A Payoff Dynamics Model for Equality-Constrained Population Games

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***Abstract*— This brief proposes a novel form of continuous-time evolutionary game dynamics for generalized Nash equilibrium seeking in equality-constrained population games. Using Lyapunov stability theory and duality theory, we provide sufficient conditions to guarantee the asymptotic stability, non-emptiness, compactness, and optimality of the equilibria set of the proposed dynamics for certain population games. Moreover, we illustrate our theoretical developments through a numerical simulation of an equality-constrained congestion game.**

I. Introduction

Population games provide an evolutionary game theoretical framework to describe the strategic interaction of a large population of players [1], [2]. Under this framework, the players are regarded as non-cooperative decision-makers that often in time are allowed to select which strategy to play. As such, the ideas of population games have received significant attention from the control community [3]. For instance, the authors in [4] exploit the theory of population games for the dynamic resource allocation of a water distribution system; the authors in [5] apply the framework of population games to design a real-time demand response controller for a smart grid; and the authors in [6] extend the conventional framework of population games of [2] to consider dynamic payoff mechanisms, also referred to as payoff dynamics models [6], [7], which allow the consideration of a wider spectrum of strategic interaction scenarios. In this paper, we explore this latter idea of payoff dynamics models [6] to design some continuous-time evolutionary game dynamics for generalized Nash equilibrium seeking in population games.

In the field of game theory, generalized Nash equilibrium seeking usually refers to the problem of achieving Nash equilibria that satisfy a set of constraints. More precisely, the decision-making process of the players is not only influenced by the payoffs of the game, but also by certain constraints that must be satisfied at any equilibrium of the game. Although several recent works have addressed such a problem from different perspectives [8], [9], [10], [11], to the best of our knowledge, limited attention has been given to such a problem from the aforementioned perspective of population games [2]. Some exceptions are the works in [12], [13], [14], [15]. Namely, the authors in [12], [13] propose a set of decision-making protocols to include some affine inequality constraints in the players' decisions; the authors in [14] propose some evolutionary game dynamics to include affine equality constraints within the game; and, finally, the authors in [15] propose a form of payoff dynamics models [6] for generalized Nash equilibrium seeking in population games under convex inequality constraints.

Motivated by the approach in [15], in this paper we propose a novel form of payoff dynamics models for generalized Nash equilibrium seeking in population games under affine equality constraints. Notice that such constraints are relevant for several practical engineering and control applications. Some examples include frequency regulation in power systems [16], automatic generation control in power networks [17], and dynamic resource allocation in water distribution systems [14], among others. Compared against the previous works on generalized Nash equilibrium seeking in population games, our proposed approach has the following novelties. In contrast with [12], [13], and [15], our method considers equality constraints instead of inequality constraints. Note that, although it is possible to eliminate explicit affine equality constraints through a change of variables [18, Section 4.2.4], the direct treatment of equality constraints might have practical advantages due to the preservation of structural properties of the problem. Moreover, such affine transformations would typically imply higher rationality levels on the players, which is undesirable under the considered population games framework of [2]. In contrast with [14], on the other hand, our method preserves the property of Nash stationarity. More precisely, the equilibria set of our dynamics coincides exactly with the set of generalized Nash equilibria of the underlying game. Such a property does not hold under the approach of [14].

Consequently, the main contribution of this paper is the formulation and analysis of a novel form of payoff dynamics model, which allows the consideration of affine equality constraints in population games. In particular, we provide sufficient conditions to guarantee the asymptotic stability and optimality of the equilibria set of the dynamics for certain classes of population games. Additionally, we provide sufficient conditions to guarantee the non-emptiness and compactness of the equilibria set of the proposed dynamics. Furthermore, our theoretical developments are illustrated through a numerical simulation of a classical congestion game that has been extended to include equality constraints. Such a game resembles several relevant engineering applica-

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tions in the context of dynamic resource allocation [3].

II. Considered Framework and Proposed Model

Consider a (large) population of players that are engaged in a game with a set of strategies given by $\mathcal{S} = \{1, 2, \ldots, n\}$, where $n \in \mathbb{Z}_{\geq 2}$. At any time, the portion of players playing the strategy $i \in \mathcal{S}$ is denoted as $x_i \in \mathbb{R}_{\geq 0}$, and the state of the population is given by the vector $\mathbf{x} = [x_i] \in \mathbb{R}^n_{\geq 0}$. Consequently, the set of all possible population states is $\Delta = \left\{\mathbf{x} \in \mathbb{R}^n_{\geq 0} \,:\, \mathbf{1}^\top \mathbf{x} = m\right\}$, where $\mathbf{1}$ is the vector of ones with appropriate dimension; and $m \in \mathbb{R}_{>0}$ is the total population mass (which is assumed constant). Moreover, each strategy $i \in \mathcal{S}$ has an associated fitness function, $f_i : \mathbb{R}^n \to \mathbb{R}$, which provides the (baseline) payoff for the strategy $i \in \mathcal{S}$ at a given population state. Hence, the population game is completely characterized by the vector $\mathbf{f}(\cdot) = [f_i(\cdot)] \in \mathbb{R}^n$, i.e., $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$. Throughout, we refer to $\mathbf{f}(\cdot)$ as the baseline population game. Notice that depending on the form of $\mathbf{f}(\cdot)$, a different class of game might be considered. Below, we present two classes of games that are relevant for our theoretical developments.

Definition 1 ([2]): The game $\mathbf{f} : \Delta \to \mathbb{R}^n$ is a stable game if $(\mathbf{z} - \mathbf{x})^\top (\mathbf{f}(\mathbf{z}) - \mathbf{f}(\mathbf{x})) \leq 0$, for all $\mathbf{x}, \mathbf{z} \in \Delta$.

Definition 2 ([2]): The game $\mathbf{f} : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n$ is a full-potential game if there exists a continuously differentiable (potential) function $\varphi : \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ such that $\nabla_{\mathbf{x}} \varphi(\mathbf{x}) = \mathbf{f}(\mathbf{x})$, for all $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$, i.e., $\partial \varphi(\cdot)/\partial x_i = f_i(\cdot)$, for all $i \in \mathcal{S}$.

Remark 1: Notice that every full-potential game with concave potential function $\varphi(\cdot)$ is also a stable game.

