Short Course for PhD Students: Recent Results on Planar and Spherical Kinematics
September 5-12, 2019, Barcelona, Spain, EU

Lecture 1

Planar Kinematics Revisited through Instant Centers (ICs)

Raffaele Di Gregorio

rdigregorio@ing.unife.it

UNIFE

Department of Engineering
University of Ferrara
Via Saragat,1
44122 FERRARA, Italy, EU

Index

1) Role of the ICs in planar mechanism design

1.1) Basic concepts

1.2) Application Examples

2) Systematic determination of all the ICs

2.1) The case of the indeterminate linkages

Refs.:

- Di Gregorio R., 2008, "An Algorithm for Analytically Calculating the Positions of the Secondary Instant Centers of Indeterminate Linkages," *ASME J. Mechanical Design*, 130(4):042303-042303-9
- Simionescu P.A., Talpasanu I., Di Gregorio R., 2010, "Instant-Center Based Force Transmissivity and Singularity Analysis of Planar Linkages," *ASME J. Mechanisms and Robotics*, 2(2):021011 (12 pages)

3) Singularity analysis with the ICs

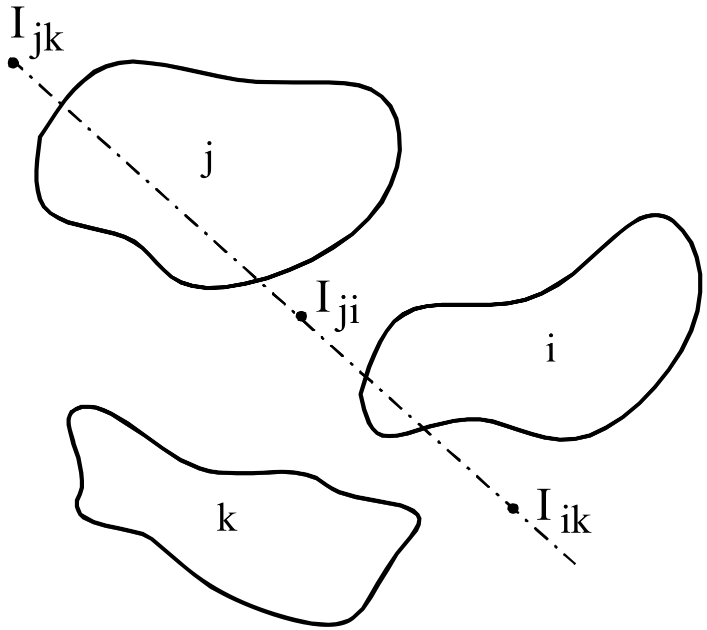
3.1) Single-DOF mechanisms

3.2) Multi-DOF mechanisms

Refs.:

- Di Gregorio R., 2007, "A novel geometric and analytic technique for the singularity analysis of one-dof planar mechanisms," *Mechanism and Machine Theory*, 42(11):1462-1483
- Di Gregorio R., 2009, "A novel method for the singularity analysis of planar mechanisms with more than one degree of freedom," *Mechanism and Machine Theory*, 44(1):83–102

Role of the ICs in planar mechanism design: Basic Concepts



Instant Center (IC) 's Equivalent Definitions:

Given two rigid bodies (*links j and i of the Figure*), **the IC**, *named* I_{ji}, **of the relative planar motion between links j and i**

is

1st: the intersection of the plane of motion and the rotation axis of that planar motion

2nd: the point at rest when considered embedded in one of the other of the two links

3rd: the point that has the same velocity, measured by any third link (link k in the Figure)**, when considered embedded in one of the other of the two links**

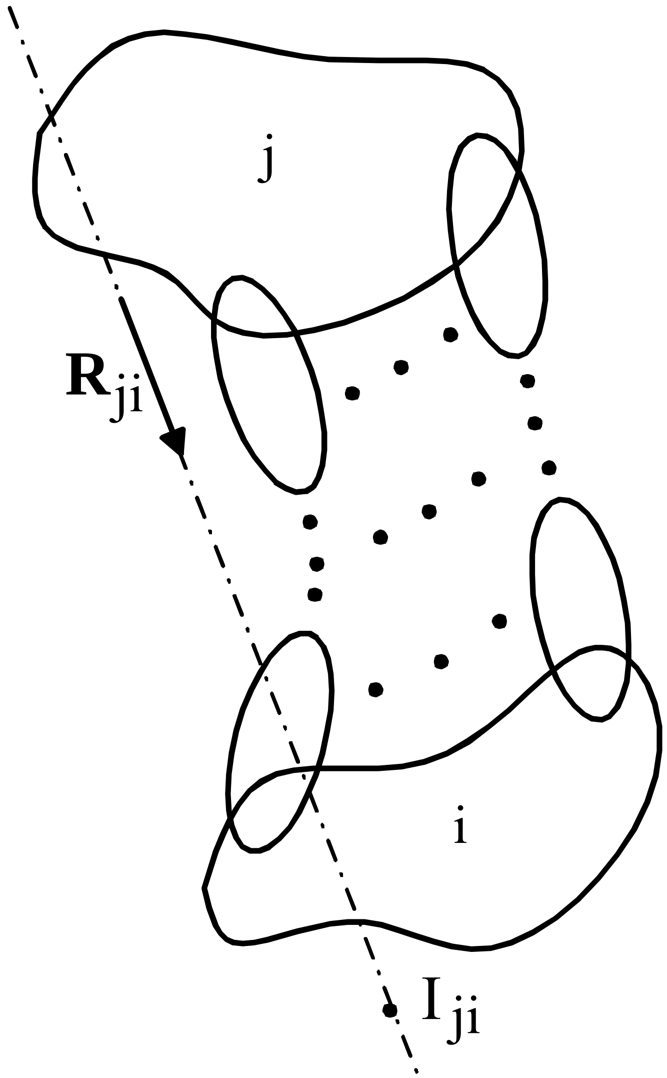
Aronhold-Kennedy (A-K) Theorem:

The three ICs (points I_{jk}, I_{ik}, and I_{ji} in the Figure) **shared by three rigid bodies in relative motion to one another <u>must lie</u> on the same straight line**

Corollary of Chasles' Theorem:

If a sliding contact joins two links, say j and i, **the instant center I_{ji} <u>must lie</u> on the common normal, at the contact point, of the two conjugate profiles**

Role of the ICs in planar mechanism design: Basic Concepts



Static Meaning of the IC:

If two links, say j and i (see the figure), **are connected to one another through a passive** (i.e., unloaded and not actuated) **frictionless planar kinematic chain, the central axis of the planar force system constituted by the reaction forces, transmitted from one link to the other through the kinematic chain, must pass through the instant center I_{ji}.**

Proof:

Since the kinematic chain is passive and frictionless, the power, w_{ji}, introduced into the mechanical system (see the Figure) by the reaction forces must be equal to zero. **Thus,** [$\mathbf{R}_{ji}$ and $M_{ji,P}$ are the resultants of forces and moments about P, respectively, transmitted from j to i; ${}^{j}\mathbf{v}_{P|i}$ and ω_{ij} are linear (referred to point P) and angular velocities of the relative motion of link i with respect to link j]:

$$w_{ji} = \mathbf{R}_{ji} \cdot {}^{j}\mathbf{v}_{Iji|i} + M_{ji,Iji}\,\omega_{ij} = 0 \quad \textit{with} \quad {}^{j}\mathbf{v}_{Iji|i}=\mathbf{0} \ \& \ \omega_{ij}\neq 0$$

$$\Downarrow$$

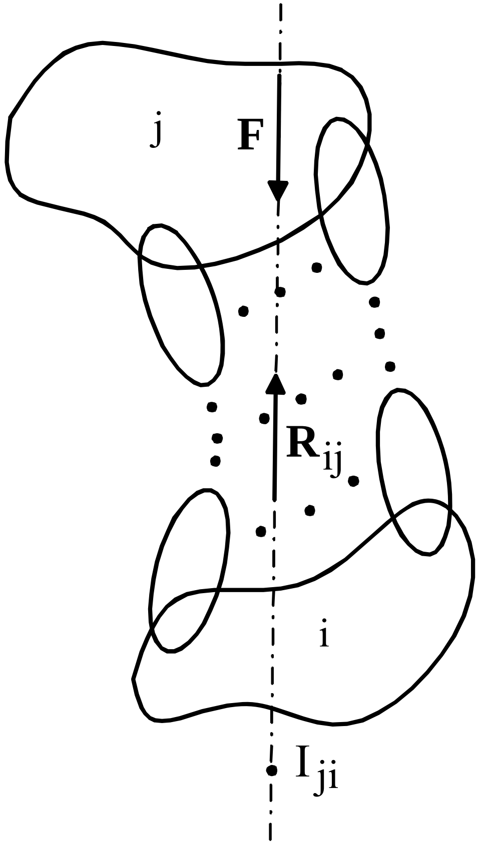
$$\mathbf{R}_{ji} \neq \mathbf{0} \quad \& \quad M_{ji,Iji} = 0$$

$$\Downarrow$$

I_{ji} belongs to the central axis. ***QED***

Role of the ICs in planar mechanism design: Basic Concepts

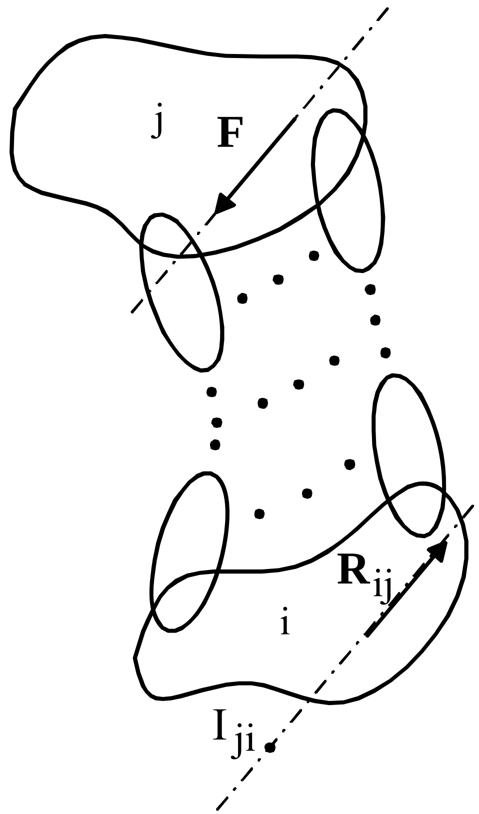
Corollaries of the IC Static Meaning:



(a)

a) If link j is loaded by a planar force system whose central axis passes through I_{ji}, the external load will be fully equilibrate by the reaction forces *and* **no relative motion between links j and i will occur** (i.e., the system is at rest whatever be the magnitude of the resultant of the external forces).

b) If link j is loaded by a planar force system whose central axis does not pass through I_{ji}, the kinematic chain will apply to link j a reaction force equal and opposite that passes through I_{ji} *and* **link j will start to rotate around I_{ji} with respect to link i.**



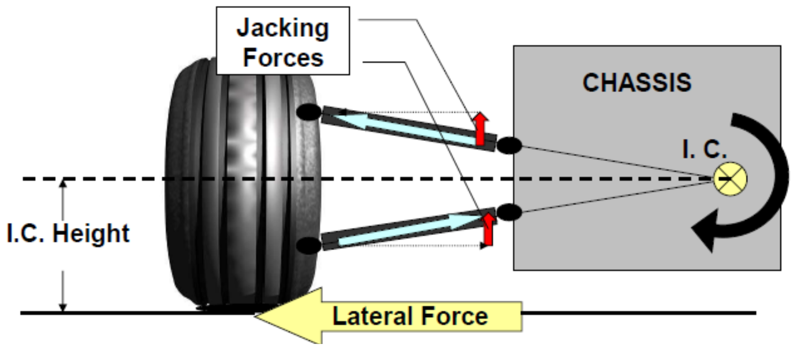
(b)

Role of the ICs in planar mechanism design: Application Examples

Vehicles' Suspensions:

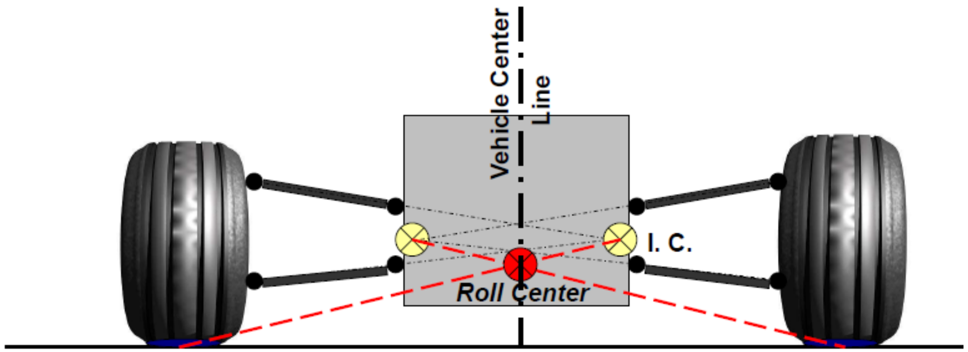
Jacking Forces

- Caused by geometrical binding of the upper and lower A-arms
- These forces are transferred from the tire to the chassis by the A-arms, and reduce the amount of force seen by the spring



Roll Center

- The roll center can be identified from this 2-D front view
  - Found at the intersection lines drawn for the Instant center to the contact patch center point, and the vehicle center line

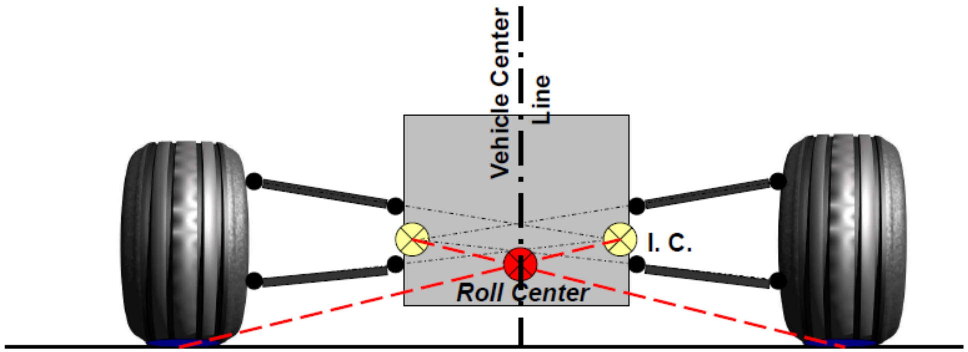


Role of the ICs in planar mechanism design: Application Examples

Vehicles' Suspensions:

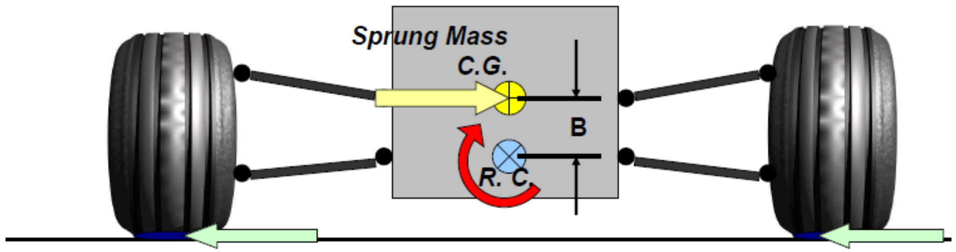
Roll Center

- The roll center can be identified from this 2-D front view
  - Found at the intersection lines drawn for the Instant center to the contact patch center point, and the vehicle center line



Roll Moment

- Present during lateral acceleration *(the cause of body roll)*
  - ***Moment Arm:***
    B = *Sprung mass C.G. height – Roll center height*
  - ***Force:***
    F = *(Sprung Mass) x (Lateral Acceleration)*

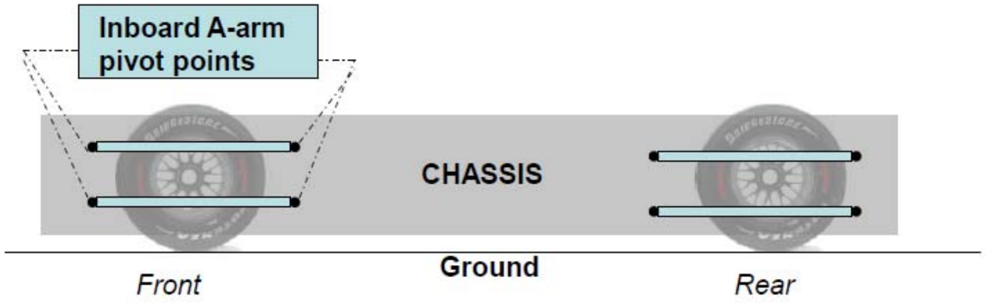


Role of the ICs in planar mechanism design: Application Examples

Vehicles' Suspensions:

