Short Course for PhD Students: Recent Results on Planar and Spherical Kinematics
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Lecture 2

Extension to Spherical Kinematics by Using Instantaneous Pole Axes (IPAs)

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1) From planar to spherical geometry: notations and background

2) Systematic determination of all the IPAs

Refs.:

- Di Gregorio R., 2011, "A general algorithm for analytically determining all the instantaneous pole axis locations in single-DOF spherical mechanisms," *Proc. IMechE Part C: J. Mechanical Engineering Science*, 225(9): 2062-2075

3) Singularity analysis of spherical mechanisms with the IPs

3.1) Single-DOF mechanisms

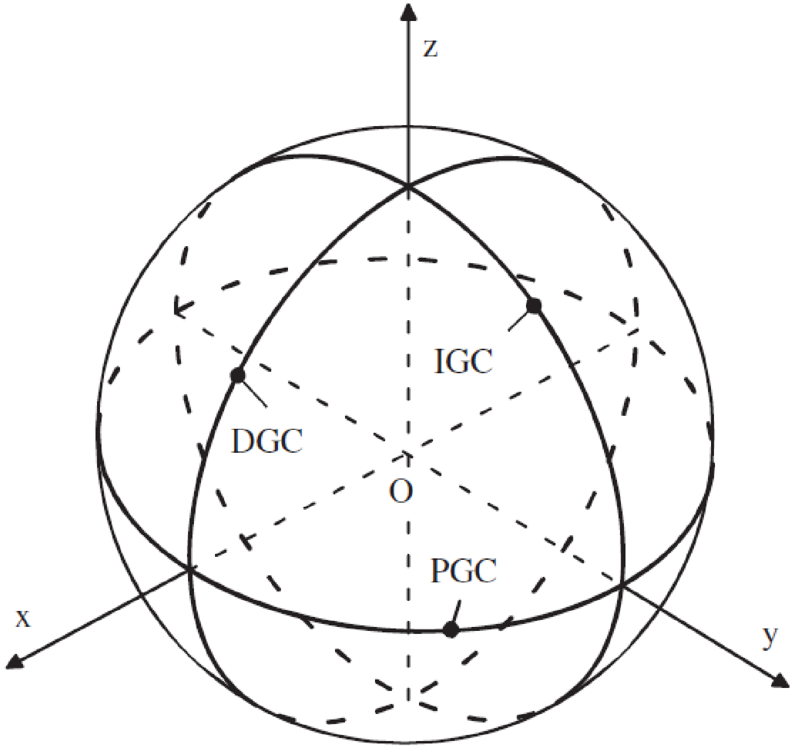
3.2) Multi-DOF mechanisms

Refs.:

- Di Gregorio R., 2013, "Analytical method for the singularity analysis and exhaustive enumeration of the singularity conditions in single-degree-of-freedom spherical mechanisms," *Proc. IMechE Part C: J. Mechanical Engineering Science*, 227(8) 1830–1840

- Di Gregorio R., 2015, "Analytic and Geometric Technique for the Singularity Analysis of Multi-Degree-of-Freedom Spherical Mechanisms," *ASME J. Mechanisms and Robotics*, 7(3): 031008 (9 pages)

From planar to spherical geometry: notations and background



RS and Cartesian Reference fixed to the frame:
IGC = Infinity Great Circle,
PGC = Primary Great Circle
DGC = Declination Great Circle

Spherical mechanisms (SMs) *are mechanisms* **where a point of the frame, named spherical-motion center,** *can be considered* **embedded in all the links**

