A robust $\mathcal{H}_\infty$ observer design for unknown input nonlinear systems: Application to fault diagnosis of a wind turbine*

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***Abstract*— The paper is devoted to the problem of designing robust unknown input observer (UIO) for fault estimation purpose. The proposed approach is based on the Takagi-Sugeno models which can be effectively applied for modelling of the wide class of nonlinear systems. It also revisits the recent results proposed in the literature and provides a less restrictive design procedure of a robust UIO. In particular, the general UIO strategy and the $\mathcal{H}_\infty$ framework are provided to design a robust fault estimation methodology. The resulting design procedure guarantees that a prescribed disturbance attenuation level is achieved with respect to the state estimation error. The main advantage of the proposed approach boils down to its simplicity because it reduces to solving a set of Linear Matrix Inequalities (LMIs). The final part of the paper presents an illustrative example devoted to the fault estimation of a three blade 1 MW variable-speed, variable-pitch wind turbine.**

I. INTRODUCTION

The problem of fault estimation, which can be also perceived as the estimation of an unknown input is widely discussed in the literature. Among many different strategies, a few of them deserve particular attention, namely: augmenting the state vector adding an unknown input [28], two–stage Kalman filter [13], minimum variance input and state estimator [7], adaptive estimation [31], sliding mode high-gain observers [25] and finally an $\mathcal{H}_\infty$ approaches [19]. Such approaches have significant practical meaning since they can be efficiently applied for fault diagnosis [11], [15], [18], [28] and Fault-Tolerant Control (FTC) [5], [17], [20], [21]. Indeed, fault estimation can be used to implement the three-step fault diagnosis procedure, which boils down to fault detection, isolation and identification [3]. Similarly, an efficient FTC is possible if and only if there is an information about the size of the fault. Without this knowledge appropriate compensation of the fault effect is impossible.

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In the paper, a novel robust fault estimation approach for nonlinear systems, which can be efficiently applied to the fault diagnosis procedure, is proposed. The developed method constitutes the extension of the general idea of an Unknown Input Observer (UIO) [6], [28]. Such an approach was initially designed to tolerate some level of a model inaccuracy in order to make a fault diagnosis more reliable [27]. It is worth to emphasis that the literature contains a numerous examples of designing of UIOs for the purpose of their applications in the model-based fault diagnosis [9], [10], [12], [28], [29]. However, it should be underlined that such methods can be effectively applied only for a narrow class of nonlinear systems e.g. Lipschitz or bilinear systems. In such case the problem of designing of novel robust UIO for the fault diagnosis purpose is still justified.

In particular, the main contribution of the paper is concerned with a novel design procedure of a robust UIO for fault estimation of nonlinear discrete-time systems modelled by Takagi-Sugeno (T-S) technique [22]. The paper revisits recent results in this emerging area [2] and provides less restrictive design procedure for the design of a robust UIO. The general UIO strategy and the $\mathcal{H}_\infty$ framework are provided to design a robust fault estimation scheme. The resulting design procedure guarantees that a prescribed disturbance attenuation level is achieved with respect to the state estimation error. The core advantage of the proposed approach boils down to its simplicity following from the fact that the problem of designing of robust UIO reduces to solving a set of LMIs. The effectiveness of the developed approach is shown on the example of fault estimation of a three blade 1 MW variable-speed, variable-pitch wind turbine.

II. PRELIMINARIES

A diagnosed system can be modelled by a T-S fuzzy model. It can be perceived as a series of locally linearised models from the nonlinear one. They can be obtained by the transformation of a nonlinear model of the diagnosed system by the application of the nonlinear sector approach [15], [22], [23]. In the case of the T-S fuzzy model a nonlinear dynamic system can be described by linear models, so-called fuzzy IF-THEN rules, representing the local system behaviour around some operating points. The proposed structure may represent a nonlinear system with control-affine state equation. Each of $i = 1, \ldots, M$ rules of the T-S model, which has p

antecedents, can be expressed as follows:

$$R^i : \text{IF } s_k^1 \text{ is } F_1^i \text{ and } \dots \text{ and } s_k^p \text{ is } F_p^i, \text{ THEN}$$

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}^i \boldsymbol{x}_k + \boldsymbol{B}^i \boldsymbol{u}_k + \boldsymbol{B}^i \boldsymbol{f}_k + \boldsymbol{W}_1^i \boldsymbol{w}_k, \tag{1}$$

$$\boldsymbol{y}_k = \boldsymbol{C}\boldsymbol{x}_k + \boldsymbol{W}_2^i \boldsymbol{w}_k, \tag{2}$$

where $\boldsymbol{x}_k \in \mathbb{R}^n$, $\boldsymbol{u}_k \in \mathbb{R}^r$ and $\boldsymbol{y}_k \in \mathbb{R}^m$ represent the state, nominal control input and system output, respectively. Moreover, $\boldsymbol{f}_k \in \mathbb{R}^r$ represents the actuator fault, F_j^i ($j = 1, \dots, p$) denotes fuzzy sets, $\boldsymbol{w}_k \in \mathbb{R}^m$ is a an exogenous disturbance vector and $\boldsymbol{s}_k = [s_k^1, s_k^2, \dots, s_k^p]^T$ is a known vector of so-called premise variables [15], [22].

