-th interval observer is denoted as $width(R^j_k)$ (because the interval hull is independent of mode switching, it is denoted as $width(R^j_k)$ not $width(R^{ij}_k)$). According to Remark 3 and Remark 4, it is known that, at infinity, $width(R^j_\infty)$ is a fixed vector [4].

According to (11) and considering the limit set $\breve{X}^{ij}$ of $\breve{X}^{ij}_k$, as k goes to infinity, $\breve{R}^{ij}_k$ will converge to

$$\breve{R}^{ij} = C\breve{X}^{ij} \oplus V \oplus (-V), \tag{15}$$

i.e., R^{ij}_k will finally enter into $\breve{R}^{ij}$ and stay inside, where $\breve{R}^{ij}$ is the limit set of $\breve{R}^{ij}_k$.

Thus, for the i-th system mode, one defines a vector

$$\breve{\mathbf{R}}^i = (\breve{R}^{i0},\ \breve{R}^{i1},\ \cdots,\ \breve{R}^{iN})$$

to describe all the limit sets of residual-bounding zonotopes (i.e., $\breve{R}^{ij}_k$) corresponding to all the interval observers. Each element of $\breve{\mathbf{R}}^i$ corresponds to one interval observer.

[4] At time instant k, R^j_k generally denotes residual zonotopes estimated by the j-th interval observer without caring about the modes.

Similarly, for the i-th mode, one defines the vector [5]

$$\mathbf{R}_k^i = (R_k^{i0},\, R_k^{i1},\, \cdots,\, R_k^{iN})$$

to denote all real-time residual zonotopes estimated by a bank of interval observers at time instant k. Thus, as k increases, each component of $\mathbf{R}_k^i$ will finally enter into that of $\breve{\mathbf{R}}^i$ and stay inside, i.e., $\mathbf{R}_k^i$ converges into $\breve{\mathbf{R}}^i$.

To summarize the discussions above, guaranteed FDI conditions are established in the following theorem.

Theorem 1. Given the plant (1), interval observers (3), Assumption 1, 2, 3 and 4, for any two modes l and m, if there exists at least one component (indexed by s) of $\breve{\mathbf{R}}^l$ and $\breve{\mathbf{R}}^m$ (s, l, $m \in \mathbb{I}$, $l \neq m$) such that

$$width(R_\infty^s) \nleq width(\Box(\breve{R}^{ls} \cap \breve{R}^{ms})), \tag{16}$$

where $\nleq$ is understood elementwise, then all the considered modes are distinguishable from each other.

Proof : If a mode l or m occurs and (16) holds, as k tends to infinity, $\mathbf{R}_k$ can finally enter in either $\breve{\mathbf{R}}^l$ or $\breve{\mathbf{R}}^m$, which identifies the mode. This implies that, if all the modes satisfy (16), by real-time testing the inclusion between $\mathbf{R}_k$ and all the candidate vectors $\breve{\mathbf{R}}^i$, the fault can be isolated by one and only one vector $\breve{\mathbf{R}}^i$ that contains $\mathbf{R}_k$. $\Box$

Remark 5. Since $\breve{R}^{ls} \cap \breve{R}^{ms}$ may not be a zonotope, the smallest box $\Box(\breve{R}^{ls} \cap \breve{R}^{ms})$ is used to replace it in (16). $\diamond$

Since R_∞^s, $\breve{R}^{ls}$ and $\breve{R}^{ms}$ in (16) can not be accurately computed but can be only approximated, Theorem 1 has only theoretical value. Practically, one uses the approximations of R_∞^s, $\breve{R}^{ls}$ and $\breve{R}^{ms}$ instead.