In the context of population games [1], [2], it is often assumed that, as strategic decision makers, the population players seek to play the strategies that lead to the highest payoff of the baseline game $\mathbf{f}(\cdot)$. Nevertheless, similar to [15], in this paper we assume that there is also a set of constraints that the players must satisfy while playing the baseline game $\mathbf{f}(\cdot)$. In contrast with [15], however, here we consider equality constraints instead of inequality constraints. More precisely, a population state $\mathbf{x} \in \Delta$ is said to be feasible if and only if $\mathbf{x} \in \Delta \cap \mathcal{Q}$, where $\mathcal{Q} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}\}$. Here, $\mathbf{A} \in \mathbb{R}^{q \times n}$; $\mathbf{b} \in \mathbb{R}^q$; and $q \in \mathbb{Z}_{\geq 1}$. To ease the forthcoming discussions, we let $\mathcal{C} = \{1, 2, \ldots, q\}$ be the set of indices of the equality constraints, and, for all $k \in \mathcal{C}$, we let $\mathbf{a}_k^\top = [a_{k1}, a_{k2}, \ldots, a_{kn}] \in \mathbb{R}^{1 \times n}$ and $b_k \in \mathbb{R}$ denote the k-th row and the k-th element of $\mathbf{A}$ and $\mathbf{b}$, respectively. Namely, $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_q]^\top$, and $\mathbf{b} = [b_1, b_2, \cdots, b_q]^\top$. Besides, we impose the following assumption on the equality constraints.

Standing Assumption 1: The set $\Delta \cap \mathcal{Q}$ is non-empty, and the matrix $\hat{\mathbf{A}} = [\mathbf{1}, \mathbf{a}_1, \cdots, \mathbf{a}_q]^\top \in \mathbb{R}^{(q+1) \times n}$ is full row rank (hence, $q \leq n - 1$).

Motivated by the ideas in [6], we consider that there is a higher level entity that provides the payoff signals to the population players (in engineering applications such an entity could be, for instance, an aggregator node). The goal of such an entity is to augment the baseline game $\mathbf{f}(\cdot)$ to include the constraints of $\mathcal{Q}$. For this reason, we propose the payoff dynamics model given by

$$\dot{y}_k = \mathbf{a}_k^\top \mathbf{x} - b_k, \quad \forall k \in \mathcal{C} \tag{1a}$$

$$p_i(\mathbf{x}, \mathbf{y}) = f_i(\mathbf{x}) - \sum_{k \in \mathcal{C}} y_k a_{ki}, \quad \forall i \in \mathcal{S}, \tag{1b}$$

and we refer to $\mathbf{p}(\cdot, \cdot) = [p_i(\cdot, \cdot)] \in \mathbb{R}^n$ as the augmented game, i.e., $\mathbf{p} : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$. By augmented we mean that the game $\mathbf{p}(\cdot, \cdot)$ considers both the baseline game $\mathbf{f}(\cdot)$ as well as the constraints of $\mathcal{Q}$.

On the other hand, following the ideas in [2], we assume that the players are equipped with a revision protocol to revise their strategies. Such a revision protocol determines the incentives to switch from one strategy to another. In this paper, we assume that, for a player playing the strategy $i \in \mathcal{S}$, the incentive to switch to the strategy $j \in \mathcal{S}$ is given by $[p_j - p_i]_+$, where $[\cdot]_+ \triangleq \max(\cdot, 0)$; and $p_i \triangleq p_i(\mathbf{x}, \mathbf{y})$, for all $i \in \mathcal{S}$. Consequently, the evolutionary dynamics model that describes the trajectory of the population state is

$$\dot{x}_i = \sum_{j \in \mathcal{S}} \Big(x_j \left[p_i - p_j\right]_+ - x_i \left[p_j - p_i\right]_+ \Big), \quad \forall i \in \mathcal{S}. \tag{2}$$

The evolutionary dynamics in (2) are well-known in the literature as the Smith dynamics [19], and a thorough deduction of these dynamics can be found in [2]. In this paper, we focus on the Smith dynamics to ease the exposition of our developments. Nevertheless, it is worth to highlight that all of the theoretical results of this paper are readily applicable to the broader family of dynamics that results from any impartial pairwise comparison revision protocol [6]. The reader might verify this claim by applying some mild modifications to our provided proofs.

As shown in Fig. 1, observe that the payoff dynamics model in (1) and the evolutionary dynamics model in (2) are interconnected in a positive feedback loop structure forming an $(n + q)$-dimensional system with state vector $[\mathbf{x}^\top, \mathbf{y}^\top]^\top$. Namely, the players receive the (augmented) payoff signal $\mathbf{p}$, determined by the payoff dynamics model in (1) (higher level entity), and the population state evolves according to the dynamics in (2). Moreover, as illustrated in Fig. 1, for the remainder of this paper we impose the following assumption on the initial conditions of the entire system.

Standing Assumption 2: $\mathbf{x}(0) \in \Delta$ and $\mathbf{y}(0) \in \mathbb{R}^q$.

III. Analysis of the Proposed Dynamics

In this section, we analyze the dynamics in (1)-(2). First, we show some invariance properties of the dynamics. Second, we characterize the equilibria set of the dynamics. Third, we provide sufficient conditions to guarantee the asymptotic stability of the equilibria set of the dynamics. Finally, we provide some additional results for the case when the baseline game $\mathbf{f}(\cdot)$ is a full-potential game [c.f., Definition 2].

To start the discussion, we first show that the set Δ is positively invariant under the dynamics in (2). Although this is a well-known result in the field of population games and evolutionary dynamics, we prove it here for completeness.

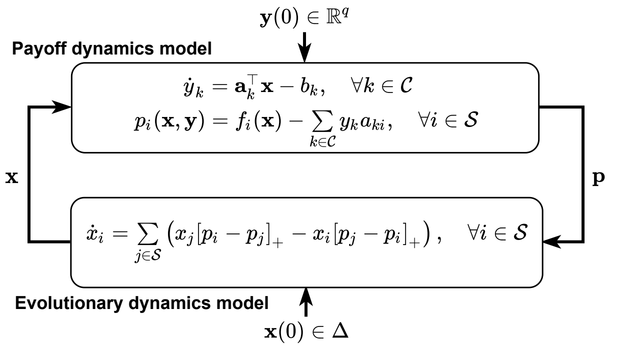

Fig. 1. Interconnection between the payoff dynamics model in (1) and the evolutionary dynamics model in (2). Here, $\mathbf{p} \triangleq \mathbf{p}(\mathbf{x},\mathbf{y})$.