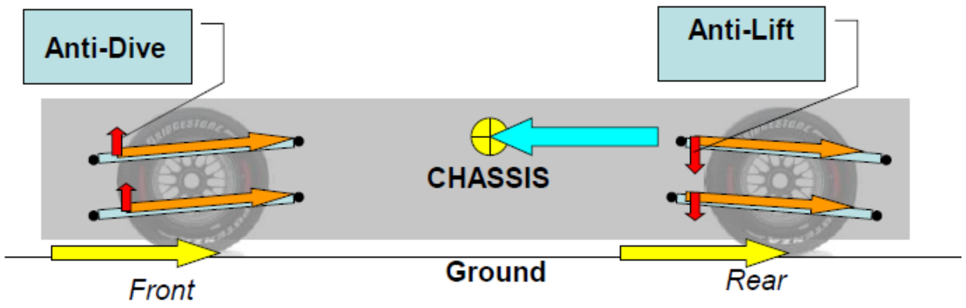
Side View

- The next step will be to consider the response of the suspension geometry to pitch situation
  - For this we will move to a 2-D side-view



Anti-Features

- By angling the A-arms from the side jacking forces are created
  - These forces can be used in the design to provide pitch resistance

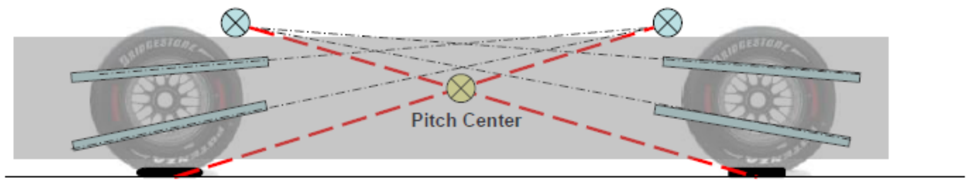


Role of the ICs in planar mechanism design: Application Examples

Vehicles' Suspensions:

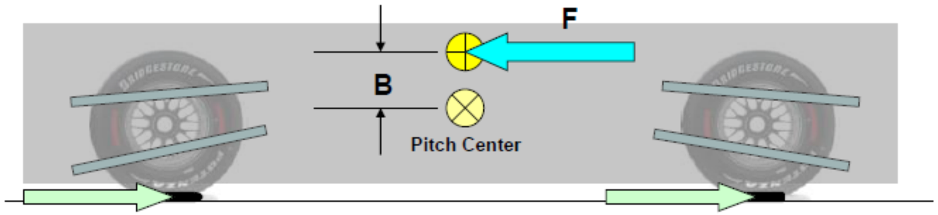
Pitch Center

- The pitch center can be identified from this 2-D side view
  - Found at the intersection lines drawn for the Instant center to the contact patch center point



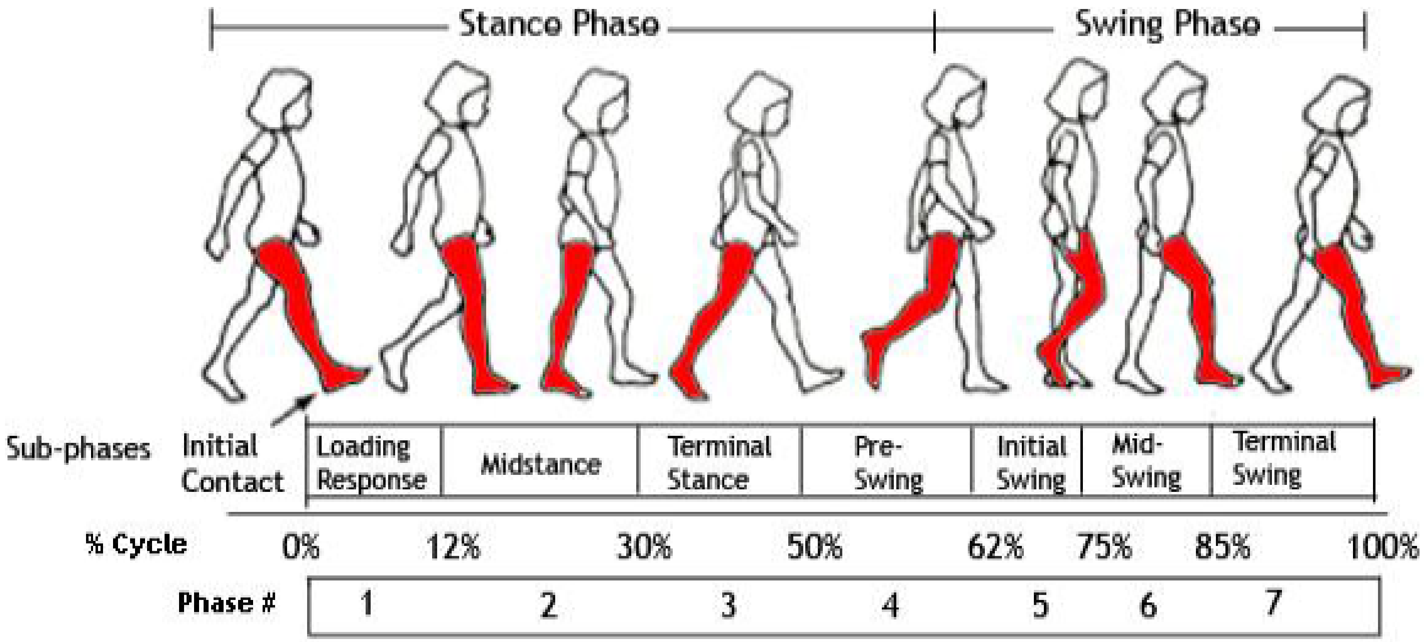
Pitch Moment

- Present during longitudinal acceleration
  - ***Moment Arm:***
  B = *Sprung mass C.G. height – Roll center height*
  - ***Force:***
  F = *(Sprung Mass) x (Longitudinal Acceleration)*



Role of the ICs in planar mechanism design: Application Examples

Lower-Limb Prostheses for Amputees:



Gait Phases

Role of the ICs in planar mechanism design: Application Examples

Lower-Limb Prostheses for Amputees:

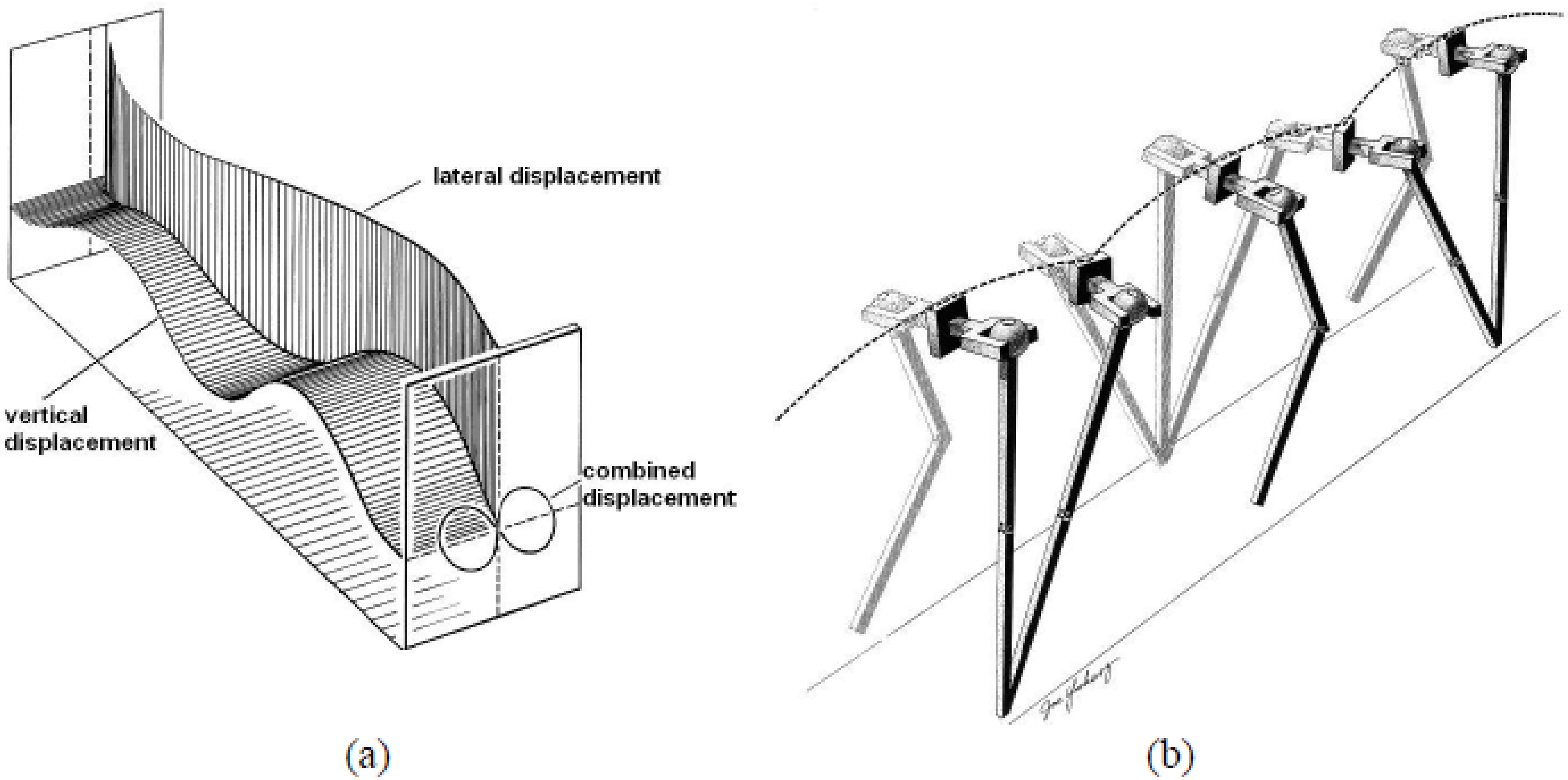


Figure 3-1: The exaggerated displacement of center of mass during one stride (a) lateral and vertical displacements in transverse and sagittal planes. Combination of these to displacements onto a plane perpendicular to the plane of progression is shown too [6]. (b) a simplified model showing bipedal locomotion; the vertical motion of the pelvis is indicated by dash lines [6].

Role of the ICs in planar mechanism design: Application Examples

Lower-Limb Prostheses for Amputees:

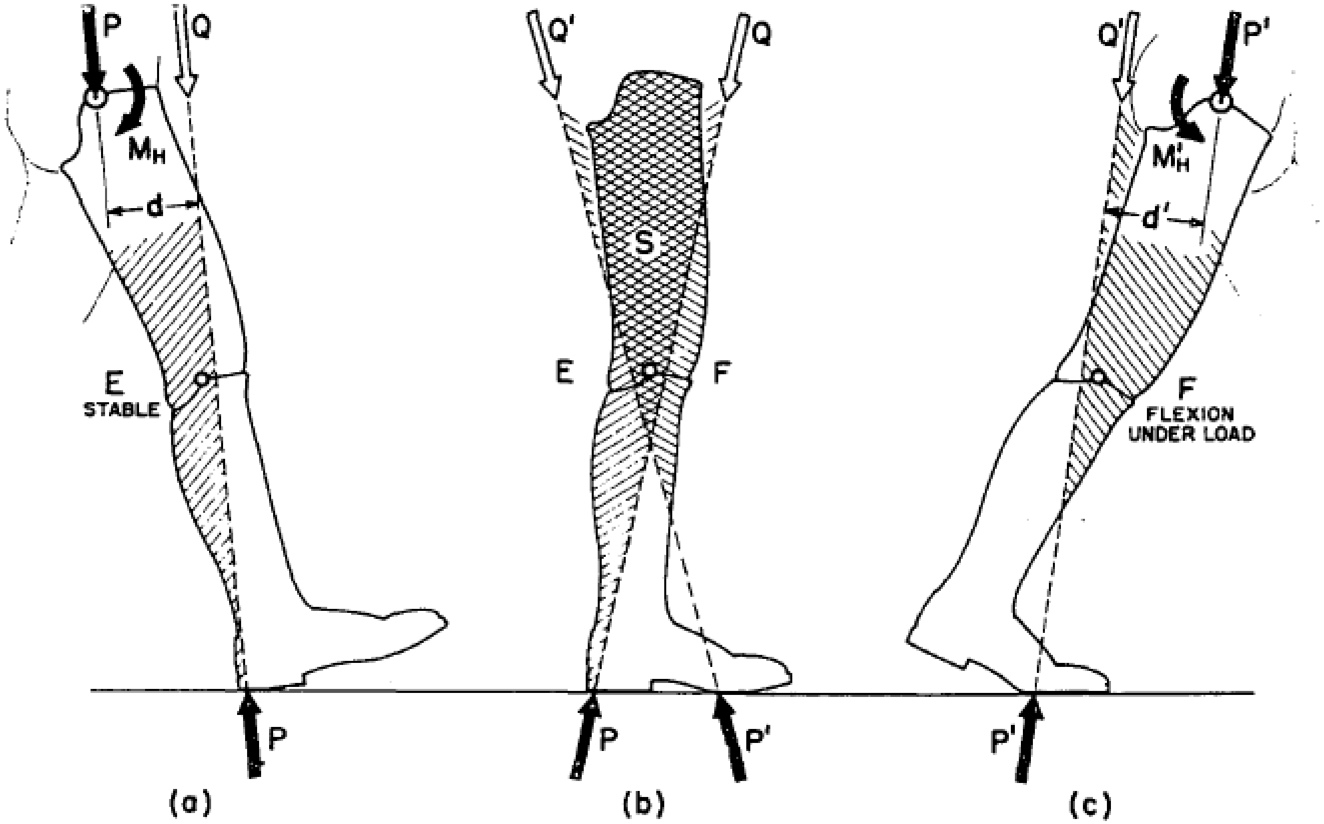


Fig. 7. The zone of voluntary stability, S.

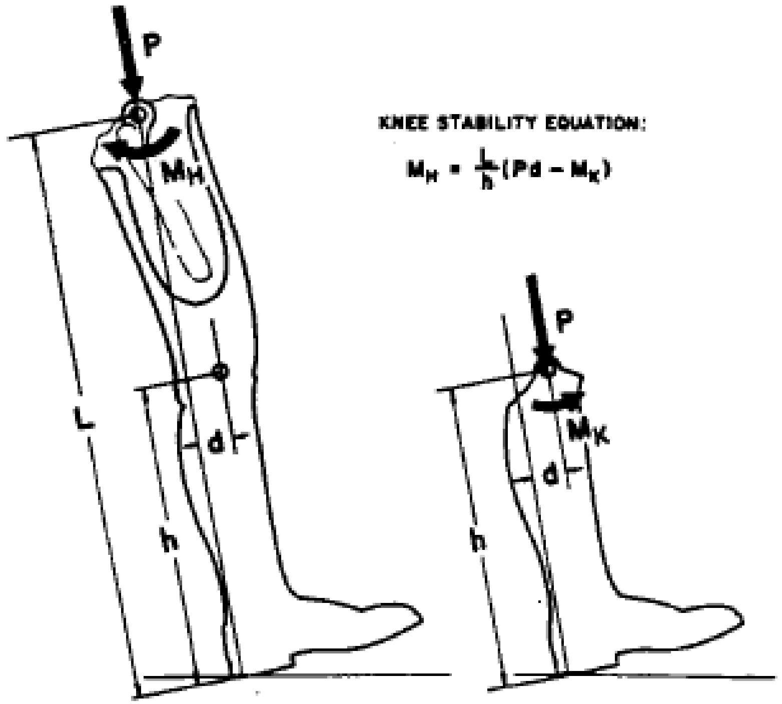


Fig. 9. Dimensions used in the knee stability equation.

Role of the ICs in planar mechanism design: Application Examples

Lower-Limb Prostheses for Amputees:

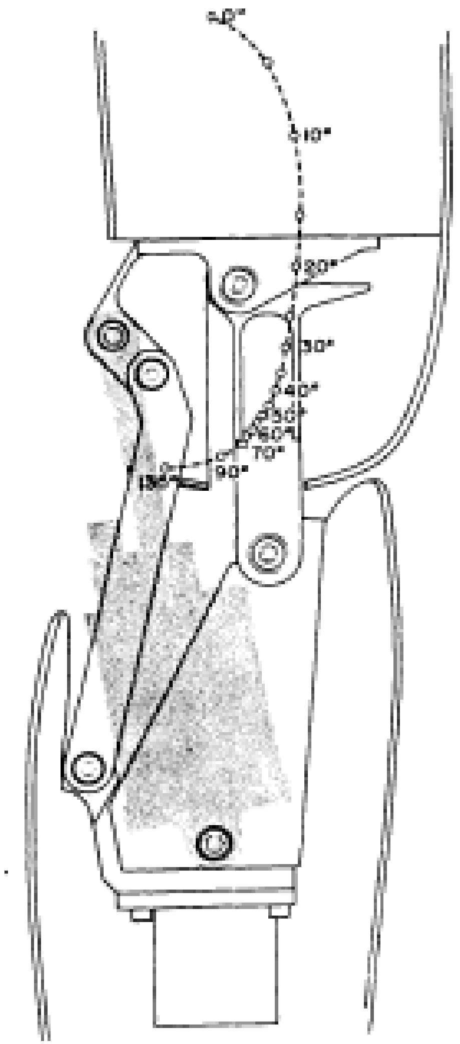


Fig. 10. Linkage arrangement and path of the instant centre for the UC-BL four-bar linkage polycentric knee.