Note that the fault $\boldsymbol{f}_k$ can be represented as unknown input along with its distribution matrices $\boldsymbol{B}^i$, $i = 1, \dots, M$, and hence, the design objective can be perceived as a twofold task:

- design of an UIO,
- fault estimation based on state estimates obtained by the UIO.

Let us start with a short review of recent results [2] regarding the first task. In the above approach, the unknown input is also present in the output equation (2), which may represent a sensor fault. Since the goal is to obtain the estimate of state and actuator fault, this term will be skipped in further deliberations. The T-S fuzzy models considered in [2] are:

$$\boldsymbol{x}_{k+1} = \sum_{i=1}^{M} h_i(\boldsymbol{s}_k)[\boldsymbol{A}^i \boldsymbol{x}_k + \boldsymbol{B}^i \boldsymbol{u}_k + \boldsymbol{B}^i \boldsymbol{f}_k + \boldsymbol{W}_1^i \boldsymbol{w}_k] \tag{3}$$

$$\boldsymbol{y}_k = \boldsymbol{C}\boldsymbol{x}_k + \boldsymbol{W}_2 \boldsymbol{w}_k, \tag{4}$$

where $h_i(\boldsymbol{s}_k)$ are normalized rule firing strengths which are defined as follows:

$$h_i(\boldsymbol{s}_k) = \frac{\mathcal{T}_{j=1}^p \mu_{F_j^i}(s_k^j)}{\sum_{i=1}^M (\mathcal{T}_{j=1}^p \mu_{F_j^i}(s_k^j))} \tag{5}$$

whereas $\mathcal{T}$ represents a t-norm (e.g., product). The expression $\mu_{F_j^i}(s_k^j)$ denotes the grade of membership of the premise variable s_k^j. Furthermore, it should be underlined that the expressions $h_i(\boldsymbol{s}_k)$ for ($i = 1, \dots, M$) satisfies:

$$\begin{cases} \sum_{i=1}^M h_i(\boldsymbol{s}_k) = 1, \\ 0 \leqslant h_i(\boldsymbol{s}_k) \leqslant 1, \qquad \forall i = 1, ..., M. \end{cases} \tag{6}$$

while the associated unknown input observer is

$$\boldsymbol{z}_{k+1} = \sum_{i=1}^{M} h_i(\boldsymbol{s}_k)[\boldsymbol{N}^i \boldsymbol{z}_k + \boldsymbol{G}^i \boldsymbol{u}_k + \boldsymbol{L}^i \boldsymbol{y}_k], \tag{7}$$

$$\hat{\boldsymbol{x}}_k = \boldsymbol{z}_k - \boldsymbol{E}\boldsymbol{y}_k. \tag{8}$$

Let us define the state estimation error $\boldsymbol{e}_k = \boldsymbol{x}_k - \hat{\boldsymbol{x}}_k$, which for (3)–(4) and (7)–(8) obeys:

$$\begin{aligned} \boldsymbol{e}_{k+1} = &\sum_{i=1}^{M} h_i(\boldsymbol{s}_k) \big[\boldsymbol{N}^i \boldsymbol{e}_k + (\boldsymbol{T}\boldsymbol{A}^i - \boldsymbol{K}^i \boldsymbol{C} - \boldsymbol{N}^i)\boldsymbol{x}_k + \\ &+ (\boldsymbol{T}\boldsymbol{B}^i - \boldsymbol{G}^i)\boldsymbol{u}_k + \boldsymbol{T}\boldsymbol{B}^i \boldsymbol{f}_k + \\ &(\boldsymbol{T}\boldsymbol{W}_1^i - \boldsymbol{K}^i \boldsymbol{W}_2)\boldsymbol{w}_k + \boldsymbol{E}\boldsymbol{W}_2 \boldsymbol{w}_{k+1}\big] \end{aligned} \tag{9}$$

with

$$\boldsymbol{T} = \boldsymbol{I} + \boldsymbol{E}\boldsymbol{C}, \quad \boldsymbol{K}^i = \boldsymbol{N}^i \boldsymbol{E} + \boldsymbol{L}^i, \tag{10}$$

which under

$$\boldsymbol{N}^i = \boldsymbol{T}\boldsymbol{A}^i - \boldsymbol{K}^i \boldsymbol{C}, \tag{11}$$

$$\boldsymbol{T}\boldsymbol{B}^i - \boldsymbol{G}^i = \boldsymbol{0}, \tag{12}$$

$$\boldsymbol{T}\boldsymbol{B}^i = \boldsymbol{0}, \tag{13}$$

$$\boldsymbol{E}\boldsymbol{W}_2 = \boldsymbol{0}, \tag{14}$$

$$\boldsymbol{T}\boldsymbol{W}_1^i - \boldsymbol{K}^i \boldsymbol{W}_2 = \boldsymbol{0} \tag{15}$$

boils down to

$$\boldsymbol{e}_{k+1} = \sum_{i=1}^{M} h_i(\boldsymbol{s}_k)\boldsymbol{N}^i \boldsymbol{e}_k. \tag{16}$$

Subsequently, the authors [2] show that the design procedure, which guarantees that $\boldsymbol{e}_k$ converges asymptotically to zero, can be reduced to solving a relatively simple set of LMIs.