Proposition 1: Consider the dynamics in (2). It follows that $\mathbf{x}(0) \in \Delta \Rightarrow \mathbf{x}(t) \in \Delta$, $\forall t \geq 0$.

Proof: From (2), notice that if $x_i = 0$, then $\dot{x}_i \geq 0$. Hence, $\mathbf{x}(0) \in \mathbb{R}^n_{\geq 0} \Rightarrow \mathbf{x}(t) \in \mathbb{R}^n_{\geq 0}$, $\forall t \geq 0$. Also, note that $\sum_{i\in\mathcal{S}} \dot{x}_i = 0$. Thus, $\mathbf{1}^\top \mathbf{x}(0) = m \Rightarrow \mathbf{1}^\top \mathbf{x}(t) = m$, $\forall t \geq 0$. Combining these observations leads to the desired result. ■

Proposition 1 and Standing Assumption 2 allow us to assert, without any additional loss of generality, that the state of the dynamics in (2) belongs to Δ for all times. Consequently, $(\mathbf{x},\mathbf{y}) \in \Delta \times \mathbb{R}^q$ for all times. It is important that the reader keeps this fact in mind throughout this section.

We now proceed to characterize the equilibria set of the $(n+q)$-dimensional system given by (1)-(2). In order to do so, we first introduce the concept of a Nash equilibrium for the augmented game $\mathbf{p}(\cdot,\cdot)$. Namely, the set of Nash equilibria of the augmented game $\mathbf{p}(\cdot,\cdot)$ is given by

$$\mathrm{NE}(\mathbf{p},\mathbf{y}) = \{\mathbf{x} \in \Delta : x_i > 0 \Rightarrow p_i = p^*, \forall i \in \mathcal{S}\}, \quad (3)$$

where $p_i \triangleq p_i(\mathbf{x},\mathbf{y})$, and $p^* \triangleq \max_{j\in\mathcal{S}} p_j(\mathbf{x},\mathbf{y})$. Hence, at a Nash equilibrium no player has incentives to change her strategy. Moreover, note that $\mathrm{NE}(\mathbf{p},\mathbf{y})$ depends on the state of the payoff dynamics model, i.e., $\mathbf{y}$. Having defined the set of Nash equilibria of the augmented game $\mathbf{p}(\cdot,\cdot)$, we now provide the following result that characterizes the equilibria set of the considered $(n+q)$-dimensional system.

Proposition 2: Consider the dynamics in (1)-(2), and let

$$\mathcal{E} = \{(\mathbf{x},\mathbf{y}) \in \Delta \times \mathbb{R}^q : \mathbf{x} \in \mathrm{NE}(\mathbf{p},\mathbf{y}) \cap \mathcal{Q}\}. \quad (4)$$

A point $(\mathbf{x}^*,\mathbf{y}^*) \in \Delta \times \mathbb{R}^q$ is an equilibrium state of the considered dynamics if and only if $(\mathbf{x}^*,\mathbf{y}^*) \in \mathcal{E}$.

Proof: First, from the Nash stationarity property of the Smith dynamics [2, Theorem 5.6.2], we conclude that $\dot{x}_i = 0, \forall i \in \mathcal{S} \Leftrightarrow \mathbf{x} \in \mathrm{NE}(\mathbf{p},\mathbf{y})$. On the other hand, from (1a) it follows that $\dot{y}_k = 0, \forall k \in \mathcal{C} \Leftrightarrow \mathbf{A}\mathbf{x} = \mathbf{b}$. In consequence, a point $(\mathbf{x}^*,\mathbf{y}^*) \in \Delta \times \mathbb{R}^q$ comprises an equilibrium state of the dynamics if and only if $\mathbf{x}^* \in \mathrm{NE}(\mathbf{p},\mathbf{y}^*) \cap \mathcal{Q}$. ■

With the aid of Proposition 2, it is now possible to provide sufficient conditions to guarantee the asymptotic stability of the equilibria set $\mathcal{E}$ under the dynamics in (1)-(2). Theorem 1 provides our results on such a matter.

Theorem 1: Consider the dynamics in (1)-(2), and the set $\mathcal{E}$ in (4). Moreover, let $\mathbf{f}(\cdot)$ be a continuously differentiable stable game. If $\mathcal{E}$ is non-empty and compact, then $\mathcal{E}$ is asymptotically stable under the considered dynamics.

Proof: Let $\mathcal{E}$ be non-empty and compact. Hence, using an appropriate Lyapunov function, it is possible to verify the stability of such a set [20, Corollary 4.7].

Consider the Lyapunov function candidate given by

$$V(\mathbf{x},\mathbf{y}) = \underbrace{\sum_{j\in\mathcal{S}}\sum_{i\in\mathcal{S}} x_i \mathrm{P}_i^j(\mathbf{x},\mathbf{y})}_{V_1(\mathbf{x},\mathbf{y})} + \underbrace{\sum_{k\in\mathcal{C}} \frac{1}{2}\left(\mathbf{a}_k^\top \mathbf{x} - b_k\right)^2}_{V_2(\mathbf{x})}, \quad (5)$$

with $\mathrm{P}_i^j(\mathbf{x},\mathbf{y}) = \int_0^{p_j(\mathbf{x},\mathbf{y})-p_i(\mathbf{x},\mathbf{y})} [\tau]_+ \, d\tau$, for all $i,j \in \mathcal{S}$. It is evident that $V(\mathbf{x},\mathbf{y}) \geq 0$, for all $(\mathbf{x},\mathbf{y}) \in \Delta \times \mathbb{R}^q$ (recall Proposition 1), and that $V_2(\mathbf{x}) = 0 \Leftrightarrow \mathbf{x} \in \mathcal{Q}$. Moreover, following a similar analysis as in [2, Theorem 7.2.9] or [15, Lemma 3.6], it is straightforward to verify that $V_1(\mathbf{x},\mathbf{y}) = 0 \Leftrightarrow \mathbf{x} \in \mathrm{NE}(\mathbf{p},\mathbf{y})$. Therefore, $V(\cdot,\cdot)$ provides a valid Lyapunov function candidate with respect to $\mathcal{E}$.