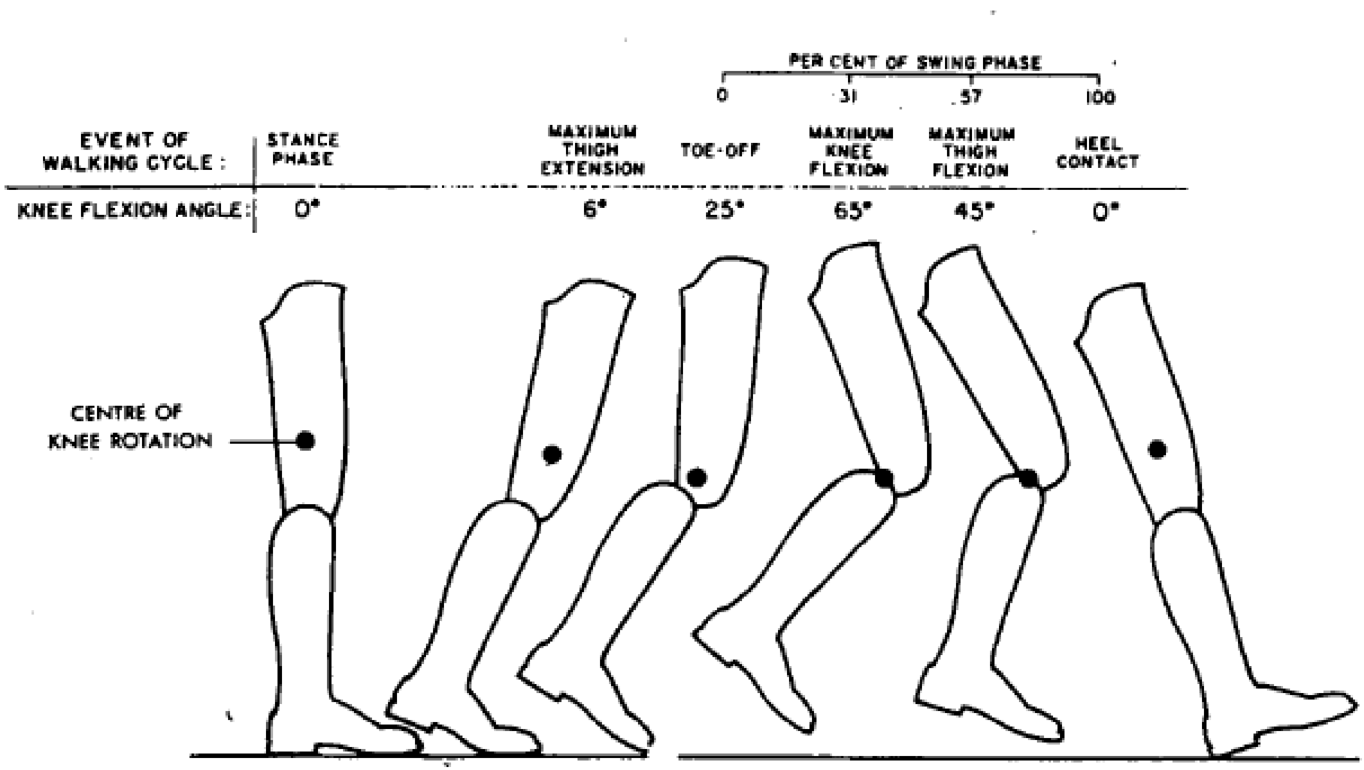


Fig. 11. Change in knee centre during walking.

Systematic Determination of all the ICs

In a mechanism with m links can be counted m(m – 1)/2 relative motions between link couples, but only (m – 1) are independent.

As a consequence, in a planar mechanism (PM) with m links, the number of ICs that can be determined is m(m – 1)/2 (i.e., one IC for each relative motion).

In single-DOF PMs, ICs' positions depend only on the mechanism configuration. So their locations can be determined just after the solution of the position analysis without using the rates of the input variable.

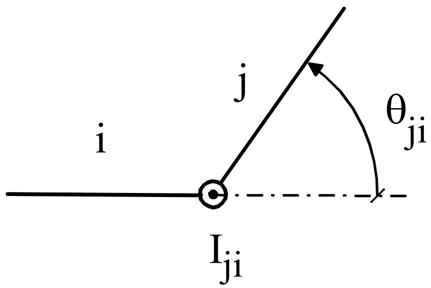
The set of PM's ICs can be splitted into two subsets: primary ICs *and* secondary ICs.

- **Primary ICs are those that are immediately identifiable by inspection.**
- **Secondary ICs are all those that are not Primary.**

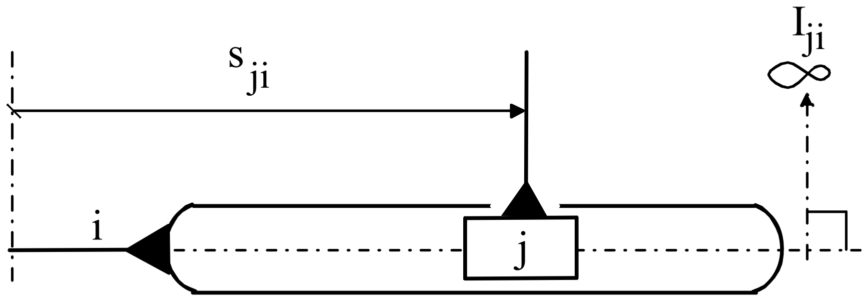
Systematic Determination of all the ICs

Determination of Primary ICs

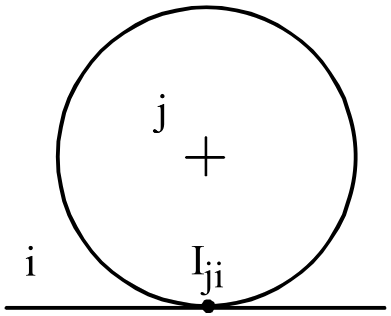
These ICs are related to the kinematic pairs. The only kinematic pairs that appear in PMs are prismatic (P) pairs, revolute (R) pairs, rolling contacts (R_c) , and sliding contacts (S_c):



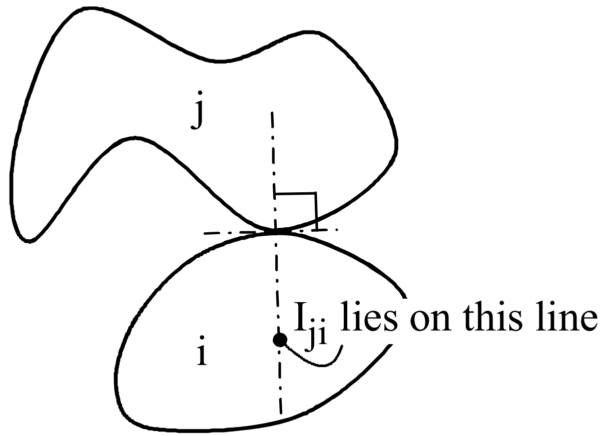
Revolute (R) Pair



Prismatic (P) Pair



Rolling Contact (R_c)



Sliding Contact (S_c)

Systematic Determination of all the ICs

Determination of Secondary ICs

DATA & TOOLS: Primary ICs + A-K Theorem + Chasles Theorem's Corollary

METHODOLOGY: 1) Find two lines the unknown IC lies on; 2) Determine (either graphically or analytically) the intersection point of those two lines; 3) Add the found IC to the data set *and* **repeat from step (1) for another unknown IC**

The CENTRAL POINT for implementing the methodology is the determination of a sequence of unknown ICs to be used. Such a sequence comes out by listing,

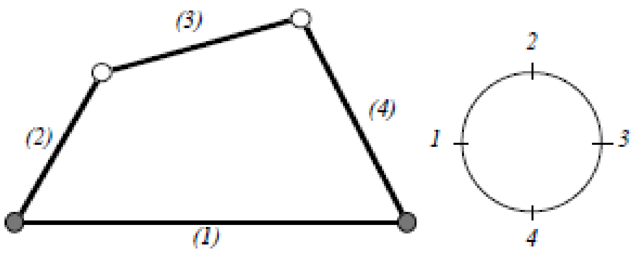
- **firstly, all the ICs that can be determined by using only the primary ICs;**
- **secondly, all the ICs that can be determined by using the primary ICs and the ICs previously determined, and so on till all the unknown ICs are included in the list**

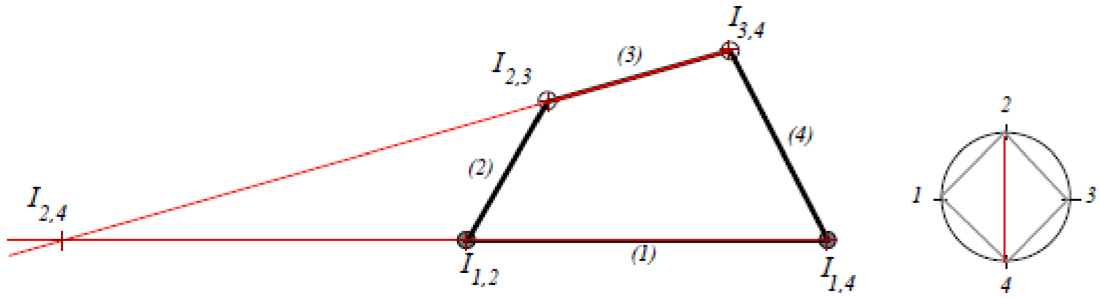
Systematic Determination of all the ICs

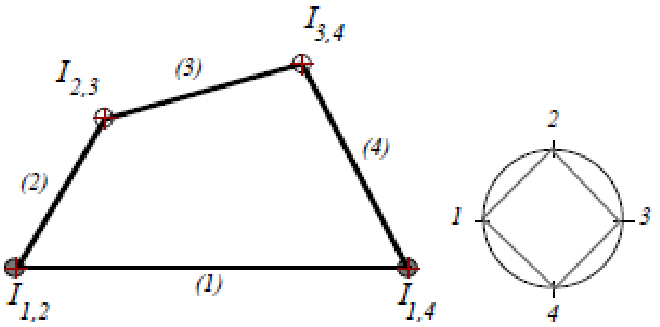
Determination of Secondary ICs (contd)

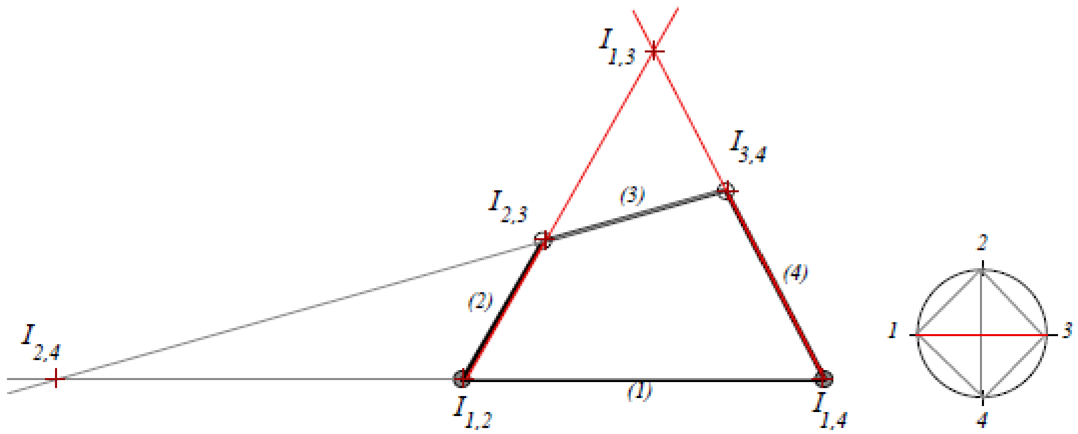
In most of single-DOF PMs, the sequence can be determined by using CIRCLE DIAGRAMS. In these PMs, the A-K theorem is sufficient to determine all the secondary ICs.

Example: four-bar linkage







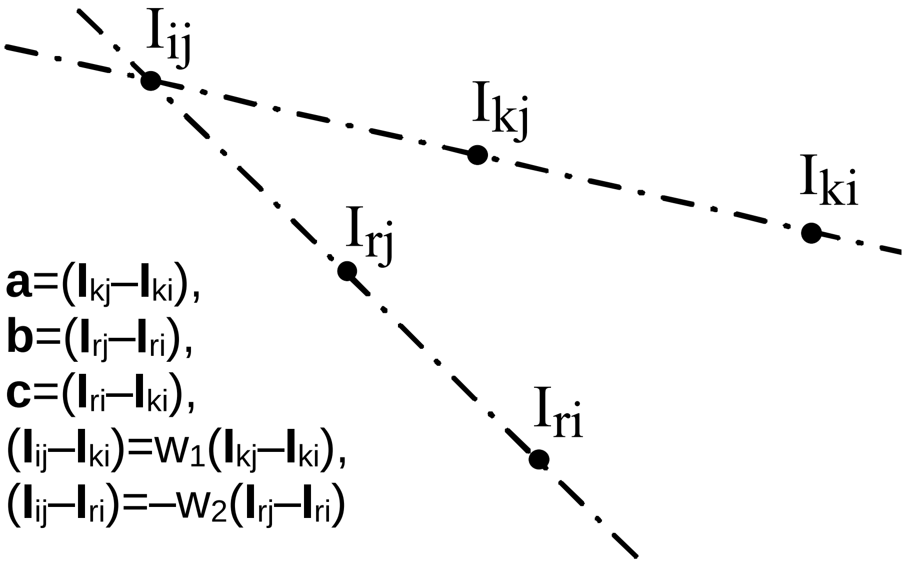


Systematic Determination of all the ICs

Determination of Secondary ICs (contd)

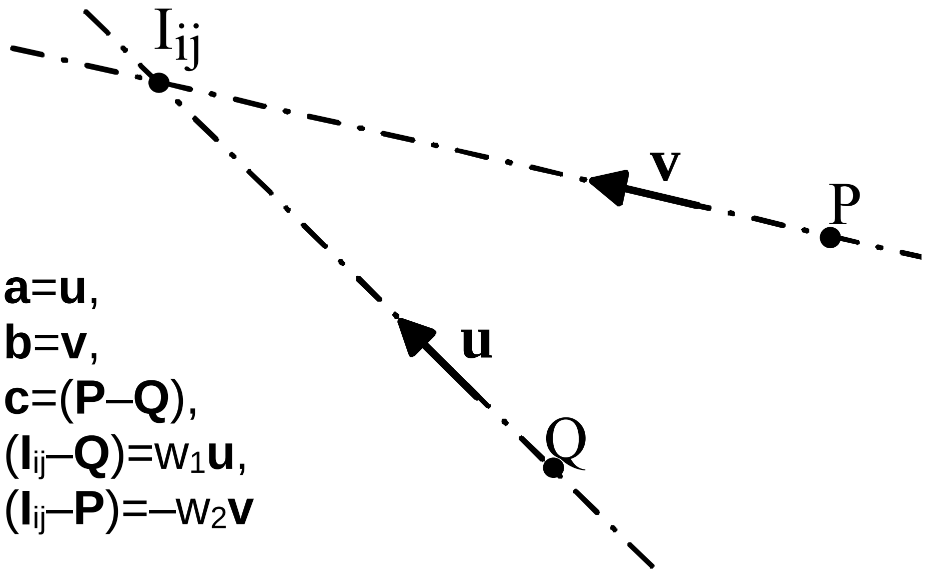
Once the sequence of lines to draw is known and the position analysis has been solved),

All secondary ICs' positions *can be sequentially determined* **as common intersection of two lines, the sought-after IC lies on** $\Rightarrow$ $w_1\,\mathbf{a} + w_2\,\mathbf{b} = \mathbf{c}$ $\Rightarrow$ $w_1 = \dfrac{\mathrm{Im}(\mathbf{c}\,\underline{\mathbf{b}})}{\mathrm{Im}(\mathbf{a}\,\underline{\mathbf{b}})}$ *and* $w_2 = \dfrac{\mathrm{Im}(\mathbf{c}\,\underline{\mathbf{a}})}{\mathrm{Im}(\underline{\mathbf{a}}\,\mathbf{b})}$



$\mathbf{a}=(\mathbf{I}_{kj}-\mathbf{I}_{ki})$,
$\mathbf{b}=(\mathbf{I}_{rj}-\mathbf{I}_{ri})$,
$\mathbf{c}=(\mathbf{I}_{ri}-\mathbf{I}_{ki})$,
$(\mathbf{I}_{ij}-\mathbf{I}_{ki})=w_1(\mathbf{I}_{kj}-\mathbf{I}_{ki})$,
$(\mathbf{I}_{ij}-\mathbf{I}_{ri})=-w_2(\mathbf{I}_{rj}-\mathbf{I}_{ri})$

(a)



$\mathbf{a}=\mathbf{u}$,
$\mathbf{b}=\mathbf{v}$,
$\mathbf{c}=(\mathbf{P}-\mathbf{Q})$,
$(\mathbf{I}_{ij}-\mathbf{Q})=w_1\mathbf{u}$,
$(\mathbf{I}_{ij}-\mathbf{P})=-w_2\mathbf{v}$

(b)

Intersection of two lines the instant center $\mathbf{I}_{ij}$ lies on (*here, complex numbers are used*):

(a) the two lines *are identified* **through the A-K theorem,**

(b) the two lines *refer to either* **a slipping contact** *or* **a prismatic pair.**

Systematic Determination of all the ICs

Determination of Secondary ICs (contd)

Circle diagram method is mainly a graphical technique, which requires that equivalent kinematic chains with only R and P pairs replace S_c pairs.