Thus, by iterating the expression of the segment matrix of residual zonotopes offline, one can obtain a sufficiently precise approximation denoted as $width(R_z^s)$ for $width(R_\infty^s)$, where z denotes the number of iterations. Furthermore, following (16), one can have the practical FDI conditions that are written as

$$width(R_z^s) \nleq width(\Box(\mathring{R}^{ls} \cap \mathring{R}^{ms})), \tag{17}$$

where

$$\mathring{R}^{ls} = C\mathring{X}^{ls} \oplus V \oplus (-V),$$
$$\mathring{R}^{ms} = C\mathring{X}^{ms} \oplus V \oplus (-V).$$

For some particular cases, one has another collection of simplified FDI conditions with respect to (17), which is

$$\mathring{R}^{ls} \cap \mathring{R}^{ms} = \emptyset, \tag{18}$$

meaning that all the corresponding components of $\mathring{R}^{ls}$ and $\mathring{R}^{ms}$ are separable from each other. As long as all the considered actuator modes satisfy either (17) or (18), all of them are detectable and isolable.

Note that the FDI conditions above are a set of sufficient conditions but not necessary. Their satisfaction guarantees FDI, while their violation does not mean that the faults are non-detectable or non-isolable through complementary computational efforts.

[5] $\mathbf{R}_k^i$ corresponding to the i-th mode is used for theoretical analysis. In practice, because residual zonotopes are obtainable in real time, $\mathbf{R}_k^i$ is rewritten as $\mathbf{R}_k = (R_k^0,\, R_k^1,\, \cdots,\, R_k^N)$ omitting the index of modes, where R_k^i denotes the residual zonotopes from the i-th interval observer at time instant k.

Remark 6. For FDI guarantees, the reduction of conservativeness of Theorem 1 is twofold. First, it is not necessary to assure one and only one interval observer can generate residual zonotopes that can contain the origin. Second, it is not necessary to guarantee, in a mode, that all residual-bounding zonotopes are disjoint at steady state. $\diamond$

4.4 Improved FDI Algorithm

The satisfaction of the FDI conditions (17) implies that residual zonotopes estimated by a bank of interval observers in different modes ultimately enter into different domains of the state space.

It is assumed that the current system is in the i-th mode, thus, a fault is detected at time instant k_d if the inclusions

$$\mathbf{R}_{k_d} \subseteq \mathring{\mathbf{R}}^i \text{ or } \mathbf{0} \in R_{k_d}^{ii},\ i \in \mathbb{I} \tag{19}$$

are violated, where the inclusion $\subseteq$ is understood elementwise, and $\mathbf{R}_{k_d}$ and $R_{k_d}^{ii}$ denote the vector of residual zonotopes estimated by a bank of interval observers and the residual zonotope estimated by the i-th interval observer at time instant k_d, respectively [6] .

As long as the inclusion of either of the two FD criteria in (19) is violated, it indicates that a new fault occurs. Otherwise, it is still considered that the system is healthy. In the on-line FD process, the FDI module selects the first FD decision out of the two in real time.

Remark 7. In (19), since the two FD criteria and all the interval observers are used for FD, comparing with the one-criterion and one-observer approach, this combination can be more sensitive to fault occurrences. $\diamond$

Algorithm 1 FDI algorithm

Require: $\hat{X}_0$, current mode $i \in \mathbb{I}$;
Ensure: Fault index f;
Initialize $N+1$ interval observers by $\hat{X}_0$;
At k: $\mathbf{R}_k \subseteq \mathring{\mathbf{R}}^i$, $\mathbf{0} \in R_k^{ii}$ and fault $\leftarrow$ FALSE;
while fault = FALSE **do**
  $k \leftarrow k+1$;
  Obtain $\mathbf{R}_k$;
  if $\mathbf{0} \notin R_k^{ii}$ or $\mathbf{R}_k \nsubseteq \mathring{\mathbf{R}}^i$ **then**
    fault $\leftarrow$ TRUE;
  end if
end while
while fault = TRUE **do**
  $k \leftarrow k+1$;
  Obtain $\mathbf{R}_k$;
  for $s \in \mathbb{I} \setminus \{i\}$ **do**
    if $\mathbf{R}_k \subseteq \mathring{\mathbf{R}}^s$ **then**
      if $\mathbf{0} \in R_k^{ss}$ **then**
        $f \leftarrow s$;
        fault $\leftarrow$ FALSE;
        terminate the algorithm;
      end if
    end if
  end for;
end while
return f;

[6] With respect to $\breve{\mathbf{R}}^i$, $\mathring{\mathbf{R}}^i$ is defined as $\mathring{\mathbf{R}}^i = (\mathring{R}^{i0},\, \mathring{R}^{i1},\, \cdots,\, \mathring{R}^{iN})$.