We now proceed to analyze the derivatives of $V(\cdot,\cdot)$. For such, let $\mathrm{P}_i^j \triangleq \mathrm{P}_i^j(\mathbf{x},\mathbf{y})$ and note that

$$\frac{\partial V(\mathbf{x},\mathbf{y})}{\partial x_s} = \sum_{j\in\mathcal{S}} \mathrm{P}_s^j + \sum_{j\in\mathcal{S}}\sum_{i\in\mathcal{S}} x_i \frac{\partial \mathrm{P}_i^j}{\partial x_s} + \sum_{k\in\mathcal{C}} \left(\mathbf{a}_k^\top \mathbf{x} - b_k\right) a_{ks}$$
$$\frac{\partial V(\mathbf{x},\mathbf{y})}{\partial y_c} = \sum_{j\in\mathcal{S}}\sum_{i\in\mathcal{S}} x_i \frac{\partial \mathrm{P}_i^j}{\partial y_c},$$

for all $s \in \mathcal{S}$ and all $c \in \mathcal{C}$. Here, let $p_i \triangleq p_i(\mathbf{x},\mathbf{y})$, $f_i \triangleq f_i(\mathbf{x})$, and observe that

$$\begin{aligned}\sum_{j,i\in\mathcal{S}} x_i \frac{\partial \mathrm{P}_i^j}{\partial x_s} &= \sum_{j,i\in\mathcal{S}} x_i \left[p_j - p_i\right]_+ \left(\frac{\partial p_j}{\partial x_s} - \frac{\partial p_i}{\partial x_s}\right) \\ &= \sum_{j,i\in\mathcal{S}} \left(x_i \left[p_j - p_i\right]_+ - x_j \left[p_i - p_j\right]_+\right) \frac{\partial p_j}{\partial x_s} \\ &= \sum_{j\in\mathcal{S}} \dot{x}_j \frac{\partial f_j}{\partial x_s} \quad \text{[using (2) and (1b)]}.\end{aligned}$$

Additionally, $\sum_{k\in\mathcal{C}} \left(\mathbf{a}_k^\top \mathbf{x} - b_k\right) a_{ks} = \sum_{k\in\mathcal{C}} \dot{y}_k a_{ks}$ [using (1a)], and

$$\begin{aligned}\sum_{j,i\in\mathcal{S}} x_i \frac{\partial \mathrm{P}_i^j}{\partial y_c} &= \sum_{j,i\in\mathcal{S}} x_i \left[p_j - p_i\right]_+ \left(\frac{\partial p_j}{\partial y_c} - \frac{\partial p_i}{\partial y_c}\right) \\ &= -\sum_{j\in\mathcal{S}} \dot{x}_j a_{cj} \quad \text{[using (2) and (1b)]}.\end{aligned}$$

By defining $\mathbf{\Gamma}_\mathrm{P} \triangleq \left[\sum_{j\in\mathcal{S}} \mathrm{P}_s^j\right] \in \mathbb{R}^n_{\geq 0}$, it follows that

$$\nabla_\mathbf{x} V(\mathbf{x},\mathbf{y}) = \mathbf{\Gamma}_\mathrm{P} + \left(\mathrm{D}\mathbf{f}(\mathbf{x})\right)^\top \dot{\mathbf{x}} + \mathbf{A}^\top \dot{\mathbf{y}}$$
$$\nabla_\mathbf{y} V(\mathbf{x},\mathbf{y}) = -\mathbf{A}\dot{\mathbf{x}},$$

where $\mathrm{D}\mathbf{f}(\mathbf{x}) \in \mathbb{R}^{n\times n}$ is the Jacobian matrix of $\mathbf{f}(\cdot)$ at $\mathbf{x}$. Therefore, by setting $\nabla_\mathbf{x} V \triangleq \nabla_\mathbf{x} V(\mathbf{x},\mathbf{y})$ and $\nabla_\mathbf{y} V \triangleq \nabla_\mathbf{y} V(\mathbf{x},\mathbf{y})$, it follows that

$$\left[\nabla_\mathbf{x} V^\top, \nabla_\mathbf{y} V^\top\right] \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{y}} \end{bmatrix} = \mathbf{\Gamma}_\mathrm{P}^\top \dot{\mathbf{x}} + \dot{\mathbf{x}}^\top \mathrm{D}\mathbf{f}(\mathbf{x})\dot{\mathbf{x}},$$

where we have used the fact that $\dot{\mathbf{y}}^\top \mathbf{A}\dot{\mathbf{x}} - \dot{\mathbf{x}}^\top \mathbf{A}^\top \dot{\mathbf{y}} = 0$ (since, as a scalar, $\dot{\mathbf{y}}^\top \mathbf{A}\dot{\mathbf{x}} = \left(\dot{\mathbf{y}}^\top \mathbf{A}\dot{\mathbf{x}}\right)^\top = \dot{\mathbf{x}}^\top \mathbf{A}^\top \dot{\mathbf{y}}$). Now, given that $\mathbf{f}(\cdot)$ is a continuously differentiable stable game, we conclude from [2, Theorem 3.3.1] that $\dot{\mathbf{x}}^\top \mathrm{D}\mathbf{f}(\mathbf{x})\dot{\mathbf{x}} \leq 0$, for all $\mathbf{x} \in \Delta$. On the other hand, notice that

$$\begin{aligned}
\mathbf{\Gamma}_{\mathrm{P}}^\top \dot{\mathbf{x}} &= \sum_{s\in\mathcal{S}} \dot{x}_s \sum_{j\in\mathcal{S}} \mathrm{P}_s^j \\
&= \sum_{s\in\mathcal{S}} \sum_{i\in\mathcal{S}} \left(x_i \left[p_s - p_i\right]_+ - x_s \left[p_i - p_s\right]_+\right) \sum_{j\in\mathcal{S}} \mathrm{P}_s^j \\
&= \sum_{s\in\mathcal{S}} \sum_{i\in\mathcal{S}} x_i \left[p_s - p_i\right]_+ \sum_{j\in\mathcal{S}} \left(\mathrm{P}_s^j - \mathrm{P}_i^j\right). \qquad (6a)
\end{aligned}$$