Unfortunately, the proposed strategy has some serious limitations:

- by comparing (2) and (4), it is obvious that $\boldsymbol{W}_2^i = \boldsymbol{W}_2$, which means that these matrices are constant, and hence, the class of nonlinear systems that can be modeled with (3)–(4) is limited;
- robustness to external disturbances $\boldsymbol{w}_k$ is achieved by eliminating them from (9), which requires that (14)–(15) are satisfied. This can be realized under perfect knowledge about $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$, which is rather vain to expect in practice;
- no solution for estimating $\boldsymbol{f}_k$ is provided in [2].

In order to solve the above-mentioned difficulties, in the subsequent section of the paper a novel design procedure, which eliminates these drawbacks will be proposed.

III. A NOVEL DESIGN PROCEDURE

The UIO structure employed in this paper is given by (7)–(8), while the system (1)–(2) can be described by

$$\boldsymbol{x}_{k+1} = \sum_{i=1}^{M} h_i(\boldsymbol{s}_k)[\boldsymbol{A}^i \boldsymbol{x}_k + \boldsymbol{B}^i \boldsymbol{u}_k + \boldsymbol{B}^i \boldsymbol{f}_k + \boldsymbol{W}_1^i \boldsymbol{w}_k] \tag{17}$$

$$\boldsymbol{y}_k = \boldsymbol{C}\boldsymbol{x}_k + \sum_{i=1}^{M} h_i(\boldsymbol{s}_k)\boldsymbol{W}_2^i \boldsymbol{w}_k, \tag{18}$$

As the structure of the UIO is given, it is possible to provide its design procedure. As it was mentioned, instead of eliminating the effect of $\boldsymbol{w}_k$, its influence on the performance of the UIO will be minimized. For that purpose, let us assume that

$$l_2 = \left\{ \mathbf{w} \in \mathbb{R}^n | \ \|\mathbf{w}\|_{l_2} < +\infty \right\}, \tag{19}$$

$$\|\mathbf{w}\|_{l_2} = \left(\sum_{k=0}^{\infty} \|\boldsymbol{w}_k\|^2 \right)^{\frac{1}{2}}. \tag{20}$$

The objective of further deliberations is to design the observer (7)–(8) in order to assure that the state estimation error $\boldsymbol{e}_k = \boldsymbol{x}_k - \hat{\boldsymbol{x}}_k$ is asymptotically convergent and the following upper bound is guaranteed

$$\|\boldsymbol{e}\|_{l_2} \leq \omega \|\mathbf{w}\|_{l_2} \tag{21}$$

where $\omega > 0$ is a prescribed disturbance attenuation level.

Using (8) and (18), it can be shown that the state estimation error obeys

$$\boldsymbol{e}_k = \boldsymbol{T}\boldsymbol{x}_k - \boldsymbol{z}_k + \sum_{j=1}^{M} h_j(\boldsymbol{s}_k)\boldsymbol{E}\boldsymbol{W}_2^j \boldsymbol{w}_k, \tag{22}$$

where $\boldsymbol{T} = \boldsymbol{I} + \boldsymbol{E}\boldsymbol{C}$, while its evolution (determined with (22), (7)–(8), (17)–(18)) is described by

$$\boldsymbol{e}_{k+1} = \sum_{i=1}^{M} h_i(\boldsymbol{s}_k) \sum_{j=1}^{M} h_j(\boldsymbol{s}_k) \sum_{l=1}^{M} h_l(\boldsymbol{s}_{k+1})[\boldsymbol{N}^i \boldsymbol{e}_k + \\ + [\boldsymbol{T}\boldsymbol{B}^i - \boldsymbol{G}^i]\boldsymbol{u}_k + \boldsymbol{T}\boldsymbol{B}^i \boldsymbol{f}_k + \bar{\boldsymbol{W}}_1^{i,j}\boldsymbol{w}_k + \bar{\boldsymbol{W}}_2^l \boldsymbol{w}_{k+1}] \tag{23}$$

where

$$\boldsymbol{N}^i\boldsymbol{T} = \boldsymbol{T}\boldsymbol{A}^i - \boldsymbol{L}^i\boldsymbol{C}, \tag{24}$$

and $\bar{W}_1^{i,j} = TW_1^i - N^iEW_2^j - L^iW_2^j$ and $\bar{W}_2^l = EW_2^l$.

To decouple the fault effect from (23) it is evident that $\boldsymbol{T}$ should be calculated to satisfy

$$\boldsymbol{T}\boldsymbol{B}^i = [\boldsymbol{I} + \boldsymbol{E}\boldsymbol{C}]\boldsymbol{B}^i = \boldsymbol{0}, \quad i = 1, \ldots, M, \tag{25}$$

and, as a result, matrix $\boldsymbol{E}$ is obtained that is one of the UIO design parameters (cf. (8)). Note that the existence conditions of the solution to (25) are provided in [2]. Finally, the influence of the input on the estimation error can be eliminated by setting another UIO design parameter:

$$\boldsymbol{G}^i = \boldsymbol{T}\boldsymbol{B}^i, \quad i = 1, \ldots, M. \tag{26}$$

Thus, equation (23) boils down to

$$\boldsymbol{e}_{k+1} = \sum_{i=1}^{M} h_i(\boldsymbol{s}_k) \sum_{j=1}^{M} h_j(\boldsymbol{s}_k) \sum_{l=1}^{M} h_l(\boldsymbol{s}_{k+1})[\boldsymbol{N}^i \boldsymbol{e}_k + \\ + \bar{\boldsymbol{W}}_1^{i,j}\boldsymbol{w}_k + \bar{\boldsymbol{W}}_2^l \boldsymbol{w}_{k+1}], \tag{27}$$

Thus, on the contrary to the methods developed in [16], [30], disturbance attenuation level ω should be achieved with respect to the fault estimation error but not the state estimation error. Unfortunately, in the observer design approaches presented in the literature [16], [30] the state estimation error depends on $\boldsymbol{w}_k$ only, while (27) contains both $\boldsymbol{w}_k$ and $\boldsymbol{w}_{k+1}$. This situation requires a different approach that will be given by the following theorem, which constitutes the preliminary results of present Section.