The FI strategy consists in searching a set vector $\mathring{\mathbf{R}}^f$ that $\mathbf{R}_k$ ultimately enters into after a fault is detected [7], where f denotes the index of a fault. Thus, at time instant k_i after FD ($k_i > k_d$), if both

$$\mathbf{R}_{k_i} \subseteq \mathring{\mathbf{R}}^f \text{ and } \mathbf{0} \in R^{ff}_{k_i},\ f \neq i,\ f \in \mathbb{I} \tag{20}$$

hold, it implies that the system is currently in the f-th mode, where the inclusion $\subseteq$ is understood elementwise.

Remark 8. In (20), $\mathbf{0} \in R^{ff}_{k_i}$ (interval observer-based FI principle) provides a guarantee for accuracy and reliability of FI decisions made by the FI criterion $\mathbf{R}_{k_i} \subseteq \mathring{\mathbf{R}}^f$ (invariant set-based FI principle). Thus, this combination can improve the reliability of the final FI decision. $\diamond$

In order to summarize the aforementioned results, Algorithm 1 is proposed in this paper for the FDI approach.

5. NUMERICAL EXAMPLE

The electric circuit example in [Ocampo-Martinez et al., 2010] illustrates the effectiveness of the approach. The performed simulations employed all the parameters in [Ocampo-Martinez et al., 2010], which are omitted here.

Since there are two actuators in the circuit, three actuator modes are considered in this example, which are denoted as F_0 (healthy), F_1 (outage of the first actuator) and F_2 (outage of the second actuator), i.e.,

$$F_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ F_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ F_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

With a sampling time 0.01s, the continuous dynamics of the circuit can be discretized into

$$A = \begin{bmatrix} 0.9804 & 0.3922 \\ -0.0002 & 1.0049 \end{bmatrix},\ B = \begin{bmatrix} 0.0196 & 0 \\ -0.0123 & 0.0125 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 \\ 0 & 20 \end{bmatrix},\ E = \begin{bmatrix} 0.0196 \\ 0.0002 \end{bmatrix}.$$

Based on the discrete-time model, three interval observers matching the three actuator modes are designed as in (3) and residual zonotopes are defined as in (5). For the discrete model and interval observers, the parameters are

- observer gains: $L_0 = L_1 = L_2 = \begin{bmatrix} 0.3334 & -0.8229 \\ 0.02 & 0.1333 \end{bmatrix}$,
- uncertainties: $\bar{\omega} = \begin{bmatrix} 0.0294 \\ 0.0004 \end{bmatrix}$, $\omega^c = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,
- initial state: $x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\hat{X}_0 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} \oplus \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \mathbf{B}^3$.

All RPI approximations of the limit sets of residual-bounding zonotopes (10) can be obtained after iterating (10) thirty steps with an initial RPI set of (12). For brevity, the numerical values of their boxes are presented here:

$\mathring{R}^{00} = ([-0.1965, 0.1965], [-0.1504, 0.1504])$,
$\mathring{R}^{01} = ([2.9722, 3.5611], [-63.4954, -59.5046])$,
$\mathring{R}^{02} = ([-0.1965, 0.1965], [60.4746, 64.5254])$,
$\mathring{R}^{10} = ([-3.5611, -2.9722], [59.5046, 63.4954])$,
$\mathring{R}^{11} = ([-0.1965, 0.1965], [-0.1504, 0.1504])$,
$\mathring{R}^{12} = ([-3.5611, -2.9722], [120.1296, 127.8704])$,
$\mathring{R}^{20} = ([-0.1965, 0.1965], [-64.5254, -60.4746])$,
$\mathring{R}^{21} = ([2.9726, 3.5607], [-127.8231, -120.1769])$,
$\mathring{R}^{22} = ([-0.1965, 0.1965], [-0.1504, 0.1504])$.