If $p_s \leq p_i$, then $[p_s - p_i]_+ = 0$. Thus, it suffices to analyze only the cases where $p_s > p_i$. In particular, observe that

$$\begin{aligned}
p_j \geq p_s > p_i &\Rightarrow \left(\mathrm{P}_s^j - \mathrm{P}_i^j\right) < 0 \\
p_s > p_j > p_i &\Rightarrow \left(\mathrm{P}_s^j - \mathrm{P}_i^j\right) = \left(0 - \mathrm{P}_i^j\right) < 0 \\
p_s > p_i \geq p_j &\Rightarrow \left(\mathrm{P}_s^j - \mathrm{P}_i^j\right) = (0 - 0) = 0.
\end{aligned}$$

In consequence, $\mathbf{\Gamma}_{\mathrm{P}}^\top \dot{\mathbf{x}} \leq 0$ for all $(\mathbf{x}, \mathbf{y}) \in \Delta \times \mathbb{R}^q$, which implies that $\mathcal{E}$ is stable in the sense of Lyapunov. That is, if $(\mathbf{x}(0), \mathbf{y}(0))$ is sufficiently close to $\mathcal{E}$, then $(\mathbf{x}(t), \mathbf{y}(t))$ is bounded for all $t \geq 0$ [20, Definition 4.10].

Now, from the Nash stationarity of (2) [2, Theorem 5.6.2], $\mathbf{x} \in \mathrm{NE}(\mathbf{p}, \mathbf{y}) \Rightarrow \dot{\mathbf{x}} = \mathbf{0} \Rightarrow \dot{\mathbf{x}}^\top \mathrm{D}\mathbf{f}(\mathbf{x})\dot{\mathbf{x}} = 0$ (here, $\mathbf{0}$ is the vector of zeros with appropriate dimension). Moreover, following the same analysis as in [2, Theorem 7.2.9], it can be shown that $\mathbf{\Gamma}_{\mathrm{P}}^\top \dot{\mathbf{x}} = 0 \Leftrightarrow \mathbf{x} \in \mathrm{NE}(\mathbf{p}, \mathbf{y})$. Thus, $\mathbf{\Gamma}_{\mathrm{P}}^\top \dot{\mathbf{x}} + \dot{\mathbf{x}}^\top \mathrm{D}\mathbf{f}(\mathbf{x})\dot{\mathbf{x}} = 0 \Leftrightarrow \mathbf{x} \in \mathrm{NE}\left(\mathbf{p}, \mathbf{y}\right)$. That is, the derivatives of the Lyaunov function $V(\mathbf{x}, \mathbf{y})$ are zero if and only if $\mathbf{x} \in \mathrm{NE}(\mathbf{p}, \mathbf{y})$. Therefore, given that $\mathcal{E}$ is Lyapunov stable, it follows from LaSalle's Theorem [20, Theorem 3.3] that if $\mathcal{E}$ is the largest invariant set of the dynamics within $\mathcal{R} = \{(\mathbf{x}, \mathbf{y}) \in \Delta \times \mathbb{R}^q : \mathbf{x} \in \mathrm{NE}(\mathbf{p}, \mathbf{y})\}$, then $\mathcal{E}$ is asymptotically stable (i.e., Lyapunov stable and attractive). We now proceed to prove such a property by contradiction.

Let $\mathcal{I} \subseteq \mathcal{R}$ be the largest invariant set of the dynamics within $\mathcal{R}$. Clearly, $\mathcal{E} \subseteq \mathcal{I}$ and $(\mathbf{x}(\tau), \mathbf{y}(\tau)) \in \mathcal{I} \Rightarrow (\mathbf{x}(t), \mathbf{y}(t)) \in \mathcal{I}$, for all $t \geq \tau$. Moreover, given that $\mathcal{I} \subseteq \mathcal{R}$, from the Nash stationarity of (2) [2, Theorem 5.6.2], $(\mathbf{x}(\tau), \mathbf{y}(\tau)) \in \mathcal{I} \Rightarrow \mathbf{x}(t) = \mathbf{x}(\tau)$, for all $t \geq \tau$. Now, suppose that $\mathcal{E} \subset \mathcal{I}$. Note that $\mathcal{E} \subset \mathcal{I} \Leftrightarrow \mathcal{I} \backslash \mathcal{E} \neq \emptyset$. Therefore, let $\mathcal{T} = \mathcal{I} \setminus \mathcal{E}$. Given that $\mathcal{T} \cap \mathcal{E} = \emptyset$ and $\mathcal{T} \subset \mathcal{I} \subseteq \mathcal{R}$, from (4) it follows that $(\mathbf{x}, \mathbf{y}) \in \mathcal{T} \Rightarrow \mathbf{x} \notin \mathcal{Q}$. Consequently, if $(\mathbf{x}(\tau), \mathbf{y}(\tau)) \in \mathcal{T}$, then: i) $\mathbf{x}(t) = \mathbf{x}(\tau)$ for all $t \geq \tau$ (since $\mathcal{T} \subset \mathcal{I}$); and ii) $\dot{\mathbf{y}}(t) = \dot{\mathbf{y}}(\tau) \neq \mathbf{0}$ for all $t \geq \tau$ (since $\dot{\mathbf{y}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{b}$ and $\mathbf{x}(t) = \mathbf{x}(\tau) \notin \mathcal{Q}$ is fixed). Moreover, ii) implies that $\|\mathbf{y}(t)\| \to \infty$ as $t \to \infty$, where $\|\cdot\|$ is any p-norm. Clearly, this contradicts the boundedness of $(\mathbf{x}(t), \mathbf{y}(t))$ provided by the Lyapunov stability of $\mathcal{E}$. Therefore, we conclude that there cannot exist a non-empty set $\mathcal{T}$ such that $\mathcal{T} = \mathcal{I} \setminus \mathcal{E}$. Consequently, $\mathcal{I} = \mathcal{E}$. ■

Theorem 1 provides sufficient conditions to guarantee the asymptotic stability of the equilibria set $\mathcal{E}$. In particular, notice that it is required for $\mathcal{E}$ to be non-empty and compact. Regarding such a matter, Theorem 2 and Proposition 3 demonstrate that these properties hold for certain full-potential games [c.f., Definition 2]. It is worth to highlight that results similar to Proposition 3 are available in the convex optimization literature [21, Proposition 5.3.1]. Nevertheless, we provide a self-contained proof for completeness.