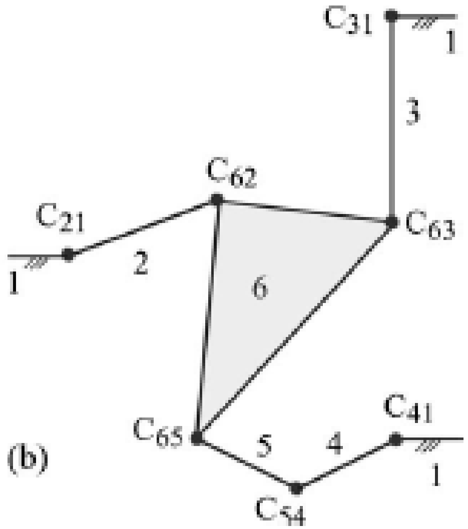
In order to devise an algorithm that computes the coordinates of all the ICs after the solution of the position analysis other techniques must be used. In

- **Di Gregorio R., 2008,** "An Algorithm for Analytically Calculating the Positions of the Secondary Instant Centers of Indeterminate Linkages," *ASME J. Mech. Design*, 130(4):042303-042303-9

has been proposed the use of the following "Table"

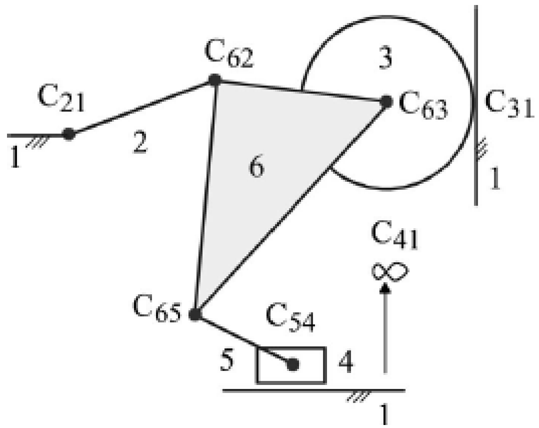
	1	2	3	4	5	6
1		×	×	×	II	I
2	×		I	III	III	×
3	×	I		III	III	×
4	×	III	III		×	II
5	II	III	III	×		×
6	I	×	×	II	×	

(a)



(b)

A linkage with six bars and seven revolute pairs:(a)table; (b) sketch of the linkage



Mechanism obtained from linkage (b) by replacing 2 R-pairs with 1 P-pair and 1 R_c-pair

Systematic Determination of all the ICs

Determination of Secondary ICs (contd)

General Case Table

	1	2	…	i	…	j	…	m	1_{sc}	2_{sc}	…	q_{sc}
1	■	×	…	×	…		…		×		…	×
2	×	■	…		…		…	×		×	…	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
i	×		…	■	…		…		×		…	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
j			…		…	■	…	×		×	…	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
m		×	…		…	×	…	■			…	×

General Case Table

Mechanism with m links and q sliding contacts [the first row (column) only contains the row (column) index and is not counted as an effective row (column)]:

Rows' No. = m (Each row refers to a link)
Columns' No. = m + q

- the last q columns refer to the sliding contacts:
in these columns, the two cells whose row indices correspond to either of the two links that touch one another at the sliding contact are filled with the cross "×"; whereas, all the other cells are left empty (the IC that lies on the common normal at the contact point is the one whose two indices are the two row indices of the two nonempty cells).

- The first m columns refer to the links:
in these columns, the black cell is the one with row index equal to column index (the black cell does not correspond to any geometric and/or kinematic parameter of the mechanism); whereas, all the other cells correspond to the ICs identified by the row and the column indices of the cell. The cells with a cross "×" correspond to the primary ICs; the empty cells correspond to the secondary ICs.

Systematic Determination of all the ICs

Determination of Secondary ICs (contd)

General Case Table

	1	2	...	i	...	j	...	m	1_{sc}	2_{sc}	...	q_{sc}
1	■	×	...	×	...		...		×		...	×
2	×	■	...		...		...	×		×	...	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
i	×		...	■	...		...		×		...	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
j			...		...	■	...	×		×	...	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
m		×	...		...	×	...	■			...	×

General Case Table

With these rules:

- the square sub-matrix constituted of the first *m* columns is symmetric;

- two filled cells in the same column mean that the positions of two ICs with a common index (the column index) **are known** (**i.e. the geometric data of the line, where the IC with indices equal to the two row indices of the two filled cells is located, are available**).

The filling rules for the empty cells of the first m columns of the table are as follows:
the empty cells must be filled with "I", if the secondary ICs they refer to can be determined by using only the primary ICs; they must be filled with "II", if the secondary ICs they refer to can be determined by using only the primary ICs and the ICs denoted with "I"; and so forth.

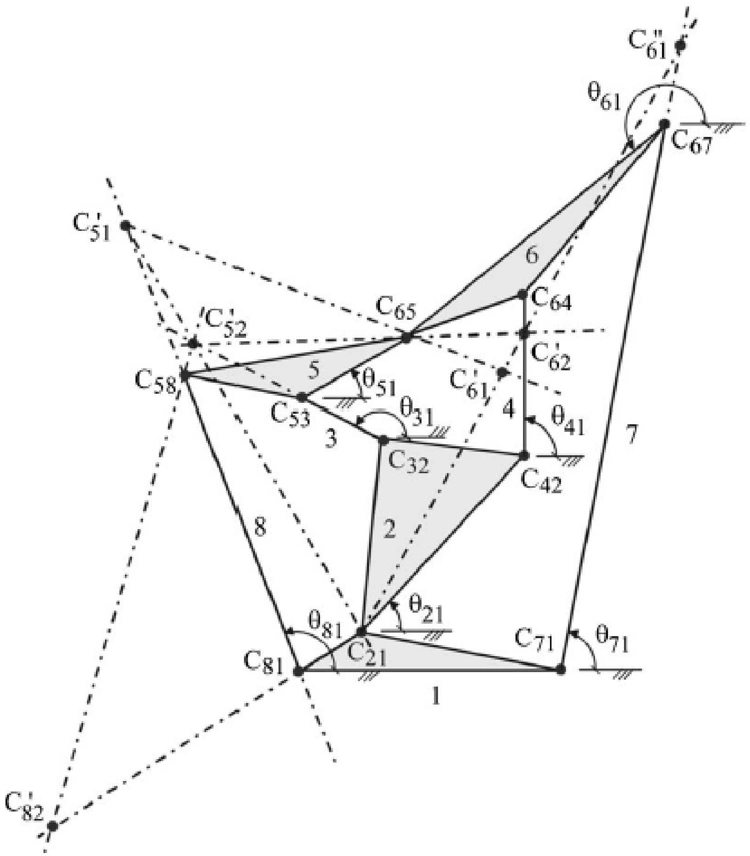
The I_{ij} IC can be determined, if there are 4 filled cells at the four intersections among the two rows i and j and two any columns of the table. (**The indices of the four filled cells give sufficient pieces of information to write the complex equation that must be solved to analytically determine the position of I_{ij}. Of course, if the (i, j) cell can be filled, the (j, i) cell can be filled too**).

Systematic Determination of all the ICs

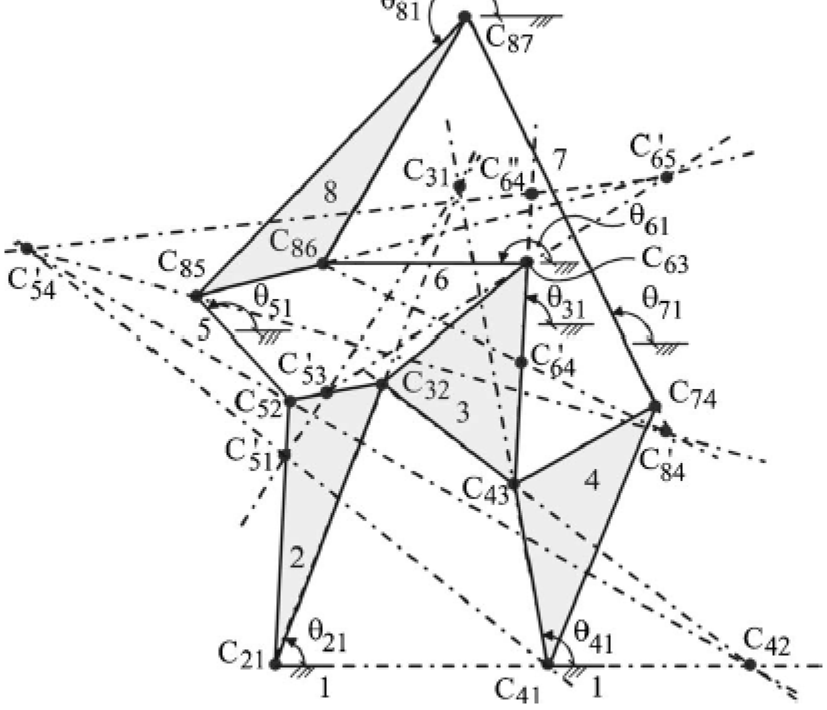
Determination of Secondary ICs: The case of the indeterminate linkages

Single-DOF PMs whose secondary ICs cannot be located by directly applying the A-K theorem are named "Indeterminate Mechanisms" (IMs).

"Double Butterfly linkage" and "Single Flier 8-bar linkage" are two of the most famous IMs



Double Butterfly linkage



Single Flier 8-bar linkage

Systematic Determination of all the ICs

Determination of Secondary ICs: The case of the indeterminate linkages

In the IMs, if one tries to find the IC sequence to compute (e.g., by using a circle diagram), he will not be able to complete it since, for some ICs, only one line is found.

This problem can be solved as follows:

i) **Choose an IC, say I_{ij}, for which one line has been identified;**

ii) **Write the position of I_{ij} as an explicit function of a scalar unknown, say x, and of the geometric data of the line;**

iii) **Coherently write a minimal set of complex equations** (i.e., of type w_1 **a** + w_2 **b** = **c**), **whosesolutions give the positions of other ICs** (i.e., the complex numbers **a**, **b**, and **c** are computed as functions of x) **and analytically solve them until an instant center, say I_{kl}, which must be at the intersection of three (instead of two) already located lines is identified** (i.e., a graphic mismatch occurs); **Since I_{kl} lies on three lines, two complex independent equations can be written for locating I_{kl}, but only three additional scalar unknowns must be introduced to write them. These two complex equations are a system of four scalar equations linear in the three additional unknowns and with coefficients that are explicit functions of x.**

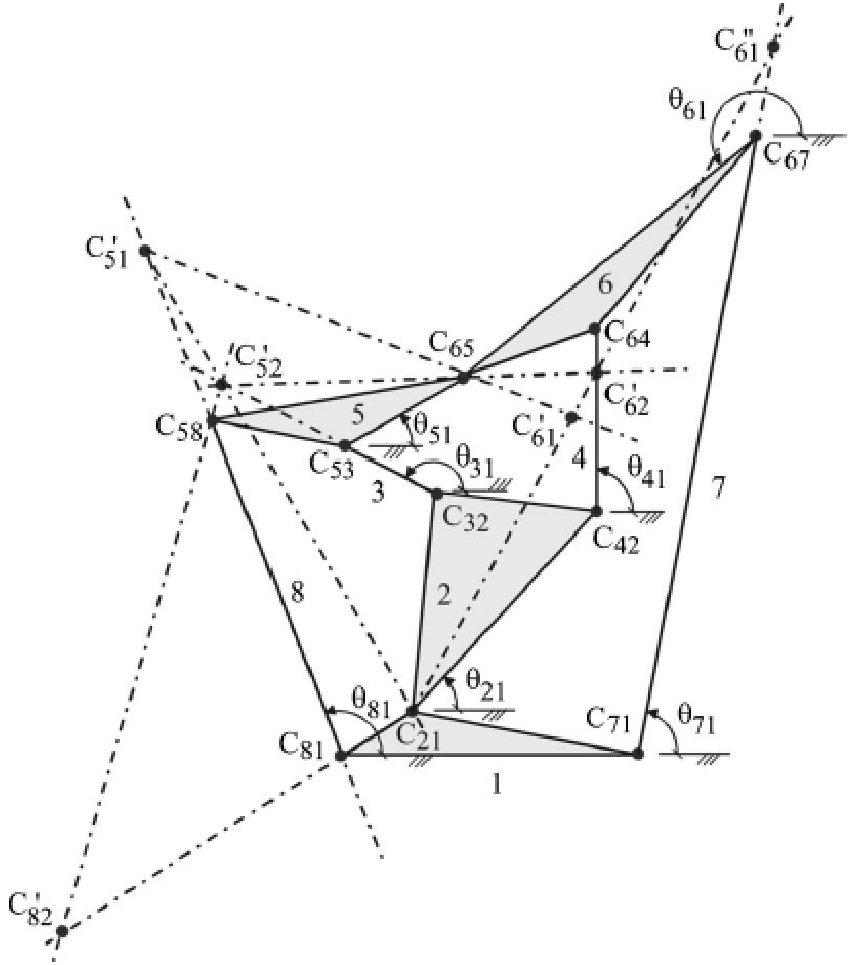
iv) **Write such a system and solve it by linearly eliminating the three additional unknowns to compute a compatibility equation whose only unknown is x;**

v) **Solve this compatibility equation to obtain the value of x;**

vi) **Determine the positions of I_{ij}, I_{kl}, and of all the other ICs used to identify I_{kl} by using the computed value of x;**

Systematic Determination of all the ICs

Determination of Secondary ICs: The case of the indeterminate linkages



Such a procedure can be easily implemented by using the "Table"

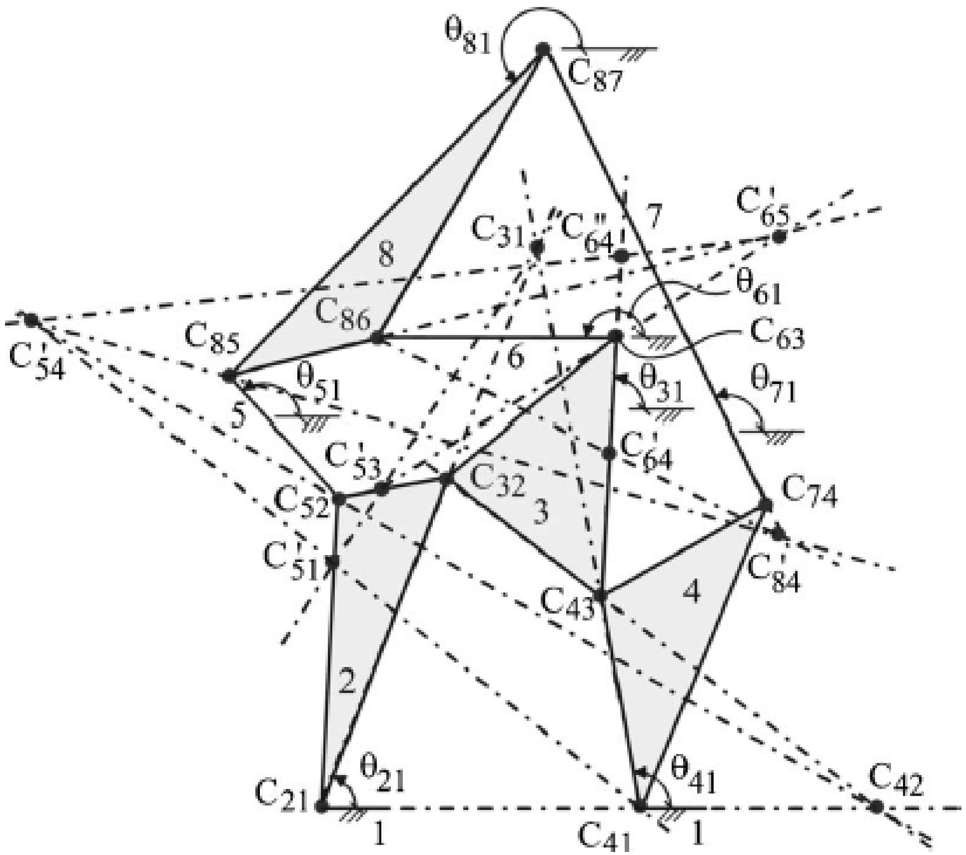
	1	2	3	4	5	6	7	8
1		×			III*	IV^+	×	×
2	×		×	×	II*	III*		I*
3		×			×			
4		×				×		
5	III*	II*	×			×		×
6	IV^+	III*		×	×		×	
7	×					×		
8	×	I*			×			

Double Butterfly linkage

Systematic Determination of all the ICs

Determination of Secondary ICs: The case of the indeterminate linkages

Such a procedure can be easily implemented by using the "Table"



	1	2	3	4	5	6	7	8
1	■	×	I	×	I*			
2	×	■	×	I	×			
3	I	×	■	×	II*	×		
4	×	I	×	■	II*	IV+	×	III*
5	I*	×	II*	II*	■	III*		×
6			×	IV+	III*	■		×
7				×			■	×
8				III*	×	×	×	■