Theorem 1: For a prescribed disturbance attenuation level $\omega > 0$ for the state estimation error (27), the $\mathcal{H}_\infty$ observer design problem for the system (17)–(18) and the observer (7)–(8) is solvable if there exist matrices $\boldsymbol{P}^i \succ \boldsymbol{0}$, $\boldsymbol{N}^i$, $\boldsymbol{L}^i$ $(i = 1, \ldots, M)$ and $\boldsymbol{U}$ such that the following constraints are satisfied

$$\sum_{j=1}^{M} h_j(\boldsymbol{s}_k) \sum_{i=1}^{M} h_i(\boldsymbol{s}_k) \sum_{l=1}^{M} h_l(\boldsymbol{s}_{k+1}) \Upsilon_{i,j}^l \prec \boldsymbol{0}, \tag{28}$$

where

$$\Upsilon_{i,j}^l = \begin{bmatrix} \boldsymbol{I} - \boldsymbol{P}^i & \boldsymbol{0} & \boldsymbol{0} & (\boldsymbol{N}^i)^T\boldsymbol{U}^T \\ \boldsymbol{0} & -\mu^2\boldsymbol{I} & \boldsymbol{0} & (\bar{\boldsymbol{W}}_1^{i,j})^T\boldsymbol{U}^T \\ \boldsymbol{0} & \boldsymbol{0} & -\mu^2\boldsymbol{I} & (\bar{\boldsymbol{W}}_2^i)^T\boldsymbol{U}^T \\ \boldsymbol{U}\boldsymbol{N}^i & \boldsymbol{U}\bar{\boldsymbol{W}}_1^{i,j} & \boldsymbol{U}\bar{\boldsymbol{W}}_2^i & \boldsymbol{P}^l - \boldsymbol{U} - \boldsymbol{U}^T \end{bmatrix} \tag{29}$$

and

$$\boldsymbol{N}^i\boldsymbol{T} = \boldsymbol{T}\boldsymbol{A}^i - \boldsymbol{L}^i\boldsymbol{C}, \quad i = 1, \ldots, M. \tag{30}$$

Proof. The task of $\mathcal{H}_\infty$ observer design [16], [30] relies on the determination of the gain matrix $\boldsymbol{K}^i$ such that

$$\lim_{k\to\infty} \boldsymbol{e}_0 = \boldsymbol{0} \quad \text{for } \boldsymbol{w}_k = \boldsymbol{0}, \tag{31}$$

$$\|\boldsymbol{e}\|_{l_2} \leq \omega\|\mathbf{w}\|_{l_2} \quad \text{for } \boldsymbol{w}_k \neq \boldsymbol{0},\ \boldsymbol{e}_0 = \boldsymbol{0}. \tag{32}$$

In this work it is proposed that is sufficient to obtain a Lyapunov function V_k for $k = 0, \ldots \infty$ for which the following inequality is fulfilled:

$$\Delta V_k + \boldsymbol{e}_k^T\boldsymbol{e}_k - \mu^2\boldsymbol{w}_k^T\boldsymbol{w}_k - \mu^2\boldsymbol{w}_{k+1}^T\boldsymbol{w}_{k+1} < 0, \tag{33}$$

where $\Delta V_k = V_{k+1} - V_k$, $\mu > 0$. Indeed, if $\boldsymbol{w}_k = \boldsymbol{0}$, $(k = 0, \ldots, \infty)$ then (33) boils down to

$$\Delta V_k + \boldsymbol{e}_k^T\boldsymbol{e}_k < 0,\ k = 0, \ldots \infty, \tag{34}$$

and hence $\Delta V_k < 0$, which leads to (31). If $\boldsymbol{w}_k \neq \boldsymbol{0}$ $(k = 0, \ldots, \infty)$, then (33) yields

$$J = \sum_{k=0}^{\infty}\left(\Delta V_k + \boldsymbol{e}_k^T\boldsymbol{e}_k - \mu^2\boldsymbol{w}_k^T\boldsymbol{w}_k - \mu^2\boldsymbol{w}_{k+1}^T\boldsymbol{w}_{k+1}\right) < 0 \tag{35}$$

which can be rewritten as follows:

$$J = -V_0 + \sum_{k=0}^{\infty}\boldsymbol{e}_k^T\boldsymbol{e}_k - \mu^2\sum_{k=0}^{\infty}\boldsymbol{w}_k^T\boldsymbol{w}_k - \mu^2\sum_{k=0}^{\infty}\boldsymbol{w}_{k+1}^T\boldsymbol{w}_{k+1} < 0 \tag{36}$$

Bearing in mind that

$$\mu^2\sum_{k=0}^{\infty}\boldsymbol{w}_{k+1}^T\boldsymbol{w}_{k+1} = \mu^2\sum_{k=0}^{\infty}\boldsymbol{w}_k^T\boldsymbol{w}_k - \mu^2\boldsymbol{w}_0^T\boldsymbol{w}_0, \tag{37}$$

the inequality (36) can be written as

$$J = -V_0 + \sum_{k=0}^{\infty}\boldsymbol{e}_k^T\boldsymbol{e}_k - 2\mu^2\sum_{k=0}^{\infty}\boldsymbol{w}_k^T\boldsymbol{w}_k + \mu^2\boldsymbol{w}_0^T\boldsymbol{w}_0 < 0 \tag{38}$$

Knowing that $V_0 = 0$ for $\boldsymbol{e}_0 = 0$, (38) leads to (32) with $\omega = \sqrt{2}\mu$.