Theorem 2: Consider the dynamics in (1)-(2), and the equilibria set $\mathcal{E}$ in (4). Moreover, let $\mathbf{f}(\cdot)$ be a full-potential game with concave potential function $\varphi(\cdot)$. It holds that $(\mathbf{x}^*, \mathbf{y}^*) \in \mathcal{E}$ if and only if $\mathbf{x}^* \in \arg\max_{\mathbf{x}\in\Delta\cap\mathcal{Q}} \varphi(\mathbf{x})$. Additionally, $\mathcal{E}$ is non-empty.

Proof: To prove this result, let us consider an optimization perspective. In particular, note that the optimization problem $\max_{\mathbf{x}\in\Delta\cap\mathcal{Q}} \varphi(\mathbf{x})$ is equivalent to

$$\max_{\mathbf{x}\in\mathbb{R}^n} \varphi(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{x} \succeq \mathbf{0}, \quad \mathbf{1}^\top \mathbf{x} = m, \quad \mathbf{A}\mathbf{x} = \mathbf{b}. \qquad (7)$$

Since $\varphi(\cdot)$ is differentiable, concave, all the constraints are affine, and the set $\Delta \cap \mathcal{Q}$ is non-empty [c.f., Standing Assumption 1], it follows that the (refined) Slater's condition holds [18, Section 5.2.3]. In consequence, the Karush-Kuhn-Tucker (KKT) optimality conditions are necessary and sufficient for the problem in (7). Now, observe that the Lagrangian dual function of the problem in (7) is given by

$$L(\mathbf{x}, \boldsymbol{\lambda}, \nu, \mathbf{y}) = \varphi(\mathbf{x}) + \boldsymbol{\lambda}^\top \mathbf{x} - \nu\left(\mathbf{1}^\top \mathbf{x} - m\right) - \mathbf{y}^\top\left(\mathbf{A}\mathbf{x} - \mathbf{b}\right), \qquad (8)$$

where $\boldsymbol{\lambda} \in \mathbb{R}^n_{\geq 0}$ and $\nu \in \mathbb{R}$. Hence, the KKT conditions for the problem in (7) are

$$\begin{aligned}
p_i\left(\mathbf{x}^*, \mathbf{y}^*\right) &= \nu^* - \lambda_i^*, \quad \forall i \in \mathcal{S} &(9a)\\
\mathbf{1}^\top \mathbf{x}^* &= m &(9b)\\
x_i^* &\geq 0, \quad \forall i \in \mathcal{S} &(9c)\\
\mathbf{a}_k^\top \mathbf{x}^* - b_k &= 0, \quad \forall k \in \mathcal{C} &(9d)\\
\lambda_i^* x_i^* &= 0, \quad \forall i \in \mathcal{S} &(9e)\\
\lambda_i^* &\geq 0, \quad \forall i \in \mathcal{S}. &(9f)
\end{aligned}$$

Here, we have used $p_i\left(\mathbf{x}^*, \mathbf{y}^*\right) = f_i\left(\mathbf{x}^*\right) - \sum_{k\in\mathcal{C}} y_k^* a_{ki}$, which follows from (1b). Therefore, let

$$\mathcal{K} = \left\{ (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^q : \begin{array}{l} (\mathbf{x}, \boldsymbol{\lambda}, \nu, \mathbf{y}) \text{ satisfies (9)} \\ \text{for some } \boldsymbol{\lambda} \in \mathbb{R}^n,\ \nu \in \mathbb{R} \end{array} \right\}.$$

To show that $(\mathbf{x}^*, \mathbf{y}^*) \in \mathcal{E} \Leftrightarrow \mathbf{x}^* \in \arg\max_{\mathbf{x}\in\Delta\cap\mathcal{Q}} \varphi(\mathbf{x})$, we must show that $(\mathbf{x}^*, \mathbf{y}^*) \in \mathcal{E} \Leftrightarrow (\mathbf{x}^*, \mathbf{y}^*) \in \mathcal{K}$.

($\Rightarrow$) Let $(\mathbf{x}^*, \mathbf{y}^*) \in \mathcal{E}$. Since $\mathbf{x}^* \in \Delta \cap \mathcal{Q}$, conditions (9b)-(9d) hold immediately. Moreover, since $\mathbf{x}^* \in \mathrm{NE}\left(\mathbf{p}, \mathbf{y}^*\right)$, conditions (9a), (9e), and (9f) are satisfied by taking $\nu^* = \max_{j\in\mathcal{S}} p_j\left(\mathbf{x}^*, \mathbf{y}^*\right)$, and $\lambda_i^* = \nu^* - p_i\left(\mathbf{x}^*, \mathbf{y}^*\right)$, for all $i \in \mathcal{S}$.

($\Leftarrow$) Let $(\mathbf{x}^*, \mathbf{y}^*) \in \mathcal{K}$. Conditions (9b)-(9d) imply that $\mathbf{x}^* \in \Delta \cap \mathcal{Q}$. Furthermore, conditions (9a) and (9e) imply that $p_i\left(\mathbf{x}^*, \mathbf{y}^*\right) = \nu^*$ for all $i \in \operatorname{supp}\left(\mathbf{x}^*\right)$, and conditions (9a) and (9f) imply that $p_j\left(\mathbf{x}^*, \mathbf{y}^*\right) = \nu^* - \lambda_j^* \leq \nu^*$ for all $j \in \mathcal{S}$. Therefore, for all $i \in \operatorname{supp}\left(\mathbf{x}^*\right)$ it holds that $p_i\left(\mathbf{x}^*, \mathbf{y}^*\right) = \max_{j\in\mathcal{S}} p_j\left(\mathbf{x}^*, \mathbf{y}^*\right)$, and so $\mathbf{x}^* \in \mathrm{NE}\left(\mathbf{p}, \mathbf{y}^*\right)$. In consequence, $(\mathbf{x}^*, \mathbf{y}^*) \in \mathcal{E}$.

Finally, the non-emptiness of $\mathcal{E}$ follows from the previous discussion in conjunction with the fact that the optimization problem in (7) has at least one solution. The latter claim follows from Standing Assumption 1 (i.e., $\Delta \cap \mathcal{Q} \neq \emptyset$), the continuity of $\varphi(\cdot)$, and the compactness of $\Delta \cap \mathcal{Q}$. ■

Proposition 3: Consider the dynamics in (1)-(2), and the equilibria set $\mathcal{E}$ in (4). Moreover, let $\mathbf{f}(\cdot)$ be a full-potential game with concave potential function $\varphi(\cdot)$. If in addition $\mathbb{R}^n_{>0} \cap \Delta \cap \mathcal{Q}$ is non-empty, then $\mathcal{E}$ is compact.