Since $\mathring{\mathbf{R}}^0 \cap \mathring{\mathbf{R}}^1$, $\mathring{\mathbf{R}}^0 \cap \mathring{\mathbf{R}}^2$ and $\mathring{\mathbf{R}}^1 \cap \mathring{\mathbf{R}}^2$ are empty (i.e., satisfying (18)), then the proposed technique can be applied to this example. The scenarios for both faults are defined as follows: from time instants 1 to 100, the system is healthy, and from time instants 101 to 200, a fault occurs. In Figure 1, Figure 2, Figure 3 and Figure 4, $R^j_k(n)$ and $\mathring{R}^{ij}_k(n)$ denote the n-th components of R^j_k and $\mathring{R}^{ij}_k$, respectively.

In the second plot of Figure 1, $0 \notin R^0_{103}(2)$ and $\mathbf{R}_{103} \nsubseteq \mathring{\mathbf{R}}^0$, which indicates a fault is detected at time instant 103. In Figure 2, both $\mathbf{0} \in R^1_{136}$ and $\mathbf{R}_{136} \subseteq \mathring{\mathbf{R}}^1$ are confirmed, which isolates the fault in the first actuator at time instant 136. Similarly, in the second figure of Figure 3, $0 \notin R^0_{103}(2)$ and $\mathbf{R}_{103} \nsubseteq \mathring{\mathbf{R}}^0$, which indicates a fault is detected at time instant 103. In Figure 4, at time instant 135, $\mathbf{0} \in R^2_{135}$ and $\mathbf{R}_{135} \subseteq \mathring{\mathbf{R}}^2$ are confirmed, which implies that the fault in the second actuator is isolated.

Notice that, $R^0_k(1)$ and $R^2_k(1)$, $\mathring{R}^{00}(1)$ and $\mathring{R}^{02}(1)$ in Figure 1, $R^0_k(1)$ and $R^2_k(1)$, $\mathring{R}^{10}(1)$ and $\mathring{R}^{12}(1)$ in Figure 2, $R^0_k(1)$ and $R^2_k(1)$, $\mathring{R}^{00}(1)$ and $\mathring{R}^{02}(1)$ in Figure 3, $R^0_k(1)$ and $R^2_k(1)$, $\mathring{R}^{20}(1)$ and $\mathring{R}^{22}(1)$ in Figure 4 coincide with each other, respectively. However, these coincidences do not affect the effectiveness of the proposed strategy.

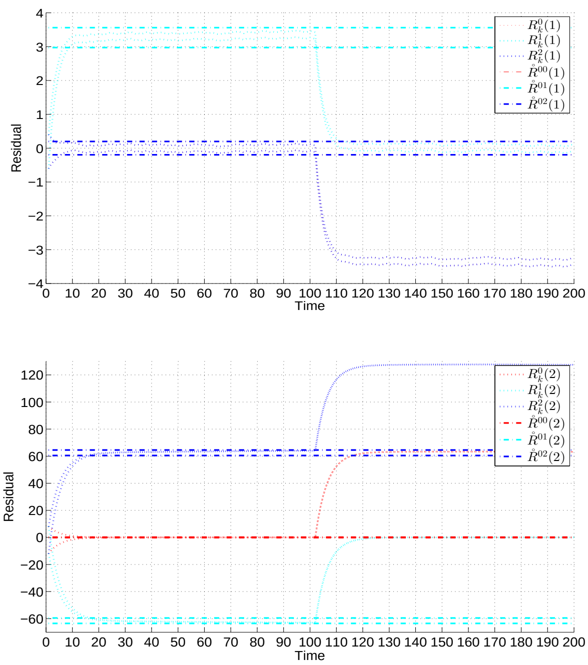

Fig. 1. FD of the fault 1

6. CONCLUSIONS

Comparing with the previous work, the algorithm proposed in this paper has at least three improvements. First,

[7] Before FI, the index f is unknown.