Single Flier 8-bar linkage

Systematic Determination of all the ICs

Determination of Secondary ICs: The case of the indeterminate linkages

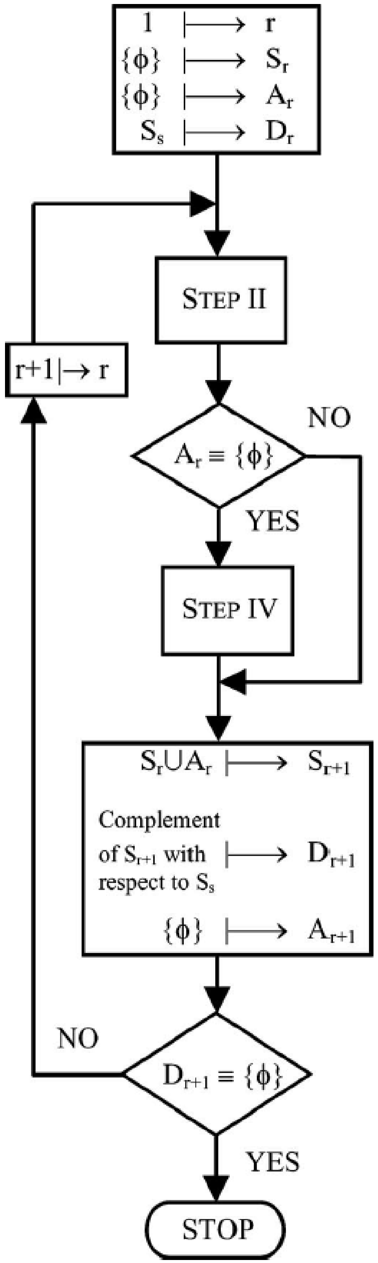
Table's filling rules for IMs:

(a) **Assume as known an IC that lies only on one line and fill the corresponding cell with the symbol "I*";**

(b) **Fill the empty cells with the symbol "II* ", if the secondary ICs they refer to can be determined by using only the filled cells that refer to already determined ICs and the cell filled with I*; fill the empty cells with the symbol "III* ", if the secondary ICs they refer to can be determined by using only the filled cells that refer to already determined ICs and the cells filled with I* or II*; and so forth until a cell that refers to an IC** (overdetermined IC) **that must lie on three previously determined lines is identified;**

(c) **Identify the minimum number of ICs to be computed as a function of the assumed one by looking at the cells that are strictly necessary to move from the I* cell to the cell that refers to the overdetermined IC;**

(d) **Write and solve the set of complex equations that give the position of the assumed IC and compute the positions of all the others.**



Systematic Determination of all the ICs

The General Algorithm

Input data of the sorting algorithm that generates the sequence S_o:

S_p = set of the primary ICs; **S_s = set of the secondary ICs;**

L = set of the lines that are common normals at the contact points of the sliding contacts

Step I (Initialization of the Dummy Variables): **Put 1 into the counter, r; put the empty set, {Φ}, into the sets S_r and A_r; put the set S_s into the set D_r.**

Step II: **For each I_{ij} of D_r, verify whether there are two lines that locate it by using only the elements of S_pU**L**US_r; if this condition occurs, write the appropriate complex equation, solve it, and add I_{ij} to the elements of the set A_r.**

Step III: **If the set A_r is the empty set, then** (the mechanism is Indeterminate) **continue, else go to "Step V."**

Step IV: **Solve the indeterminacy as explained above**

Step V: **Put the set S_rUA_r into the set S_{r+1}** (note that **$S_{r+1} \subset D_r$**)**; put the complement of S_{r+1} with respect to S_s into the set D_{r+1}; put the empty set, {Φ}, into the set A_{r+1}.**

Step VI: **If the set D_{r+1} is the empty set, then** (all the secondary ICs have been computed) **"Stop" the algorithm, else put r+1 into the counter r and go to "Step II".**

Singularity analysis with the ICs: Single-DOF Mechanisms

Definitions of Singularities:

[Gosselin and Angeles, 1990]: "Singularities are mechanism configurations where the instantaneous kinematics becomes indeterminate (i.e. where at least one out of the input–output instantaneous relationships that can be defined fails)"

[Hunt, 1978]: "Singularities are configurations where the mechanism locally either loses (stationary configurations) or gains (uncertainty configurations) at least one DOF"

These definitions express two different point of view, but they are equivalent.

The whole velocity analysis of a one-dof planar mechanism can be implemented through the ICs.

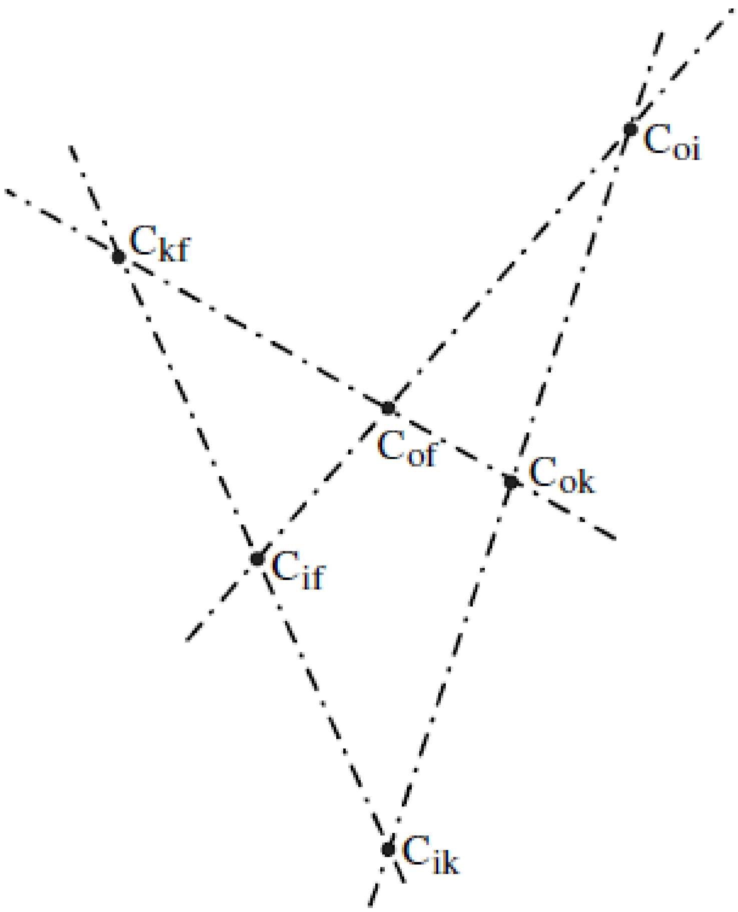
In particular [Di Gregorio, 2007]:

- the explicit expression of any input–output instantaneous relationship can be written by using the ICs;

- such relationship can be written by using only the ICs of the relative motions among 4 links: input link (i), output link (o), reference link (f) used to evaluate the rate of the input variable, and reference link (k) used to evaluate the rates of all the output variables.

[The input variable is a geometric parameter that defines the pose (position and orientation) of link "i" with respect to link "f "; whereas, the output variables are geometric parameters which define the pose of link "o" with respect to link "k"]

Singularity analysis with the ICs: Single-DOF Mechanisms (contd)



Among the four links "i", "o", "f " and "k", six different relative motions exist: "if ", "ok","ik", "kf ", "oi" and "of " [the second letter indicates the link from which the motion of the link, denoted by the first letter, is observed].

C_{mn},**and** $\mathbf{C}_{mn}$ **with** mn $\in$ {if, ok, ik, kf, oi, of } **will denote the the ICs and the position vector that locates them in the plane of motion, respectively, of these six motions.**

According to the A–K theorem, these ICs must lie on the straight lines shown in the Figure.

The following relationship holds:

$$ {}^{r}\mathbf{v}_{mn|t} = \mathrm{j}\omega_{tr}(\mathbf{C}_{mn} - \mathbf{C}_{tr}) \qquad (1) $$

where $\mathrm{j} = \sqrt{-1}$,

ω_{tr} **= signed magnitude** (positive if ccw) **of the angular velocity of link t with respect to link r**

${}^{r}\mathbf{v}_{mn|t}$ **= velocity** (in complex-number format) **of** C_{mn} **when considered fixed to link t and observed by link r**

Singularity analysis with the ICs: Single-DOF Mechanisms (contd)

Single-DOF mechanisms have a single input.
Thus, without loosing generality, a single-input–single-output system (SISO system) **can be considered as the general case since the formulas regarding a one-dof mechanism with n output variables** (i.e. a single-input-multiple–output system (SIMO system)) **can be obtained by considering n independent SISO systems working in parallel.**

In a SISO system the general form of the instantaneous input-output relationship is [$\dot{x}$ and $\dot{y}$ are the rates of the input, x, **and the output,** y, **variables, respectively; whereas,** a **and** b **are coefficients that are functions of** x]

$$b\dot{y} = a\dot{x} \tag{2}$$

$$\Downarrow$$

a=0 ⇒ dead center position of the output link (i.e., type-I singularity)

b=0 ⇒ dead center position of the input link (i.e., type-II singularity)

a=b=0 ⇒ uncertainty configuration (i.e., type-III singularity)

Singularity analysis with the ICs: Single-DOF Mechanisms (contd)

The input (output) variable is a motion characteristic of the relative motion "if " ("ok").

According to the type of joint connecting link "i" ("o") to link "f" ("k"), it can be either a rotation angle or a translation along a known direction.

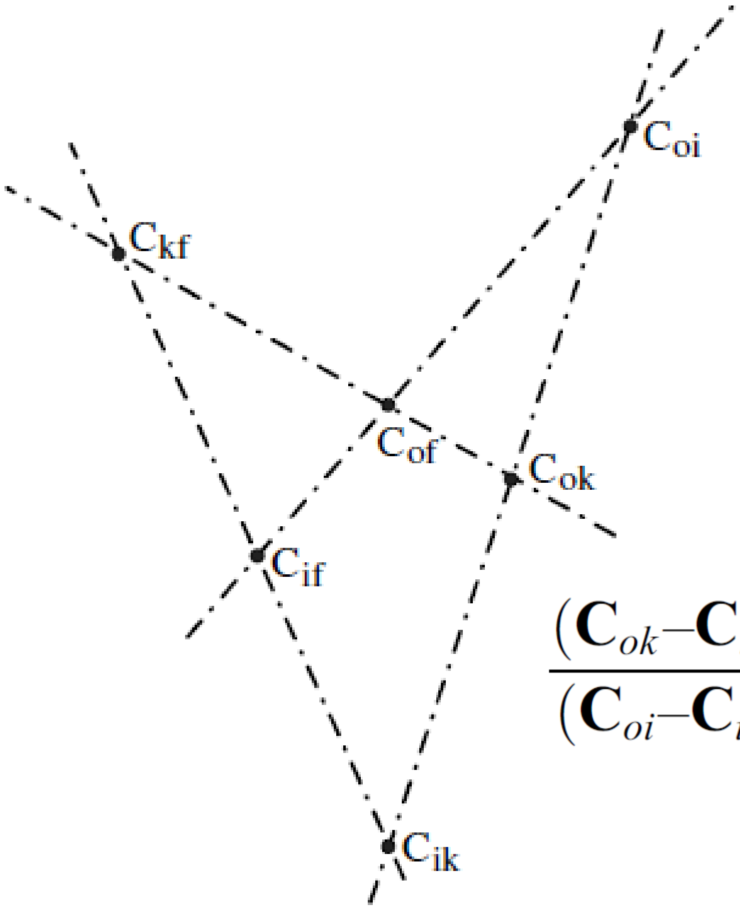
Therefore, there are only four cases to consider:

- **(i) rot–rot case:** both the input and the output variables are rotation angles
- **(ii) rot–tra case:** the input variable is a rotation angle **and** the output variable is a translation along a line
- **(iii) tra–rot case:** the input variable is a translation along a line **and** the output variable is a rotation angle
- **(iv) tra–tra case:** both input and output variables are translations along lines

Singularity analysis with the ICs: Single-DOF Mechanisms (contd)

Rot-Rot case:

$\dot{x}=\omega_{if}$ & $\dot{y}=\omega_{ok}$



Kinematic relationships:

$$ {}^{f}\mathbf{v}_{oi|o} = {}^{f}\mathbf{v}_{oi|i} \qquad {}^{k}\mathbf{v}_{oi|o} = {}^{k}\mathbf{v}_{oi|i} $$

$$ {}^{k}\mathbf{v}_{oi|o} = {}^{k}\mathbf{v}_{oi|i} \qquad {}^{f}\mathbf{v}_{ik|i} = {}^{f}\mathbf{v}_{ik|k} $$

$$ \omega_{if} + \omega_{ok} = \omega_{of} + \omega_{ik}; \qquad \omega_{if} = \omega_{ik} + \omega_{kf} $$

$$ \Downarrow \qquad \Downarrow $$

Input-Output relationship

$$ \frac{(\mathbf{C}_{ok}-\mathbf{C}_{ik})}{(\mathbf{C}_{oi}-\mathbf{C}_{ik})}\omega_{ok} = \frac{(\mathbf{C}_{of}-\mathbf{C}_{if})}{(\mathbf{C}_{oi}-\mathbf{C}_{of})}\omega_{if}; \quad \frac{(\mathbf{C}_{oi}-\mathbf{C}_{ok})}{(\mathbf{C}_{oi}-\mathbf{C}_{ik})}\omega_{ok} = \frac{(\mathbf{C}_{if}-\mathbf{C}_{kf})}{(\mathbf{C}_{ik}-\mathbf{C}_{kf})}\omega_{if} $$

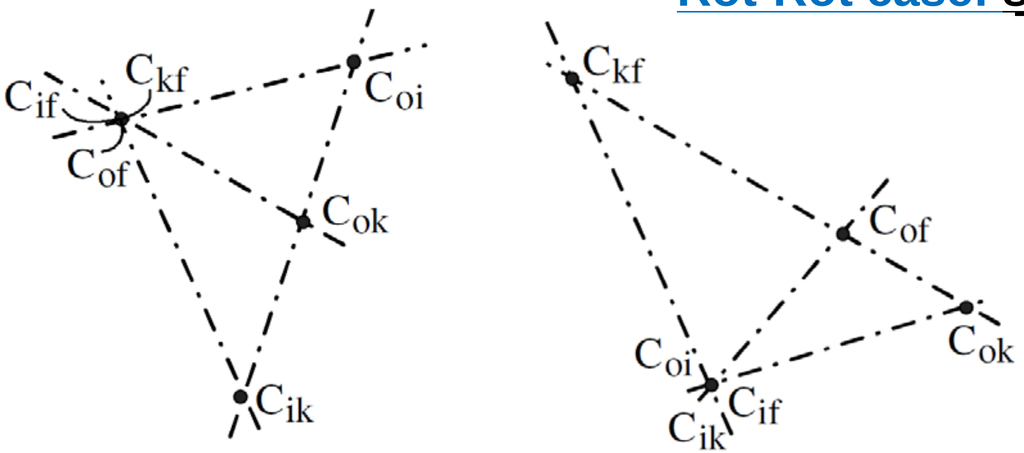
$$ \Downarrow $$

Geometric relationship among the six ICs

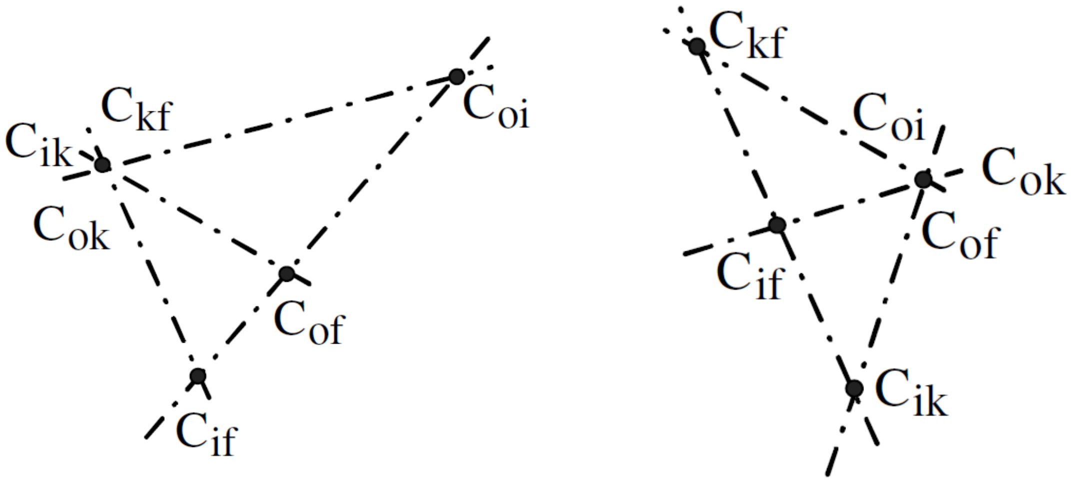
$$ \frac{(\mathbf{C}_{ok}-\mathbf{C}_{ik})}{(\mathbf{C}_{oi}-\mathbf{C}_{ok})} = \frac{(\mathbf{C}_{of}-\mathbf{C}_{if})}{(\mathbf{C}_{oi}-\mathbf{C}_{of})}\frac{(\mathbf{C}_{ik}-\mathbf{C}_{kf})}{(\mathbf{C}_{if}-\mathbf{C}_{kf})} $$

Singularity analysis with the ICs: Single-DOF Mechanisms (contd)

Rot-Rot case: singularity conditions



Type-I singularity conditions



Type-II singularity conditions

$$a = (\mathbf{C}_{of} - \mathbf{C}_{if})(\mathbf{C}_{oi} - \mathbf{C}_{ik})$$

$$b = (\mathbf{C}_{ok} - \mathbf{C}_{ik})(\mathbf{C}_{oi} - \mathbf{C}_{of})$$

$$\Downarrow$$

Type-I singularity conditions (a=0):

$$\mathbf{C}_{of} = \mathbf{C}_{if}$$

$$\mathbf{C}_{oi} = \mathbf{C}_{ik}$$

Type-II singularity conditions (b=0):