Proof: First, note that $\mathcal{E}$ is closed because it is the preimage of the closed set $\{0\}$ under the continuous map $V(\cdot,\cdot)$ provided in (5) [c.f., proof of Theorem 1].

To show that $\mathcal{E}$ is bounded, on the other hand, consider the optimization perspective described in the proof of Theorem 2. We have that strong duality holds for the problem in (7), and that $\varphi^* = \max_{\mathbf{x}\in\Delta\cap\mathcal{Q}} \varphi(\mathbf{x})$ exists and is finite. Consequently, letting $\hat{\mathbf{x}} \in \mathbb{R}^n_{>0} \cap \Delta \cap \mathcal{Q}$, it follows that

$$\begin{aligned}\varphi^* &= \max_{\mathbf{x}\in\mathbb{R}^n} \quad L\left(\mathbf{x}, \boldsymbol{\lambda}^*, \nu^*, \mathbf{y}^*\right) \quad \text{[using (8)]} \\ &\geq \varphi\left(\hat{\mathbf{x}}\right) + \sum_{j\in\mathcal{S}} \lambda_j^* \hat{x}_j \quad [\text{since } \hat{\mathbf{x}} \in \Delta \cap \mathcal{Q}] \\ &\geq \varphi\left(\hat{\mathbf{x}}\right) + \left(\min_{s\in\mathcal{S}} \hat{x}_s\right) \sum_{j\in\mathcal{S}} \lambda_j^*,\end{aligned}$$

where $\boldsymbol{\lambda}^* \in \mathbb{R}^n_{\geq 0}$, $\nu^* \in \mathbb{R}$, and $\mathbf{y}^* \in \mathbb{R}^q$, are the optimal Lagrange multipliers. Hence, since $\min_{s\in\mathcal{S}} \hat{x}_s > 0$,

$$\sum_{j\in\mathcal{S}} \lambda_j^* \leq \frac{\varphi^* - \varphi\left(\hat{\mathbf{x}}\right)}{\min_{s\in\mathcal{S}} \hat{x}_s} \in [0, \infty).$$

More precisely, the set of optimal Lagrange multipliers associated to the inequality constraints in (7) is bounded. Now, condition (9a) is equivalent to $\mathbf{f}\left(\mathbf{x}^*\right) - \hat{\mathbf{A}}^\top \hat{\mathbf{y}}^* = -\boldsymbol{\lambda}^*$, where $\hat{\mathbf{A}} \in \mathbb{R}^{(q+1)\times n}$ is as in Standing Assumption 1; and $\hat{\mathbf{y}}^* = \left[\nu^*, y_1^*, y_2^*, \cdots, y_q^*\right]^\top \in \mathbb{R}^{q+1}$. Since, $\hat{\mathbf{A}}$ is full row rank, it follows that $\hat{\mathbf{y}}^* = \left(\hat{\mathbf{A}}\hat{\mathbf{A}}^\top\right)^{-1} \hat{\mathbf{A}}\left(\boldsymbol{\lambda}^* + \mathbf{f}\left(\mathbf{x}^*\right)\right)$. Hence, $\hat{\mathbf{y}}^*$ is the image of an uniformly continuous map (in this case, a linear map) applied to $\boldsymbol{\lambda}^* + \mathbf{f}\left(\mathbf{x}^*\right)$. Since $\boldsymbol{\lambda}^* + \mathbf{f}\left(\mathbf{x}^*\right)$ is bounded ($\boldsymbol{\lambda}^*$ is bounded from the previous discussion, and $\mathbf{f}\left(\mathbf{x}^*\right)$ is bounded because $\mathbf{f}(\cdot)$ is continuous and $\mathbf{x}^*$ belongs to the compact set $\Delta \cap \mathcal{Q}$), it follows that $\hat{\mathbf{y}}^*$ is bounded. In consequence, $\mathbf{y}^*$ is bounded and so is $\mathcal{E}$. This completes the proof. ■

Theorem 2 and Proposition 3 provide sufficient conditions to guarantee the non-emptiness and compactness of $\mathcal{E}$ for full-potential games. Therefore, for continuously differentiable full-potential games satisfying the conditions of Proposition 3, the result of Theorem 1 follows immediately. Furthermore, Theorem 2 shows that, for full-potential games, the equilibria set of the dynamics in (1)-(2) is aligned with the maximizers of the underlying (constrained) potential-maximization problem. This property is especially appealing for optimization-based game theoretical applications. In the following section, we provide an illustration of such a result.

IV. An Illustrative Example

In this section, we illustrate our theoretical developments on a classical congestion game that has been extended to consider equality constraints. Note that congestion games, as relevant engineering problems [22], have been also considered in some of the previous works on payoff dynamics models [6], [7], and, for that reason, we consider such an example game in this paper as well.

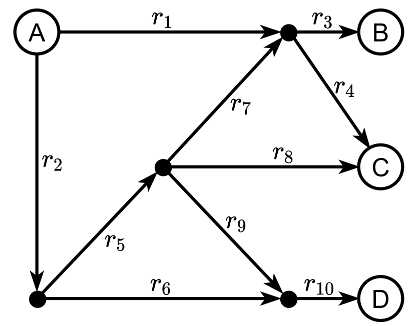

Fig. 2. Considered topology for the congestion game.