Since the general framework for designing the robust observer is given, then the following form of the Lyapunov function is proposed:

$$V_k = \sum_{i=1}^{M} h_i(\boldsymbol{s}_k)\boldsymbol{e}_k^T\boldsymbol{P}^i\boldsymbol{e}_k, \quad \boldsymbol{P}^i \succ \boldsymbol{0}. \tag{39}$$

Thus, by defining $\boldsymbol{v}_k = [\boldsymbol{e}_k^T, \boldsymbol{w}_k^T, \boldsymbol{w}_{k+1}^T]^T$ it can be shown that the condition (33) is equivalent to

$$\sum_{j=1}^{M} h_j(\boldsymbol{s}_k) \sum_{i=1}^{M} h_i(\boldsymbol{s}_k) \sum_{l=1}^{M} h_l(\boldsymbol{s}_{k+1})\boldsymbol{v}_k^T\Phi_{i,j}^l\boldsymbol{v}_k < 0 \tag{40}$$

where $\Phi^l_{i,j}$ is:

$$\begin{bmatrix} (\boldsymbol{N}^i)^T\boldsymbol{P}^l\boldsymbol{N}^i+\boldsymbol{I}-\boldsymbol{P}^i & (\boldsymbol{N}^i)^T\boldsymbol{P}^l\bar{\boldsymbol{W}}_1^{i,j} & (\boldsymbol{N}^i)^T\boldsymbol{P}^l\bar{\boldsymbol{W}}_2^l \\ (\bar{\boldsymbol{W}}_1^{i,j})^T\boldsymbol{P}^l\boldsymbol{N}^i & (\bar{\boldsymbol{W}}_1^{i,j})^T\boldsymbol{P}^l\bar{\boldsymbol{W}}_1^{i,j}-\mu^2\boldsymbol{I} & (\bar{\boldsymbol{W}}_1^{i,j})^T\boldsymbol{P}^l\bar{\boldsymbol{W}}_2^l \\ (\bar{\boldsymbol{W}}_2^l)^T\boldsymbol{P}^l\boldsymbol{N}_i & (\bar{\boldsymbol{W}}_2^l)^T\boldsymbol{P}^l\bar{\boldsymbol{W}}_1^{i,j} & (\bar{\boldsymbol{W}}_2^l)^T\boldsymbol{P}^l\bar{\boldsymbol{W}}_2^l-\mu^2\boldsymbol{I} \end{bmatrix} \tag{41}$$

Let us remind the following lemma [4]:

Lemma 1: The undermentioned statements are equivalent

1) There exists $\boldsymbol{X} \succ 0$ such that

$$\boldsymbol{V}^T\boldsymbol{X}\boldsymbol{V}-\boldsymbol{W} \prec 0 \tag{42}$$

2) There exists $\boldsymbol{X} \succ 0$ such that

$$\begin{bmatrix} -\boldsymbol{W} & \boldsymbol{V}^T\boldsymbol{U}^T \\ \boldsymbol{U}\boldsymbol{V} & \boldsymbol{X}-\boldsymbol{U}-\boldsymbol{U}^T \end{bmatrix} \prec 0. \tag{43}$$

Applying Lemma 1 to (41) leads to (29), which completes the proof. Note that (28) requires further relaxation procedure in order to be tractable within the effective LMI framework. A basic sufficient solution to this problem were described in [26] and further improved by many researchers [8] and the references therein. As indicated in [8], the conditions provided by [24] lead to a good compromise between conservatism and complexity, which in the case (28) leads to the following lemma:

Lemma 2: Condition (28) is fulfilled providing the following conditions hold:

$$\Upsilon^l_{i,i} \prec \boldsymbol{0},\ i \in \{1,\dots,M\}, \tag{44}$$

$$\frac{2}{M-1}\Upsilon^l_{i,i}+\Upsilon^l_{i,j}+\Upsilon^l_{j,i} \prec \boldsymbol{0},\ i,j,l \in \{1,\dots,M\},\ i \neq j \tag{45}$$

IV. Fault estimation strategy

Since the design procedure of the UIO is provided, it is possible to propose a fault estimation strategy that is based on the obtained state estimates. For that purpose, let us rewrite the system (17)–(18):

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}(\boldsymbol{s}_k)\boldsymbol{x}_k+\boldsymbol{B}(\boldsymbol{s}_k)\boldsymbol{u}_k+\boldsymbol{B}(\boldsymbol{s}_k)\boldsymbol{f}_k+ \\ +\boldsymbol{W}_1(\boldsymbol{s}_k)\boldsymbol{w}_k, \tag{46}$$

$$\boldsymbol{y}_{k+1} = \boldsymbol{C}\boldsymbol{x}_{k+1}+\boldsymbol{W}_2(\boldsymbol{s}_{k+1})\boldsymbol{w}_{k+1}, \tag{47}$$