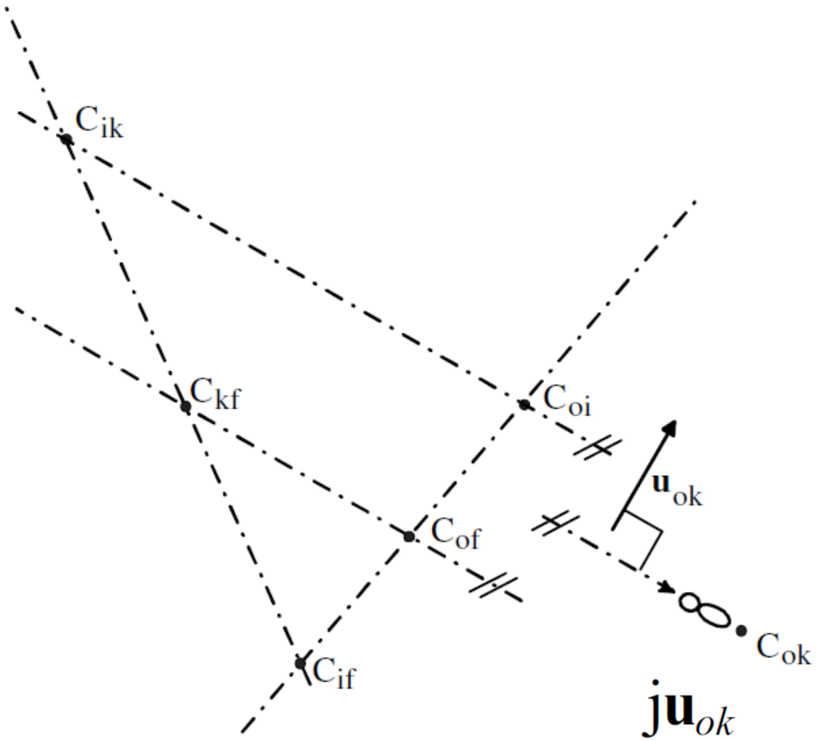
$$\mathbf{C}_{ok} = \mathbf{C}_{ik}$$

$$\mathbf{C}_{oi} = \mathbf{C}_{of}$$

Singularity analysis with the ICs: Single-DOF Mechanisms (contd)

Rot-Tra case:

$\dot{x}=\omega_{if}$ & $\dot{y}=\dot{s}_{ok}$



Kinematic relationships:

$$ {}^{f}\mathbf{v}_{oi|o} = {}^{f}\mathbf{v}_{oi|i} \qquad \dot{s}_{ok}\mathbf{u}_{ok} = {}^{k}\mathbf{v}_{oi|i} $$

$$ \dot{s}_{ok}\mathbf{u}_{ok} = {}^{k}\mathbf{v}_{oi|i} \qquad {}^{f}\mathbf{v}_{ik|i} = {}^{f}\mathbf{v}_{ik|k}, $$

$$ \omega_{if} = \omega_{of} + \omega_{ik}; \qquad \omega_{if} = \omega_{ik} + \omega_{kf} $$

$\Downarrow$ **Input-Output relationship** $\Downarrow$

$$ \frac{\mathbf{j}\mathbf{u}_{ok}}{(\mathbf{C}_{oi}-\mathbf{C}_{ik})}\dot{s}_{ok} = \frac{(\mathbf{C}_{of}-\mathbf{C}_{if})}{(\mathbf{C}_{oi}-\mathbf{C}_{of})}\omega_{if}; \qquad \frac{\mathbf{j}\mathbf{u}_{ok}}{(\mathbf{C}_{oi}-\mathbf{C}_{ik})}\dot{s}_{ok} = \frac{(\mathbf{C}_{kf}-\mathbf{C}_{if})}{(\mathbf{C}_{ik}-\mathbf{C}_{kf})}\omega_{if} $$

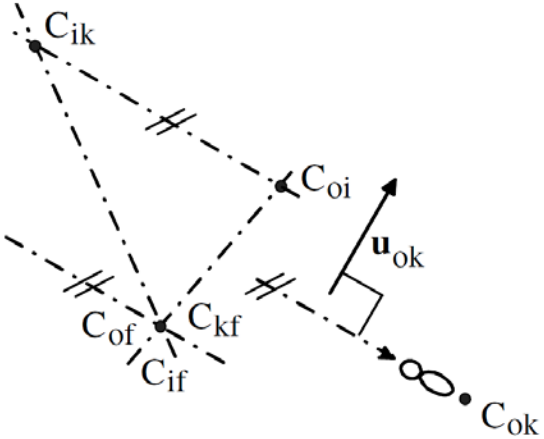
$\Downarrow$

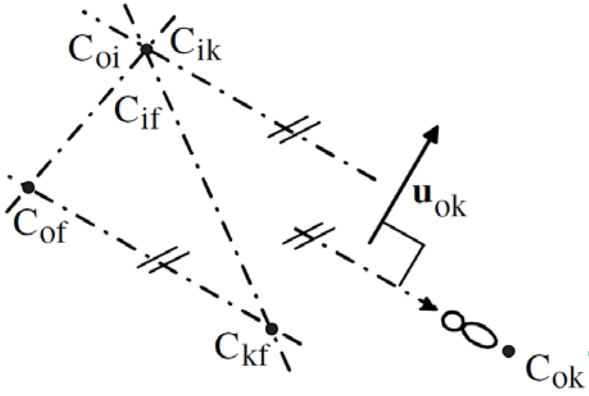
Geometric relationship among the six ICs

$$ 1 = \frac{(\mathbf{C}_{of}-\mathbf{C}_{if})}{(\mathbf{C}_{oi}-\mathbf{C}_{of})}\frac{(\mathbf{C}_{ik}-\mathbf{C}_{kf})}{(\mathbf{C}_{kf}-\mathbf{C}_{if})} $$

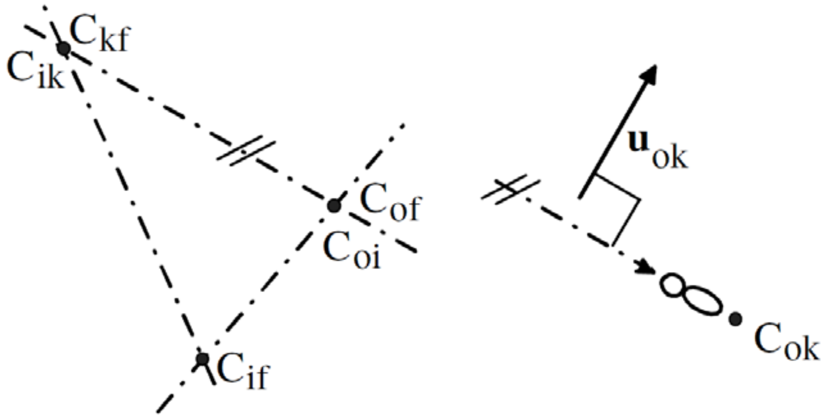
Singularity analysis with the ICs: Single-DOF Mechanisms (contd)

Rot-Tra case: singularity conditions





Type-I singularity conditions



Type-II singularity conditions

$$a = (\mathbf{C}_{of} - \mathbf{C}_{if})(\mathbf{C}_{oi} - \mathbf{C}_{ik})$$

$$b = j\mathbf{u}_{ok}(\mathbf{C}_{oi} - \mathbf{C}_{of})$$

$$\Downarrow$$

Type-I singularity conditions (a=0):

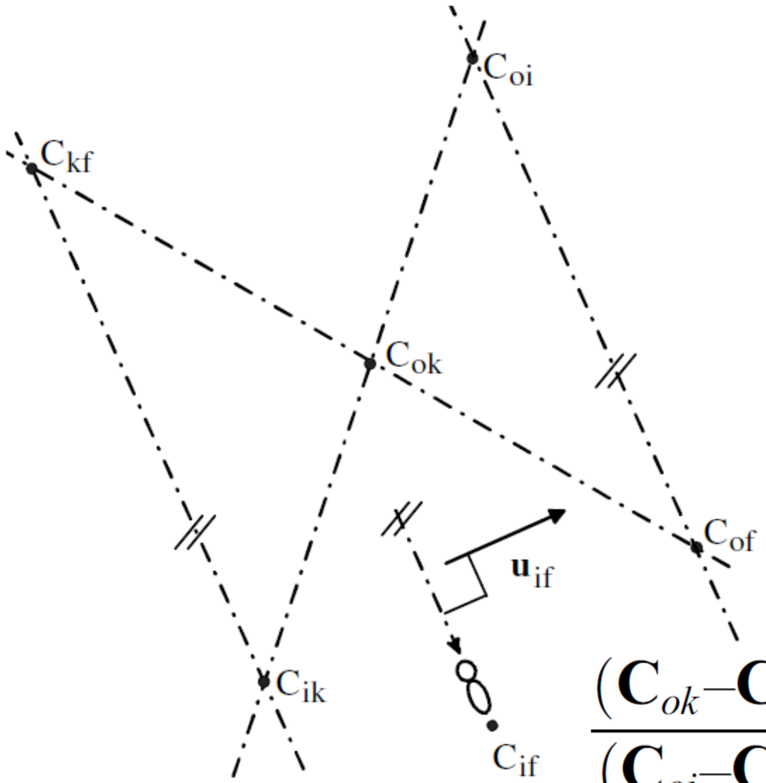
$$\mathbf{C}_{of} = \mathbf{C}_{if}$$

$$\mathbf{C}_{oi} = \mathbf{C}_{ik}$$

Type-II singularity conditions (b=0):

$$\mathbf{C}_{oi} = \mathbf{C}_{of}$$

Singularity analysis with the ICs: Single-DOF Mechanisms (contd)



Tra-Rot case:

$\dot{x} = \dot{s}_{if}$ & $\dot{y} = \omega_{ok}$

Kinematic relationships:

$$^{f}\mathbf{v}_{oi|o} = \dot{s}_{if}\mathbf{u}_{if} \qquad ^{k}\mathbf{v}_{oi|o} = {}^{k}\mathbf{v}_{oi|i}$$

$$^{k}\mathbf{v}_{oi|o} = {}^{k}\mathbf{v}_{oi|i} \qquad \dot{s}_{if}\mathbf{u}_{if} = {}^{f}\mathbf{v}_{ik|k}$$

$$\omega_{ok} = \omega_{of} + \omega_{ik}; \qquad 0 = \omega_{ik} + \omega_{kf}$$

$\Downarrow$ **Input-Output relationship** $\Downarrow$

$$\frac{(\mathbf{C}_{ok}-\mathbf{C}_{ik})}{(\mathbf{C}_{oi}-\mathbf{C}_{ik})}\omega_{ok} = \frac{\mathbf{j}\mathbf{u}_{if}}{(\mathbf{C}_{of}-\mathbf{C}_{oi})}\dot{s}_{if}; \quad \frac{(\mathbf{C}_{oi}-\mathbf{C}_{ok})}{(\mathbf{C}_{oi}-\mathbf{C}_{ik})}\omega_{ok} = \frac{\mathbf{j}\mathbf{u}_{if}}{(\mathbf{C}_{ik}-\mathbf{C}_{kf})}\dot{s}_{if}$$

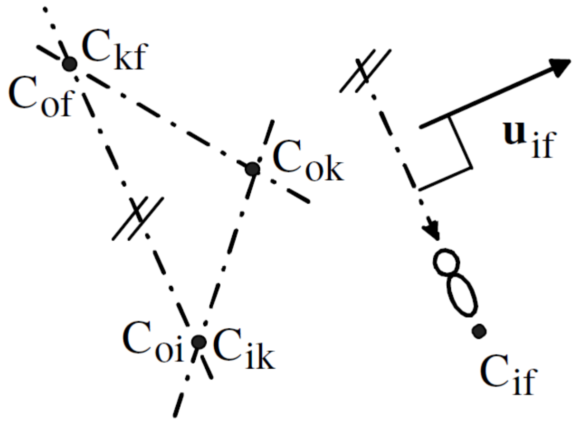
$\Downarrow$

Geometric relationship among the six ICs

$$\frac{(\mathbf{C}_{ok}-\mathbf{C}_{ik})}{(\mathbf{C}_{oi}-\mathbf{C}_{ok})} = \frac{(\mathbf{C}_{ik}-\mathbf{C}_{kf})}{(\mathbf{C}_{of}-\mathbf{C}_{oi})}$$

Singularity analysis with the ICs: Single-DOF Mechanisms (contd)

Tra-Rot case: singularity conditions



Type-I singularity conditions

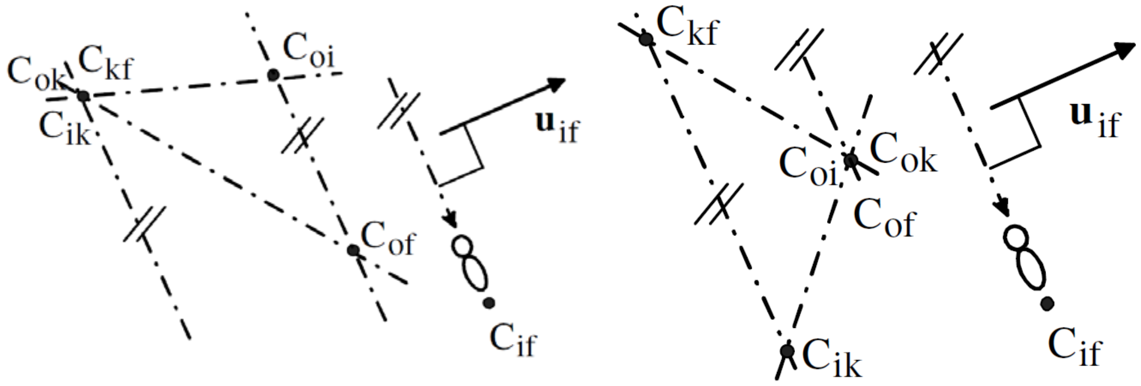
$$a = j\mathbf{u}_{if}(\mathbf{C}_{oi} - \mathbf{C}_{ik})$$

$$b = (\mathbf{C}_{ok} - \mathbf{C}_{ik})(\mathbf{C}_{of} - \mathbf{C}_{oi})$$

$$\Downarrow$$

Type-I singularity conditions (a=0):

$$\mathbf{C}_{oi} = \mathbf{C}_{ik}$$



Type-II singularity conditions (b=0):

$$\mathbf{C}_{ok} = \mathbf{C}_{ik}$$

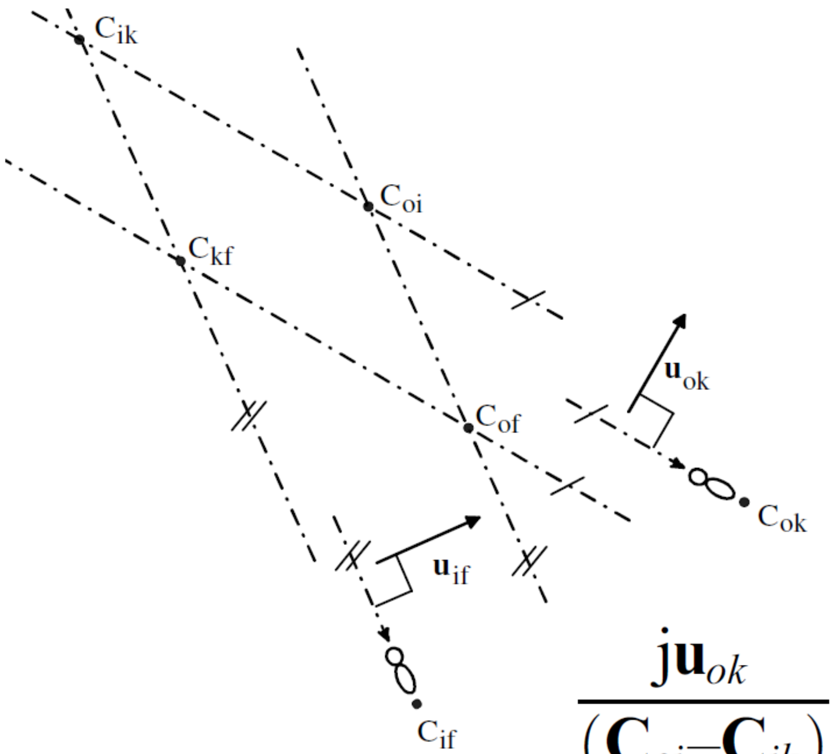
$$\mathbf{C}_{oi} = \mathbf{C}_{of}$$

Type-II singularity conditions

Singularity analysis with the ICs: Single-DOF Mechanisms (contd)

Tra-Tra case:

$\dot{x} = \dot{s}_{if}$ & $\dot{y} = \dot{s}_{ok}$



Kinematic relationships:

$${}^{f}\mathbf{v}_{oi|o} = \dot{s}_{if}\mathbf{u}_{if} \qquad \dot{s}_{ok}\mathbf{u}_{ok} = {}^{k}\mathbf{v}_{oi|i}$$

$$\dot{s}_{ok}\mathbf{u}_{ok} = {}^{k}\mathbf{v}_{oi|i} \qquad \dot{s}_{if}\mathbf{u}_{if} = {}^{f}\mathbf{v}_{ik|k}$$

$$0 = \omega_{of} + \omega_{ik}; \qquad 0 = \omega_{ik} + \omega_{kf}$$

$\Downarrow$ **Input-Output relationship** $\Downarrow$

$$\frac{\mathbf{j}\mathbf{u}_{ok}}{(\mathbf{C}_{oi}-\mathbf{C}_{ik})}\dot{s}_{ok} = \frac{\mathbf{j}\mathbf{u}_{if}}{(\mathbf{C}_{of}-\mathbf{C}_{oi})}\dot{s}_{if}; \qquad \frac{\mathbf{j}\mathbf{u}_{ok}}{(\mathbf{C}_{oi}-\mathbf{C}_{ik})}\dot{s}_{ok} = \frac{\mathbf{j}\mathbf{u}_{if}}{(\mathbf{C}_{kf}-\mathbf{C}_{ik})}\dot{s}_{if}$$

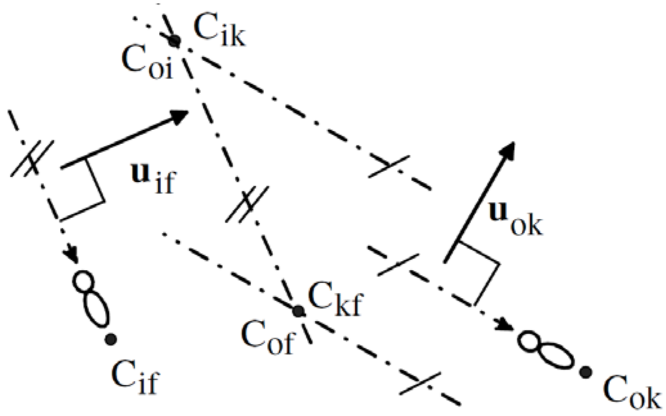
$\Downarrow$

Geometric relationship among the six ICs

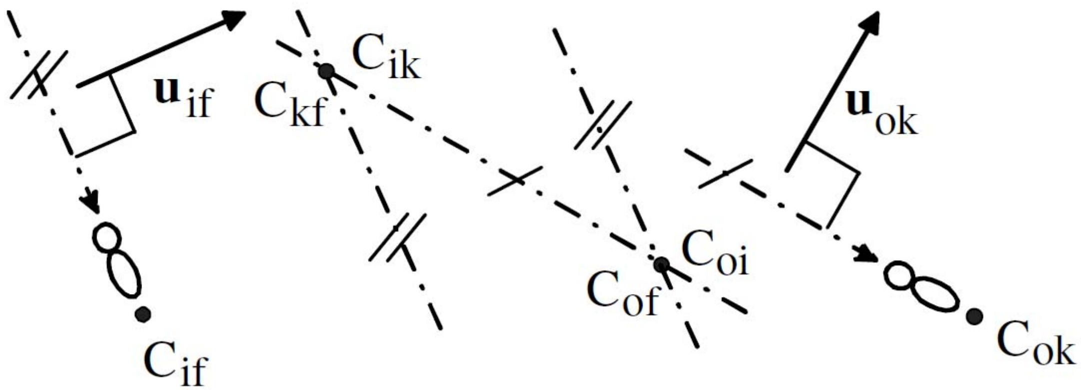
$$1 = \frac{(\mathbf{C}_{kf}-\mathbf{C}_{ik})}{(\mathbf{C}_{of}-\mathbf{C}_{oi})}$$

Singularity analysis with the ICs: Single-DOF Mechanisms (contd)

Tra-Tra case: singularity conditions



Type-I singularity conditions



Type-II singularity conditions

$$a = j\mathbf{u}_{if}(\mathbf{C}_{oi} - \mathbf{C}_{ik})$$

$$b = j\mathbf{u}_{ok}(\mathbf{C}_{of} - \mathbf{C}_{oi})$$

$$\Downarrow$$

Type-I singularity conditions (a=0):

$$\mathbf{C}_{oi} = \mathbf{C}_{ik}$$

Type-II singularity conditions (b=0):

$$\mathbf{C}_{oi} = \mathbf{C}_{of}$$

Singularity analysis with the ICs: Single-DOF Mechanisms (contd)

Discussion:

For Single-DOF PMs:

1) A general algorithm exists which analytically determines all the IC positions

2) The singularity analysis can always be transformed into geometric conditions on the positions of some ICs

3) By exploiting the above-recalled general algorithm (1), the geometric conditions that identify the singularities become analytic conditions

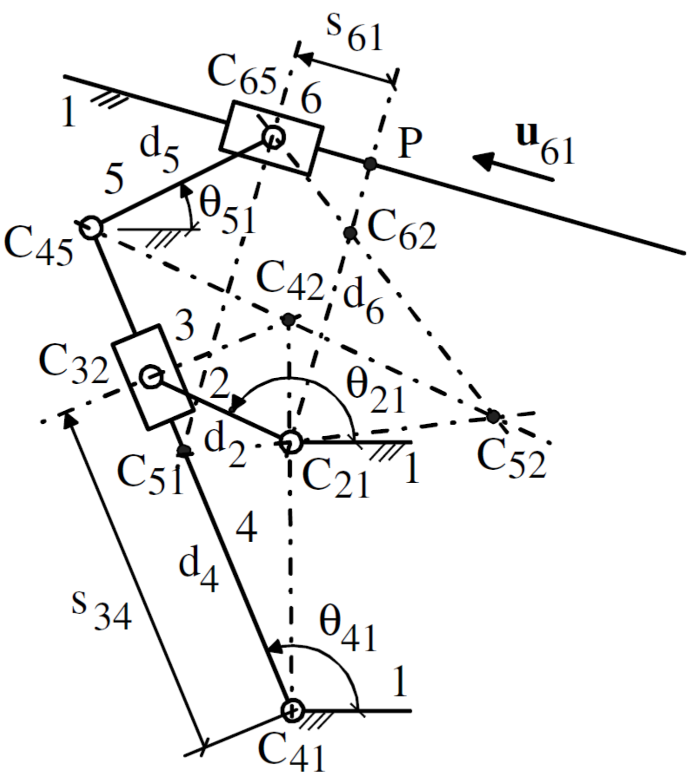
$\Downarrow$

The above-presented singularity analysis is both geometric and analytic.
Also, the use of complex numbers in representing position vectors makes it possible to write simple algebraic equations to solve when implementing such analysis.

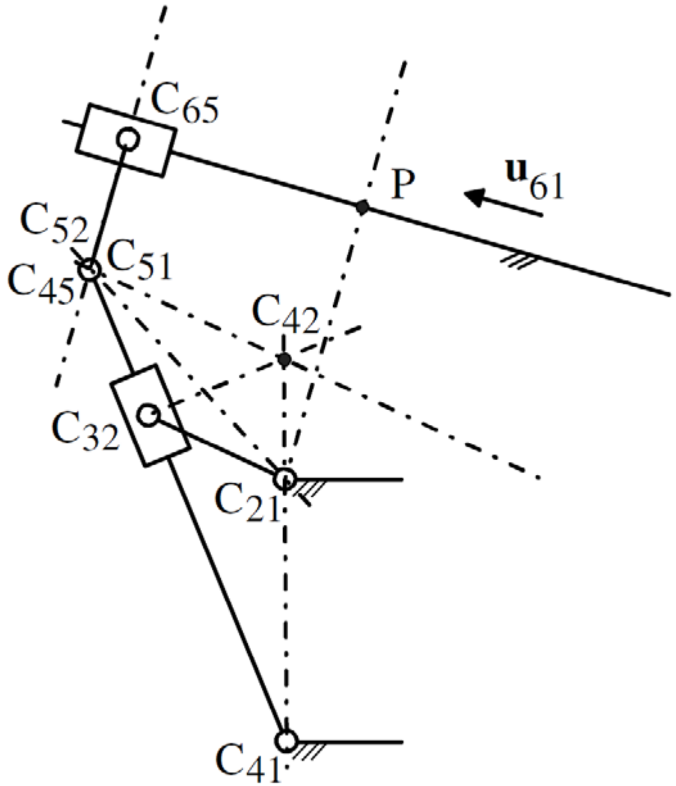
Singularity analysis with the ICs: Single-DOF Mechanisms (contd)

Case Study: shaper mechanism

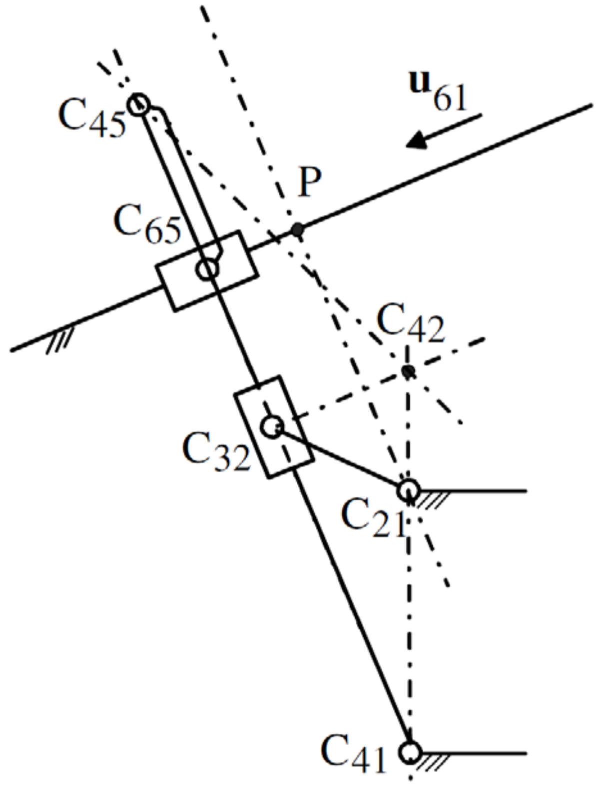
- **Rot-Tra Case:** $x=\theta_{21}$, $y=s_{61}$; "o"=6, "i"=2, "f"="k"=1;
- **Type-II singularity condition:** $C_{oi}=C_{of} \Rightarrow C_{62}=C_{61}$
- **Type-I singularity condition:** $C_{of}=C_{if}$ or $C_{oi}=C_{ik} \Rightarrow C_{61}=C_{21}$ or $C_{62}=C_{21}$



Shaper Mechanism

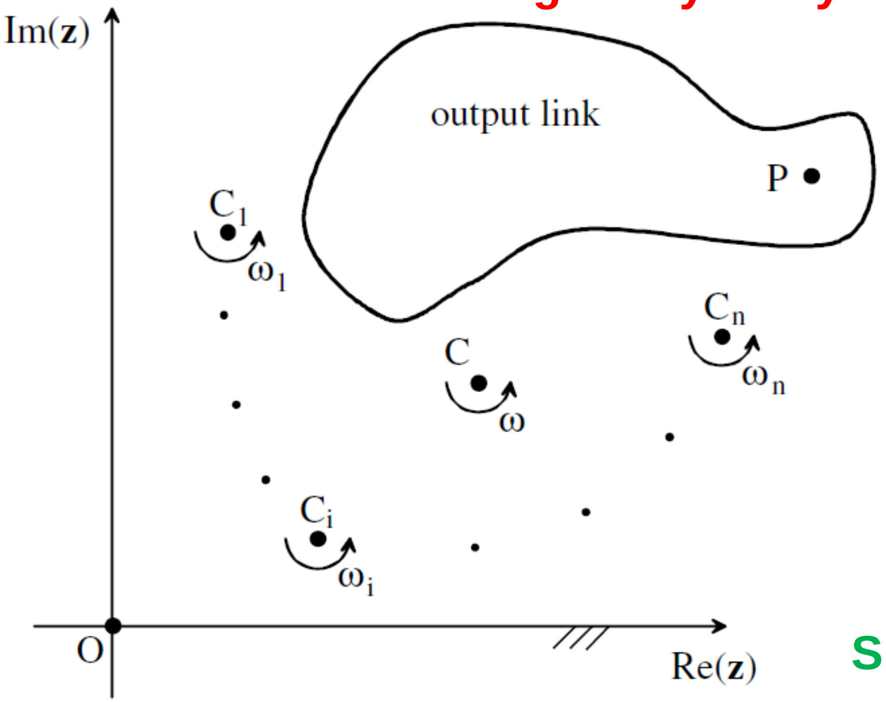


Type-II singularity: $C_{62}=C_{61}$



Type-III singularity: C_{51}, C_{52}, and C_{62} are indetermined

Singularity analysis with the ICs: Multi-DOF Mechanisms



n= DOF number = number of input variables

C_i, ω_i, v_P^i = IC, angular velocity, *and* point P's velocity, respectively, of output link's motion relative to the frame when all the inputs are locked but the i-th one

C, ω, v_P = **IC**, **angular velocity**, *and* **point P's velocity**, **respectively, of output link's actual motion with respect to the frame**

$$\mathbf{v}_P = j\omega(\mathbf{P}-\mathbf{C}),$$
$$\mathbf{v}_P^i = j\omega_i(\mathbf{P}-\mathbf{C}_i)$$

Superposition Principle (superposition of n single-DOF PMs):

$$\boldsymbol{\omega} = \sum_{i=1,n} \boldsymbol{\omega}_i, \qquad \omega = \sum_{i=1,n} \omega_i, \qquad \omega = \sum_{i=1,n} \omega_i$$

$$\mathbf{v}_P = \sum_{i=1,n} \mathbf{v}_P^i \Rightarrow j\omega(\mathbf{P}-\mathbf{C}) = \sum_{i=1,n} j\omega_i(\mathbf{P}-\mathbf{C}_i) \Rightarrow \omega\mathbf{C} = \sum_{i=1,n} \omega_i \mathbf{C}_i$$

THUS

Each ω_i is a signed weight associated to the IC C_i, and the actual IC, C, is the centroid of the set of the C_i heavy points

Singularity analysis with the ICs: Multi-DOF Mechanisms (contd)

$$\omega = \sum_{i=1,n} \omega_i \,; \qquad \omega \mathbf{C} = \sum_{i=1,n} \omega_i \mathbf{C}_i$$

$\Downarrow$

Each ω_i is a signed weight associated to the IC C_i, *and* the actual IC, C, is the centroid of the set of the C_i heavy points

$\Downarrow$

The geometric properties of centroids can be applied to find geometric relationships between the positions of the actual IC, C, and of the n single-DOF PMs' ICs, C_i for i = 1,. . . ,n.

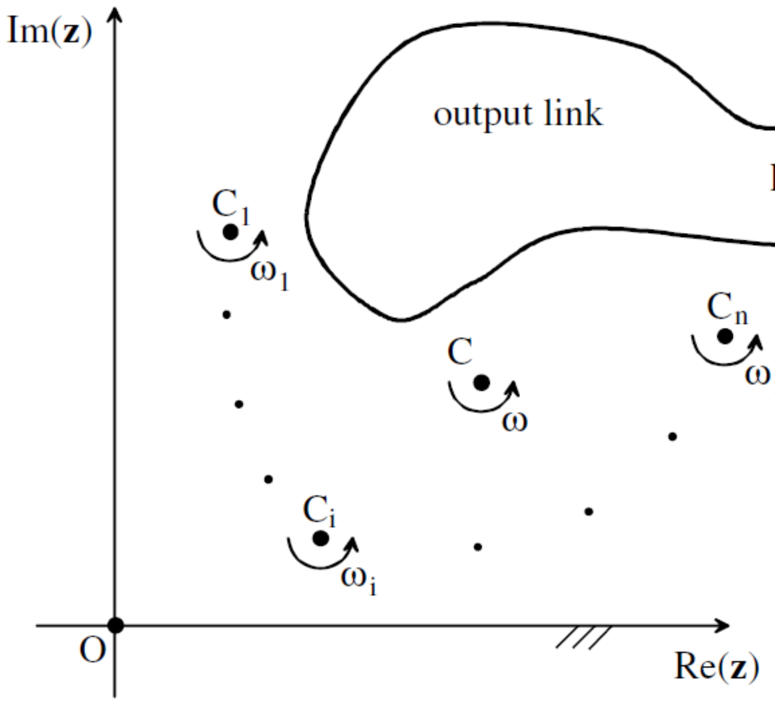
$\Downarrow$

For instance, if all the C_i lie on a straight line, C must lie on the same line; etc.