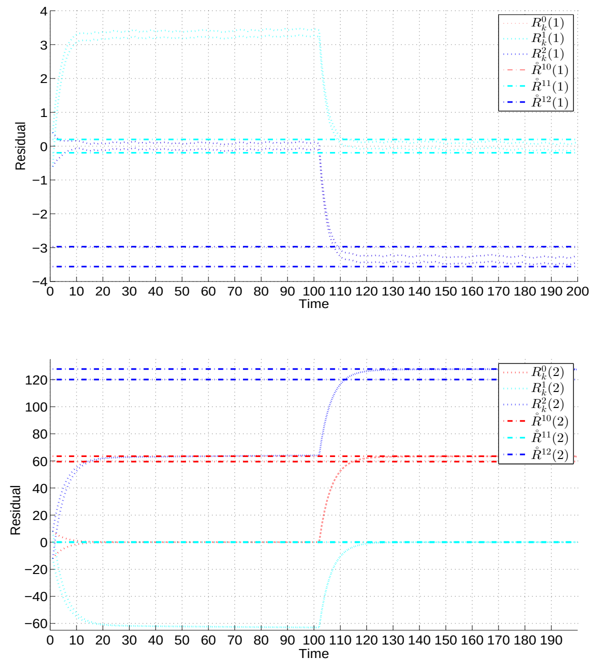

Fig. 2. FI of the fault 1

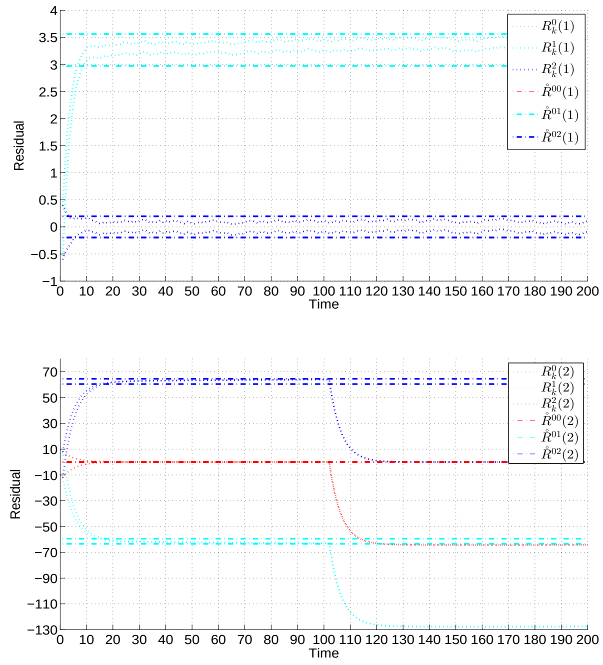

Fig. 3. FD of the fault 2

the system information provided by all interval observers is used for FDI. Second, both interval observers and invariant sets are used for FDI. Third, the explicit definition of a waiting time is avoided, instead, invariant sets are used to measure the waiting time implicitly.

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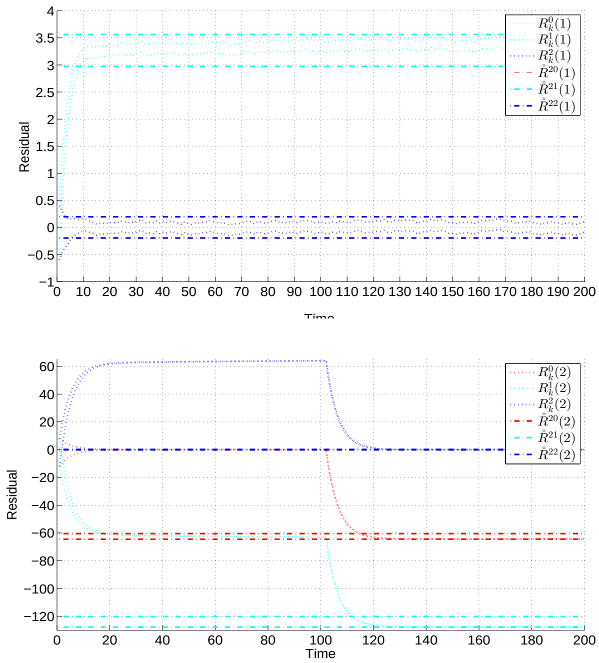

Fig. 4. FI of the fault 2

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