Following [7], [28], to obtain $\boldsymbol{f}_k$ from (46)–(47) it is necessary to satisfy

$$\text{rank}(\boldsymbol{C}\boldsymbol{B}(\boldsymbol{s}_k)) = \text{rank}(\boldsymbol{B}(\boldsymbol{s}_k)) = r, \tag{48}$$

The problem boils down to checking the full rank property of all convex combinations of matrixes $\boldsymbol{B}^i, i=1,\dots,M$ as well as $\boldsymbol{C}\boldsymbol{B}^i$ $i=1,\dots,M$. Let us start with checking the first condition concerning the property of $\boldsymbol{B}^i, i=1,\dots,M$. Note that the task of checking the full rank property of $\boldsymbol{C}\boldsymbol{B}^i$ $i=1,\dots,M$, can be realized in the same way.

Let us also assume that the system is observable and the matrix $\boldsymbol{B}^M$ is a full rank one (if not, it can be already concluded that the full rank property does not hold). Let us define

$$\boldsymbol{Q}_{p,p} = \boldsymbol{B}^{pT}\boldsymbol{B}^p, \qquad p=1,\dots,M \tag{49}$$

$$\boldsymbol{Q}_{p,a} = \boldsymbol{B}^{pT}\boldsymbol{B}^a+\boldsymbol{B}^{aT}\boldsymbol{B}^p- \\ \boldsymbol{B}^{aT}\boldsymbol{B}^a-\boldsymbol{B}^{pT}\boldsymbol{B}^p \qquad \text{for } p<a \tag{50}$$

$$\boldsymbol{R}^p_{a,b} = \begin{cases} \boldsymbol{Q}_{p,p} & \text{if } (a,b)=(1,1) \\ \boldsymbol{Q}_{b-1,p} & \text{if } a=1 \wedge b=2,\dots,p \\ \boldsymbol{I} & \text{if } a=b \wedge 1<b<k \\ -\boldsymbol{I} & \text{if } b=1 \wedge a=p+1 \\ \boldsymbol{0} & \text{otherwise} \end{cases} \tag{51}$$

Theorem 2: The undermentioned statements are equivalent

(a) All convex combinations of $\boldsymbol{B}^1,\dots,\boldsymbol{B}^M$ have full rank.

(b) $\boldsymbol{B}^M$ has full row rank and the $(M-1)Mn$-by-$(M-1)Mn$ matrix.

$$\boldsymbol{V} = \begin{bmatrix} \boldsymbol{R}_1\boldsymbol{R}_M^{-1} & \boldsymbol{V}_{1,2} & \boldsymbol{V}_{1,3} & \dots & \boldsymbol{V}_{1,4} \\ -\boldsymbol{I}_{Mn} & \boldsymbol{I}_{Mn} & \boldsymbol{0}_{Mn} & \dots & \boldsymbol{0}_{Mn} \\ \boldsymbol{0}_{Mn} & -\boldsymbol{I}_{Mn} & \boldsymbol{I}_{Mn} & \dots & \boldsymbol{0}_{Mn} \\ \dots & \dots & \dots & \dots & \dots \\ \boldsymbol{0}_{Mn} & \dots & \boldsymbol{0}_{Mn} & -\boldsymbol{I}_{Mn} & \boldsymbol{I}_{Mn} \end{bmatrix} \tag{52}$$

where $\boldsymbol{V}_{1,2} = (\boldsymbol{R}_2-\boldsymbol{R}_1)\boldsymbol{R}_M^{-1}$, $\boldsymbol{V}_{1,3} = (\boldsymbol{R}_3-\boldsymbol{R}_2)\boldsymbol{R}_M^{-1}$ and $\boldsymbol{V}_{1,4} = (\boldsymbol{R}_{M-1}-\boldsymbol{R}_{M-2})\boldsymbol{R}_M^{-1}$ is a block P-matrix [14] with respect to the partition $\{\boldsymbol{F}_1,\dots,\boldsymbol{F}_{M-1}\}$ of $\{1,\dots,(M-1)Mn\}$, with $\boldsymbol{F}_i = \{(M-1)Mn+1,\dots,iMn\}$, $i=1,\dots,M-1$.

Proof. Proof can be derived by a direct application of *Theorem 2* given in [14].

Remark 1: Following [14], a sufficient condition for a real matrix V to be block P-matrix with respect to any partition is that all its principal minors are positive. This feature makes it possible to easily check the condition of Theorem 2

Since the practical way of verifying (48) is provided, it is possible to calculate

$$\boldsymbol{H}(\boldsymbol{s}_k) = \left(\sum_{i=1}^{M} h_i(\boldsymbol{s}_k)\boldsymbol{C}\boldsymbol{B}^i\right)^+. \tag{53}$$

where expression $(\cdot)^+$ stands for the left inverse of its argument. Multiplying (47) by expression $\boldsymbol{H}(\boldsymbol{s}_k)$, and substituting (46), the following equation defining the fault $\boldsymbol{f}_k$ can be obtained

$$\boldsymbol{f}_k = \boldsymbol{H}(\boldsymbol{s}_k)[\boldsymbol{y}_{k+1}-\boldsymbol{C}\boldsymbol{A}(\boldsymbol{s}_k)\boldsymbol{x}_k+ \\ -\boldsymbol{C}\boldsymbol{W}_1(\boldsymbol{s}_k)\boldsymbol{w}_k-\boldsymbol{W}_2(\boldsymbol{s}_{k+1})\boldsymbol{w}_{k+1}]-\boldsymbol{u}_k. \tag{54}$$