A corollary of this statement is that,

for 2-dof PMs, the complex equation $\omega \mathrm{C} = \sum_{i=1,n} \omega_i \mathrm{C}_i$ **does not provide 2 independent scalar equations since C must lie on the straight line located by C_1 and C_2.**

Singularity analysis with the ICs: Multi-DOF Mechanisms (contd)



$$\mathbf{v}_P = \mathrm{j}\omega(\mathbf{P} - \mathbf{C}),$$
$$\mathbf{v}_P^i = \mathrm{j}\omega_i(\mathbf{P} - \mathbf{C}_i)$$
$$\Downarrow$$
$$\omega\mathbf{C} = \mathrm{j}\mathbf{v}_O,$$
$$\omega_i\mathbf{C}_i = \mathrm{j}\mathbf{v}_O^i$$
$$\Downarrow$$

If the i-th input makes the ouput link translate, then

$$\omega_i\mathbf{C}_i = \mathrm{j}\mathbf{v}^i_O = \dot{s}_i\,\mathbf{u}_i$$

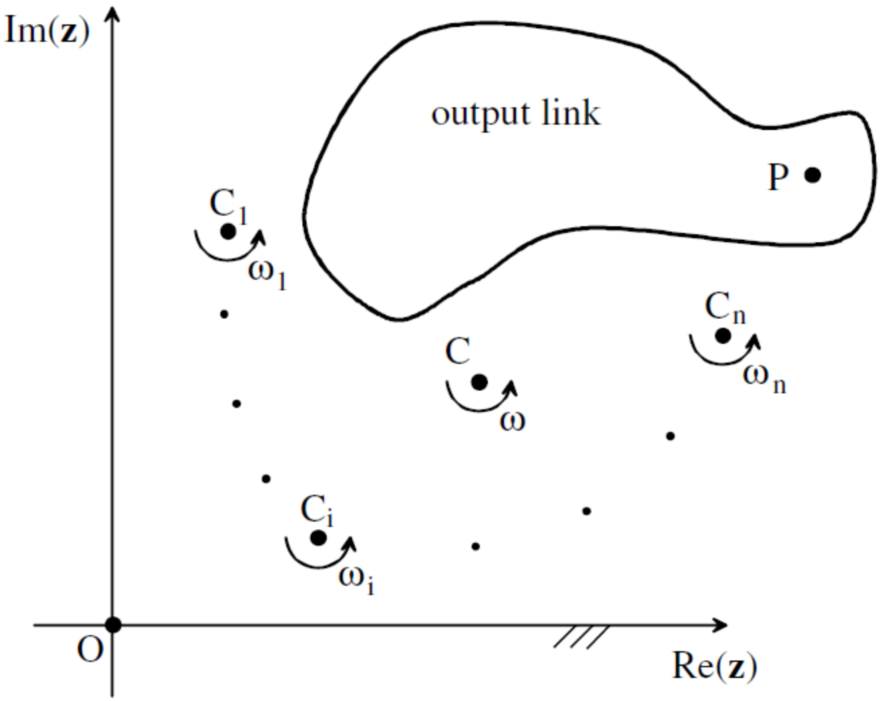
$$\Downarrow \qquad \Leftarrow \qquad \omega\mathbf{C} = \sum_{i=1,n} \omega_i\mathbf{C}_i$$

[m single-DOF PMs make the output link rotate, and n – m single-DOF PMs make the output link translate]

$$\omega = \sum_{p=1,m} \omega_p,$$

$$\omega\mathbf{C} = \sum_{p=1,m} \omega_p\mathbf{C}_p + \mathrm{j} \sum_{k=m+1,n} \dot{s}_k\mathbf{u}_k$$

Singularity analysis with the ICs: Multi-DOF Mechanisms (contd)



q={$q_1,\ldots,q_i,\ldots,q_n$}^{T}= **n-tuple collecting the n input variables of the n-DOF PM**

By locking all the inputs but one, n single-DOF PMs are generated whose input-output instantaneous relationships are

$$b_i\dot{y}_i = a_i\dot{q}_i, \quad i = 1,\ldots,n \qquad \textbf{(8)}$$

where a_i and b_i have the above-discussed explicit expressions according to the case (Rot-Rot, Rot-Tra, Tra-Rot, Tra-Tra) that applies.

$$\left.\begin{aligned}
&\omega = \sum_{p=1,m}\omega_p,\\
&\omega\mathbf{C} = \sum_{p=1,m}\omega_p\mathbf{C}_p + \mathrm{j}\sum_{k=m+1,n}\dot{s}_k\mathbf{u}_k\\
&b_i\dot{y}_i = a_i\dot{q}_i, \quad \overset{+}{i} = 1,\ldots,n
\end{aligned}\right\}\Rightarrow
\begin{aligned}
&\omega\prod_{p=1,m}b_p = \sum_{p=1,m}a_p\left(\prod_{\substack{t=1,m\\t\neq p}}b_t\right)\dot{q}_p,\\
&(\omega\mathbf{C})\prod_{i=1,n}b_i = \sum_{p=1,m}a_p\left(\prod_{\substack{i=1,n\\i\neq p}}b_i\right)\mathbf{C}_p\dot{q}_p + j\sum_{k=m+1,n}a_k\left(\prod_{\substack{i=1,n\\i\neq k}}b_i\right)\mathbf{u}_k\dot{q}_k
\end{aligned}$$

Singularity analysis with the ICs: Multi-DOF Mechanisms (contd)

$$\omega \prod_{p=1,m} b_p = \sum_{p=1,m} a_p \left(\prod_{\substack{t=1,m \\ t\neq p}} b_t \right) \dot{q}_p, \tag{9a}$$

$$(\omega \mathbf{C}) \prod_{i=1,n} b_i = \sum_{p=1,m} a_p \left(\prod_{\substack{i=1,n \\ i\neq p}} b_i \right) \mathbf{C}_p \dot{q}_p + j \sum_{k=m+1,n} a_k \left(\prod_{\substack{i=1,n \\ i\neq k}} b_i \right) \mathbf{u}_k \dot{q}_k \tag{9b}$$

System (9) is constituted of 3 scalar equations [1 scalar equation (i.e., (9a))+1 complex equation (i.e., (9b)].

System (9) is a general expression of the instantaneous input-output relationship, which is applicable to any n-DOF PM.

In this expression:

- output link's velocity field is given through the 3-tuple $\{\omega, \{\omega\mathbf{C}\}^T\}^T$

- the coefficients both of the input rates and of the output 3-tuple's entries are suitable combinations of ICs' positions in the n single-DOF PMs generated from the n-DOF PM

[i.e., the coefficients a_i and b_i for i=1,…,n, and the IC positions (finite or at infinity (ideal)), $\mathbf{C}_p$ for p=1,…,m, and $\mathbf{u}_k$ for k=m+1,…,n]

Singularity analysis with the ICs: Multi-DOF Mechanisms (contd)

$$\omega \prod_{p=1,m} b_p = \sum_{p=1,m} a_p \left(\prod_{\substack{t=1,m \\ t\neq p}} b_t \right) \dot{q}_p, \tag{9a}$$

$$(\omega \mathbf{C}) \prod_{i=1,n} b_i = \sum_{p=1,m} a_p \left(\prod_{\substack{i=1,n \\ i\neq p}} b_i \right) \mathbf{C}_p \dot{q}_p + j \sum_{k=m+1,n} a_k \left(\prod_{\substack{i=1,n \\ i\neq k}} b_i \right) \mathbf{u}_k \dot{q}_k \tag{9b}$$

1) at a non-singular configuration,

1a) for n≥3, system (9) consists of 3 independent scalar equations; whereas,

1b) for n = 2, it reduces itself to only two independent scalar equations, [in this case, either **C must lie on a straight line** (the one through C_1 and C_2 for m = 2 or the one through C_1 and parallel to $j\mathbf{u}_2$ for m = 1), or **Eq. (9a) becomes an identity for m = 0** (i.e., the output link can only translate)]

2) type-I singularities occur if system (9) cannot provide a unique solution for a number of input rates equal to the minimum between n **and** 3 **when the position of C** (or the complex number ω**C**), **the angular velocity** ω, **and**, for n ≥ 3, **the remaining input rates are assigned**.

3) type-II singularities occur if system (9) cannot provide a unique solution for the position of C (or the complex number ω**C**), **and the angular velocity** ω **when the input rates are all assigned**

Singularity analysis with the ICs: Multi-DOF Mechanisms (contd)

$$\omega \prod_{p=1,m} b_p = \sum_{p=1,m} a_p \left(\prod_{\substack{t=1,m \\ t \neq p}} b_t \right) \dot{q}_p, \tag{9a}$$

$$(\omega \mathbf{C}) \prod_{i=1,n} b_i = \sum_{p=1,m} a_p \left(\prod_{\substack{i=1,n \\ i \neq p}} b_i \right) \mathbf{C}_p \dot{q}_p + j \sum_{k=m+1,n} a_k \left(\prod_{\substack{i=1,n \\ i \neq k}} b_i \right) \mathbf{u}_k \dot{q}_k \tag{9b}$$

Type-I Singularities

Equations (9) reveal that,
at a given configuration, **if $a_r = 0$**, r $\in$ {1, 2, . . .,n}, **then the r-th input rate $\dot{q}_r$ is undetermined**.

$\Downarrow$

Statement 1: **the union of the n sets of type-I singularities of the single-DOF PMs, generated from a n-dof PM, is a subset of the set of all the type-I singularities of that n-DOF PM**

Theorem 1: **For n≥3** *and provided that* **all the $\mathbf{a}_i$ and $\mathbf{b}_i$ coefficients are different from zero, a type-I singularity occurs if and only if 3 ICs**, **$\mathbf{C}_i$**, **are either aligned** *or* **all ideal points**

Singularity analysis with the ICs: Multi-DOF Mechanisms (contd)

$$\omega \prod_{p=1,m} b_p = \sum_{p=1,m} a_p \left(\prod_{\substack{t=1,m \\ t\neq p}} b_t \right) \dot{q}_p, \tag{9a}$$

$$(\omega \mathbf{C}) \prod_{i=1,n} b_i = \sum_{p=1,m} a_p \left(\prod_{\substack{i=1,n \\ i\neq p}} b_i \right) \mathbf{C}_p \dot{q}_p + j \sum_{k=m+1,n} a_k \left(\prod_{\substack{i=1,n \\ i\neq k}} b_i \right) \mathbf{u}_k \dot{q}_k \tag{9b}$$

Type-II Singularities

Equations (9) reveal that,

at a given configuration, **the motion of the output link is undetermined even though the input rates are all assigned, if and only if at least one** b_i **coefficient is equal to zero**

$\Downarrow$

Statement 2: **the set of all the type-II singularities of a n-DOF PM is the union of the n sets of type-II singularities of the single-DOF PMs generated from that n-dof PM**

Theorem 2: **the coincidence of all the C_i ICs for i = 1,…, n, (i.e., included those C_i that are ideal points) identifies a particular type-II singularity where all the b_i coefficients vanish**

Singularity analysis with the ICs: Multi-DOF Mechanisms (contd)

$$\omega \prod_{p=1,m} b_p = \sum_{p=1,m} a_p \left(\prod_{\substack{t=1,m \\ t\neq p}} b_t \right) \dot{q}_p, \tag{9a}$$

$$(\omega \mathbf{C}) \prod_{i=1,n} b_i = \sum_{p=1,m} a_p \left(\prod_{\substack{i=1,n \\ i\neq p}} b_i \right) \mathbf{C}_p \dot{q}_p + j \sum_{k=m+1,n} a_k \left(\prod_{\substack{i=1,n \\ i\neq k}} b_i \right) \mathbf{u}_k \dot{q}_k \tag{9b}$$

Type-III Singularities

Equations (9) reveal that,

at a given configuration, **if at least one out of the b_i coefficients together with at least one out of the a_i coefficients are equal to zero, a type-III singularity will occur**

$\Downarrow$

Statement 3: **the union of the n sets of type-III singularities of the single-DOF PMs, generated from a n-dof PM, is a subset of the set of all the type-III singularities of that n-DOF PM**

Singularity analysis with the ICs: Multi-DOF Mechanisms (contd)

$$\omega \prod_{p=1,m} b_p = \sum_{p=1,m} a_p \left(\prod_{\substack{t=1,m \\ t\neq p}} b_t \right) \dot{q}_p, \tag{9a}$$

$$(\omega \mathbf{C}) \prod_{i=1,n} b_i = \sum_{p=1,m} a_p \left(\prod_{\substack{i=1,n \\ i\neq p}} b_i \right) \mathbf{C}_p \dot{q}_p + j \sum_{k=m+1,n} a_k \left(\prod_{\substack{i=1,n \\ i\neq k}} b_i \right) \mathbf{u}_k \dot{q}_k \tag{9b}$$

Discussion

The above results proved that there, in n-DOF PMs, there are two classes of singularities:

Class (a): singularities that are also singularities of at least one single-DOF PM generated from the n-DOF PM,

For these singularities the equations to solve are:

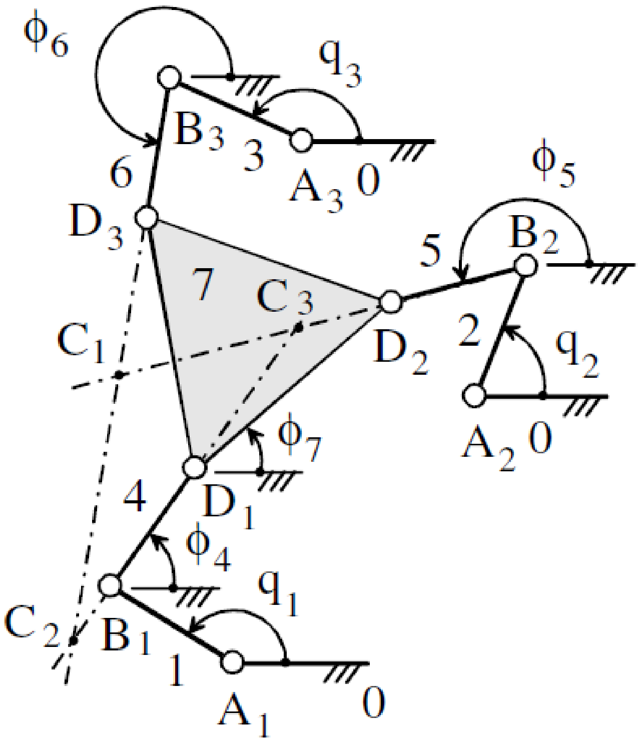
$$a_i(q_1, \ldots, q_n) = 0, \quad i = 1, \ldots, n$$
$$b_i(q_1, \ldots, q_n) = 0, \quad i = 1, \ldots, n$$

Class (b): singularities that are NOT singularities of any single-DOF PM generated from the n-DOF PM,

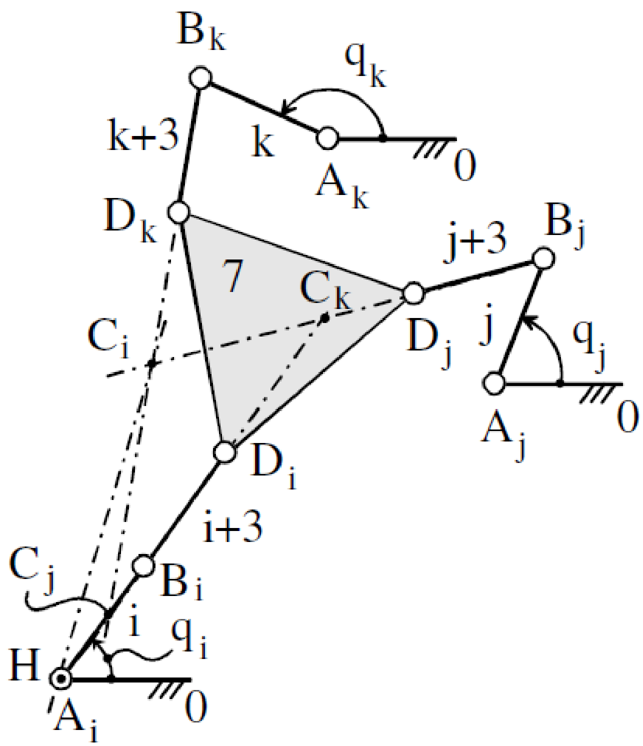
These singularities are only those identified by theorem 1 $\Rightarrow$ the equations to solve are obtained by imposing the alignment (real or ideal) of C_i triplets [the generic coordinates of each C_i can be easily computed with the above-presented general method for locating ICs' positions in single-DOF PMs]

Singularity analysis with the ICs: Multi-DOF Mechanisms (contd)

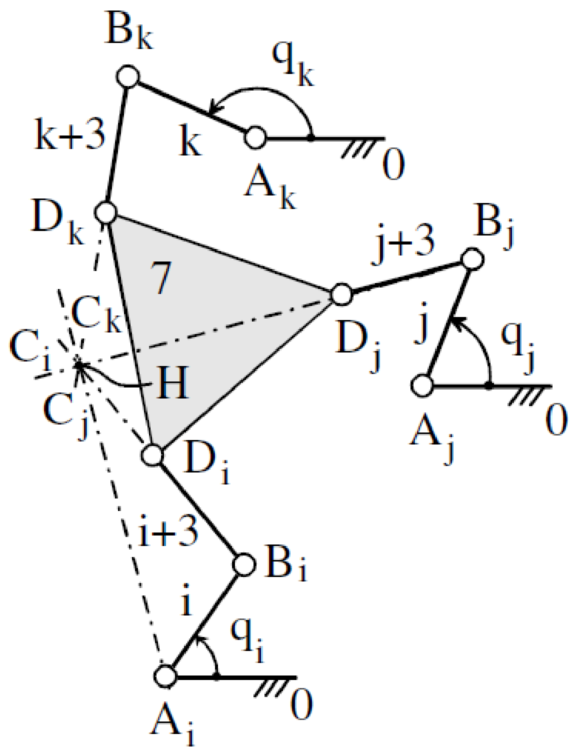
Case Study: 3-RRR PM



3-RRR PM



Example of type-I singularity



Example of type-II singularity

Further details (proofs of the theorems, computation algorithm, etc.) on Multi-DOF PMs are reported in

- Di Gregorio R., 2009, “A novel method for the singularity analysis of planar mechanisms with more than one degree of freedom,” *Mechanism and Machine Theory*, 44(1):83–102