Consider the transportation network shown in Fig. 2, and assume that there is a large number of players that must travel from point A to the terminals B, C, and D. Since the number of players is large, we represent the 100% of the players as a mass $m = 1$. According to Fig. 2, observe that there are 7 possible strategies that the players might take, i.e., $\mathcal{S} = \{1, 2, \ldots, 7\}$. Namely, identifying $i \to s_i$, for all $i \in \mathcal{S}$, the possible strategies are $s_1 = \{r_1, r_3\}$, $s_2 = \{r_2, r_5, r_7, r_3\}$, $s_3 = \{r_1, r_4\}$, $s_4 = \{r_2, r_5, r_7, r_4\}$, $s_5 = \{r_2, r_5, r_8\}$, $s_6 = \{r_2, r_5, r_9, r_{10}\}$, and $s_7 = \{r_2, r_6, r_{10}\}$. Clearly, to go from A to B players might choose s_1 or s_2; to go from A to C they might choose s_3, s_4, or s_5; and to go from A to D the might choose s_6 or s_7. Besides, suppose that terminal B requires exactly a portion $m_{\mathrm{B}} \in (0,1)$ of the total players, and that terminal C requires exactly a portion $m_{\mathrm{C}} \in (0,1)$ of the total players, with $m_{\mathrm{B}} + m_{\mathrm{C}} < 1$. Hence, the underlying equality constraints are $x_1 + x_2 = m_{\mathrm{B}}$ and $x_3 + x_4 + x_5 = m_{\mathrm{C}}$, i.e., $\mathcal{C} = \{1, 2\}$ and Standing Assumption 1 holds. Without loss of generality, for our experiments we set $m_{\mathrm{B}} = 0.2$ and $m_{\mathrm{C}} = 0.6$, i.e., terminals B and C require 20% and 60% of the total players, respectively. Furthermore, we assume that each of the roads r_z, for all $z \in \{1, 2, \ldots, 10\}$, has a linear congestion cost given by $c_z u_z$, where $c_z \in \mathbb{R}_{>0}$ and $u_z \in \mathbb{R}_{\geq 0}$ are the weight of congestion and usage level of the z-th road, respectively. For simplicity, yet without loss of generality, we (randomly) set $\mathbf{c} = [10, 15, 16, 14, 11, 5, 14, 7, 9, 12]^\top$. The goal of the players is thus to travel from point A to the end terminals, while minimizing the congestion of the roads (and while seeking to satisfy the requirement constraints of terminals B and C as well). As such, the potential function for the described congestion game is given by

$$\begin{aligned}\varphi(\mathbf{x}) = &-\frac{c_1}{2}\left(x_1 + x_3\right)^2 - \frac{c_2}{2}\left(x_2 + x_4 + x_5 + x_6 + x_7\right)^2 \\ &-\frac{c_3}{2}\left(x_1 + x_2\right)^2 - \frac{c_4}{2}\left(x_3 + x_4\right)^2 \\ &-\frac{c_5}{2}\left(x_2 + x_4 + x_5 + x_6\right)^2 - \frac{c_6}{2}x_7^2 - \frac{c_7}{2}\left(x_2 + x_4\right)^2 \\ &-\frac{c_8}{2}x_5^2 - \frac{c_9}{2}x_6^2 - \frac{c_{10}}{2}\left(x_6 + x_7\right)^2.\end{aligned}$$

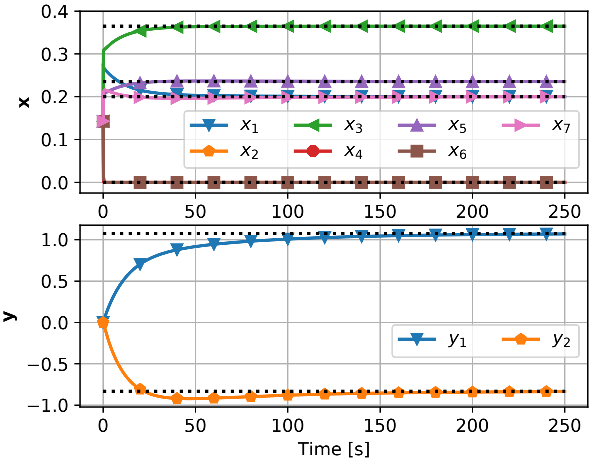

Fig. 3. Evolution of the $(7+2)$-dimensional system for the considered congestion game. The dotted lines depict the optimal values for the primal-dual variables (i.e., $\mathbf{x}^*$ and $\mathbf{y}^*$) of the underlying constrained potential-maximization problem [c.f., (7)]. Such values are obtained using [23] and are $\mathbf{x}^* \approx [0.2,\ 0,\ 0.365,\ 0,\ 0.235,\ 0,\ 0.2]^\top$, and $\mathbf{y}^* \approx [1.08,\ -0.832]^\top$.

Thus, with $f_i(\mathbf{x}) = \partial\varphi(\mathbf{x})/\partial x_i$, for all $i \in \mathcal{S}$, it follows that $p_i(\mathbf{x},\mathbf{y}) = f_i(\mathbf{x}) - y_1$, for all $i \in \{1,2\}$; $p_i(\mathbf{x},\mathbf{y}) = f_i(\mathbf{x}) - y_2$, for all $i \in \{3,4,5\}$; and $p_i(\mathbf{x},\mathbf{y}) = f_i(\mathbf{x})$, for all $i \in \{6,7\}$. Under the considered framework, it can be verified that $\varphi(\cdot)$ is a concave twice continuously differentiable potential function. Hence, the underlying baseline game $\mathbf{f}(\cdot)$ is both a full-potential game and a continuously differentiable stable game [c.f., Remark 1]. Therefore, from Theorem 2 we conclude that the corresponding equilibria set $\mathcal{E}$ is non-empty and is aligned with the maximizers of the potential function $\varphi(\cdot)$ within $\Delta \cap \mathcal{Q}$. Furthermore, observe that

$$\hat{\mathbf{x}} = [0.1,\ 0.1,\ 0.2,\ 0.2,\ 0.2,\ 0.1,\ 0.1]^\top \in \mathbb{R}^7_{>0} \cap \Delta \cap \mathcal{Q}.$$

Thus, the set $\mathbb{R}^n_{>0} \cap \Delta \cap \mathcal{Q}$ is non-empty, and, from Proposition 3, we conclude that $\mathcal{E}$ is compact. Consequently, it follows from Theorem 1 that $\mathcal{E}$ is asymptotically stable.

In Fig. 3, we present some illustrative numerical simulation of the dynamics in (1)-(2) under the considered congestion game. As initial conditions we set $x_i(0) = 1/7$, for all $i \in \mathcal{S}$, and $y_k = 0$, for all $k \in \mathcal{C}$ (so that Standing Assumption 2 is satisfied). As shown in Fig. 3, the dynamics indeed asymptotically converge to $\mathcal{E}$.

V. Concluding Remarks and Future Work

This paper has proposed a novel form of payoff dynamics models for equality-constrained population games. In particular, we have provided sufficient conditions to guarantee the asymptotic stability of the equilibria set of the proposed dynamics. Moreover, we have provided sufficient conditions to guarantee the non-emptiness, compactness, and optimality of such an equilibria set for certain full-potential population games. Future work should extend our results on the non-emptiness and compactness properties of the equilibria set to the more general family of stable games.

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