Thus, the fault estimate is given by

$$\hat{\boldsymbol{f}}_k = \boldsymbol{H}(\boldsymbol{s}_k,\boldsymbol{s}_{k+1})[\boldsymbol{y}_{k+1}-\boldsymbol{C}(\boldsymbol{s}_{k+1})\boldsymbol{A}(\boldsymbol{s}_k)\hat{\boldsymbol{x}}_k]-\boldsymbol{u}_k. \tag{55}$$

where $\hat{\boldsymbol{x}}_k$ is the state estimated with the UIO (7)–(8).

V. Illustrative example

Let us consider a three blade 1 MW variable-speed, variable-pitch wind turbine [1], which is nonlinear MIMO system. To derive a T-S model, the nonlinear functions of pitch angle and wind speed were linearized in the extreme points $[\boldsymbol{\beta}_{min},\boldsymbol{\nu}_{min}]$ and $[\boldsymbol{\beta}_{max},\boldsymbol{\nu}_{max}]$ resulting in a T-S model (a detailed description is given in [1]). For sampling time $\boldsymbol{T}_s = 0.01$ s a T-S fuzzy discrete-time model of Wind Energy Conversion System (WECS) [1] was obtained. Where the state vector $\boldsymbol{x} = [\boldsymbol{\theta}_s,\boldsymbol{\Omega}_g,\boldsymbol{\Omega}_r,\boldsymbol{\beta}]^T$ is in turn torsion angle, angular velocity of generator, angular velocity of rotor and actuation of pitch received from pitch controller. While the inputs of the system are defined as $\boldsymbol{u} = [\boldsymbol{\beta}_d, \boldsymbol{\omega}_z, \mathrm{v}]^T$, where $\boldsymbol{\beta}_d$ stands for demanded control action from pitch controller, $\boldsymbol{\omega}_z$ denotes the applied control action to the electromechanical system, while v represents the wind speed varying from 0 to 25 m/s. The system meets condition (48) and for a given attenuation level $\mu = 0.38$ and decay rate $\tau = 0.9$ the T-S UIO was obtained.

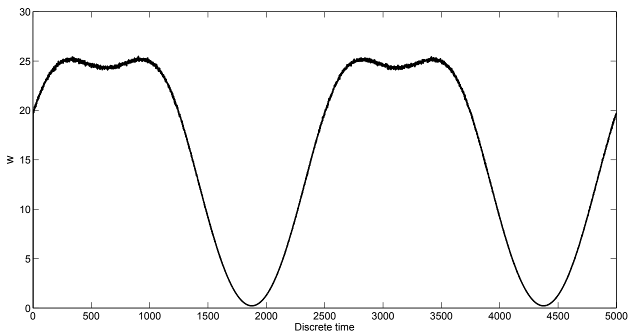

Fig. 1. Function of wind speed input w_k

The analysis starts from fault-free scenario ($\boldsymbol{f}_k = 0$) and input is given by $\boldsymbol{u} = [2.5, 0, \mathrm{v}]$ where initial conditions of the observer and system are $\hat{\boldsymbol{x}}_0 = \boldsymbol{x}_0$, $\boldsymbol{x}_0 = [0, 1, 1, 0]^T$, while exogenous disturbance input $\boldsymbol{w}_k \sim 0.01\,\mathcal{N}(0, 0.1^2\boldsymbol{I})$. Figure 1 portrays the evolution of the wind. Having all parameters of the system it is possible to compute (35), which is depicted in Fig. 2. This results clearly indicates that the prescribed disturbance attenuation level is achieved. Second

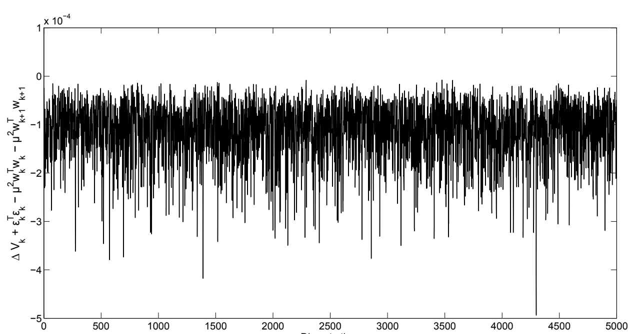

Fig. 2. Evolution of $\Delta V_k + \boldsymbol{\varepsilon}_{f,k}^T \boldsymbol{\varepsilon}_{f,k} - \mu^2 \boldsymbol{w}_k^T \boldsymbol{w}_k - \mu^2 \boldsymbol{w}_{k+1}^T \boldsymbol{w}_{k+1}$

scenario is performed for $\boldsymbol{w}_k = 0$ and $\boldsymbol{x}_0 \neq \hat{\boldsymbol{x}}_0$. The results are presented in Fig. 3 and shows the convergence of the observer. As it can be seen, the state estimation error vanishes to zero very quickly. Since the fault-free performance of the UIO is verified, it is possible to proceed to the faulty scenario. For this purpose, a fault related to the third input was simulated. This fault can be perceived as a wind sensor fault, and hence the sensor provides wrong measurements into the system. This scenario has three different phases,

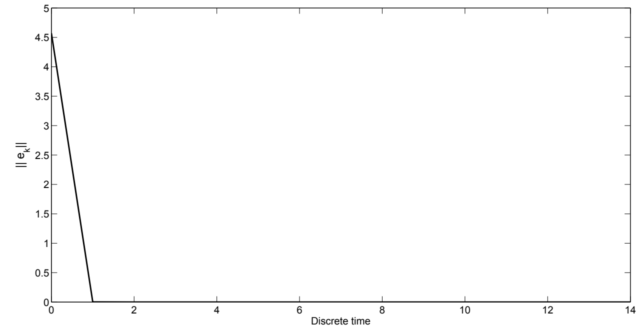

Fig. 3. Evolution of $\|\boldsymbol{e}_k\|$ (for $k = 0, \ldots, 14$)

which are defined as follows:

$$\boldsymbol{f}_{3,k} = \begin{cases} -0.001, & \text{for } 1000 \geq k \geq 1200, \\ 0.5, & \text{for } 2000 \geq k \geq 3000, \\ -2, & \text{for } 1500 \geq k \geq 1700, \\ 0, & \text{otherwise.} \end{cases} \tag{56}$$

Figure (4) presents state x_2 and its estimate. As can be seen,

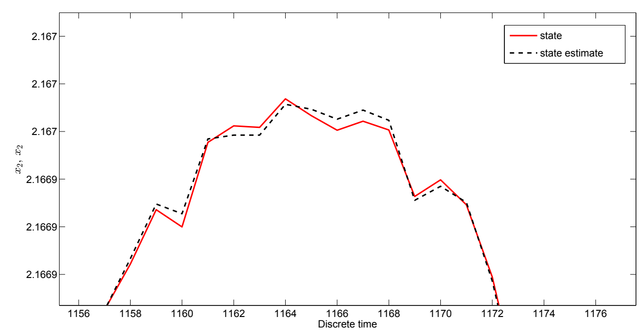

Fig. 4. State $\boldsymbol{x}_2$ (solid line) and its estimate $\hat{\boldsymbol{x}}_2$ (dotted line)

the UIO is tracking the inaccessible state with a very good quality. Faults and their estimates are presented in Figs. 5, 6, and 7, respectively. The obtained results clearly indicate

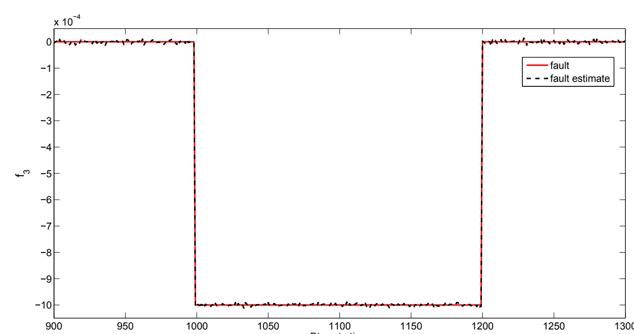

Fig. 5. Scenario 1: Fault (solid line) and its estimate $\hat{\boldsymbol{f}}_1$ (dotted line)

that the faults were obtained with a high precision, which confirms that the proposed procedure can be applied to highly nonlinear systems such as WECS.

VI. Conclusions

The primary goal of the paper was to provide a new design procedure for an UIO for T-S fuzzy systems, and

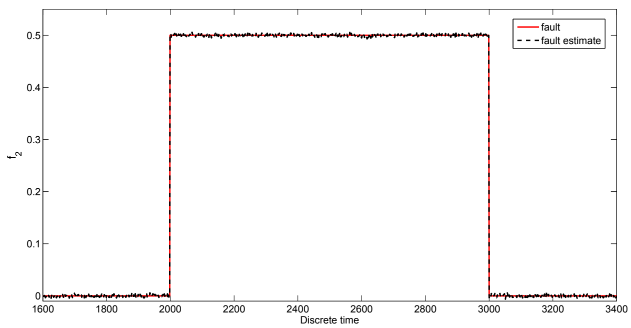

Fig. 6. Scenario 2: Fault (solid line) and its estimate $\hat{f}_1$ (dotted line)

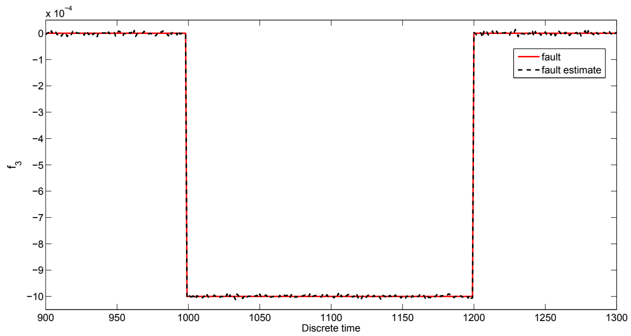

Fig. 7. Scenario 3: Fault (solid line) and its estimate $\hat{f}_1$ (dotted line)

to show how to use it for the purpose of fault estimation. The proposed procedure can be used to design a fault identification scheme in such a way that a prescribed disturbance attenuation level is achieved with respect to the state estimation error, while guaranteeing the convergence of the observer. Moreover, the developed methodology is less restrictive than the recent developments presented in the literature. The effectiveness of the proposed approach was shown on example of the fault estimation of a three blade 1 MW variable-speed, variable-pitch wind turbine.

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