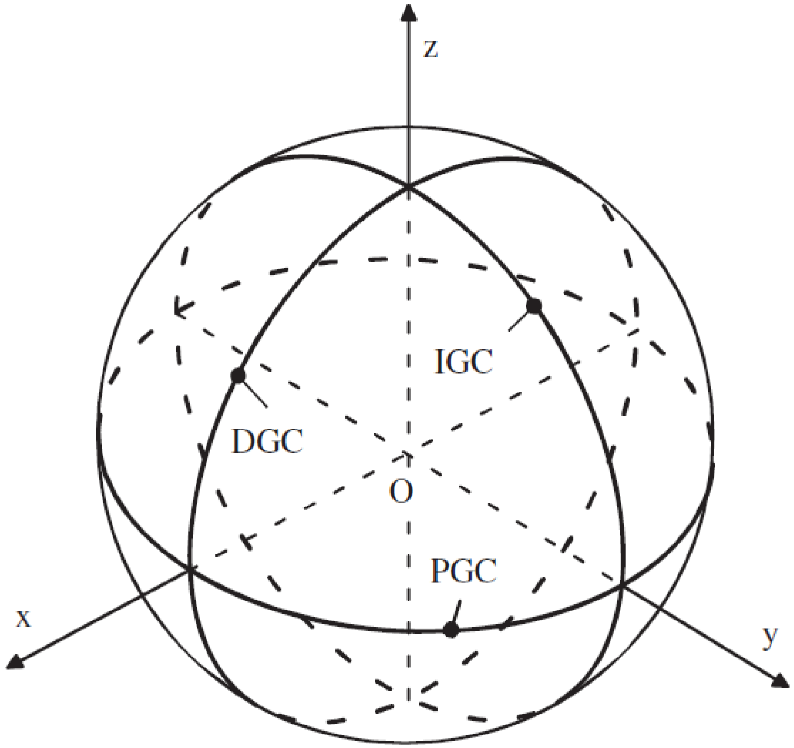
⇓

1) All the mechanism points *perform* **trajectories that lie on concentric spheres whose center** *is* **the spherical-motion center**

2) The instantaneous motion between two links *is always* **a rotation around an instantaneous rotation axis (instantaneous pole axis (IPA)) passing through the spherical-motion center. IPAs for SMs play the same role as ICs for planar mechanisms**

3) SMs *can be studied* **by projecting them through the spherical-motion center onto a sphere (reference sphere (RS)) with center at the spherical-motion center.**
So doing, the IPAs become points of the RS and the links become spherical laminae sliding on the RS

From planar to spherical geometry: notations and background (contd)



RS and Cartesian Reference fixed to the frame:
IGC = Infinity Great Circle,
PGC = Primary Great Circle
DGC = Declination Great Circle

In spherical geometry:

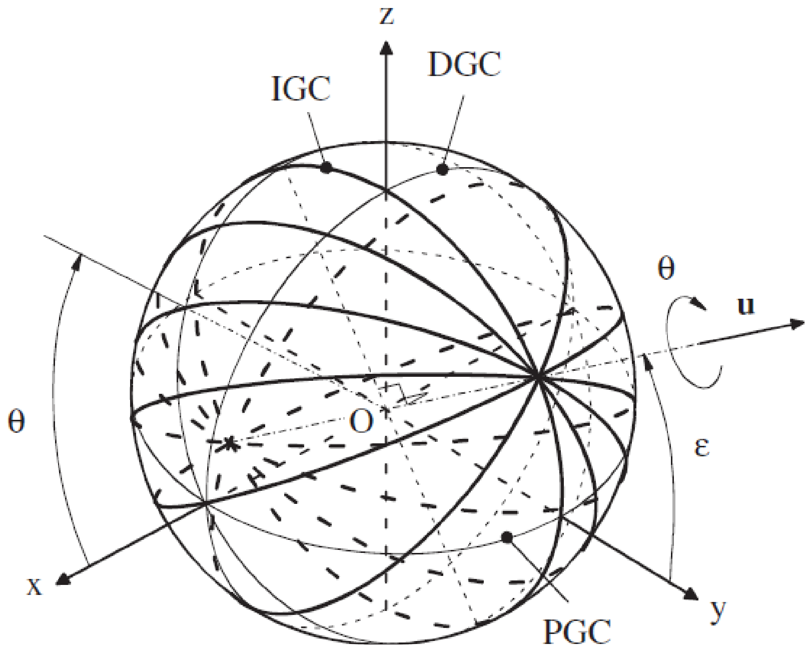
- **only one great circle (GC)** *passes through* **two points**

- **the minimum GC arc** *that joins two points on a sphere* **is the geodesic line** *and* **provides the minimum distance between those points (i.e., it plays the same role as the straight-line segment that joins two points plays in plane geometry)**

- **the pose of a GC arc fixed to a spherical lamina identifies the pose of the lamina (i.e., of the links it represents) on the sphere it belongs to**

- **3 geometric parameters are necessary to locate a GC arc on a sphere**

From planar to spherical geometry: notations and background (contd)



Positive Hemisphere: $\{(x,y,z)^T \in RS \mid x>0\}$

Negative Hemisphere: $\{(x,y,z)^T \in RS \mid x<0\}$

Positive Shell:

$\{(x,y,z)^T \in RS \mid x>0\}$ U $\{(0,y,z)^T \in IGC \mid y>0$ or $(y=0\ \&\ z>0)\}$

Negative Shell:

$\{(x,y,z)^T \in RS \mid x<0\}$ U $\{(0,y,z)^T \in IGC \mid y<0$ or $(y=0\ \&\ z<0)\}$

Pencil of Meridians = set of GCs intersecting one another on the IGC in the same points

ε = Slope of the Meridians belonging to the Pencil,

θ = Angle that locates one Meridian in the Pencil

2 GCs with different slopes have only one intersection in the positive (negative) hemisphere;

2 GCs with the same slope (i.e., that belong to the same pencil of meridians) do not intersect each other in the positive (negative) hemisphere.

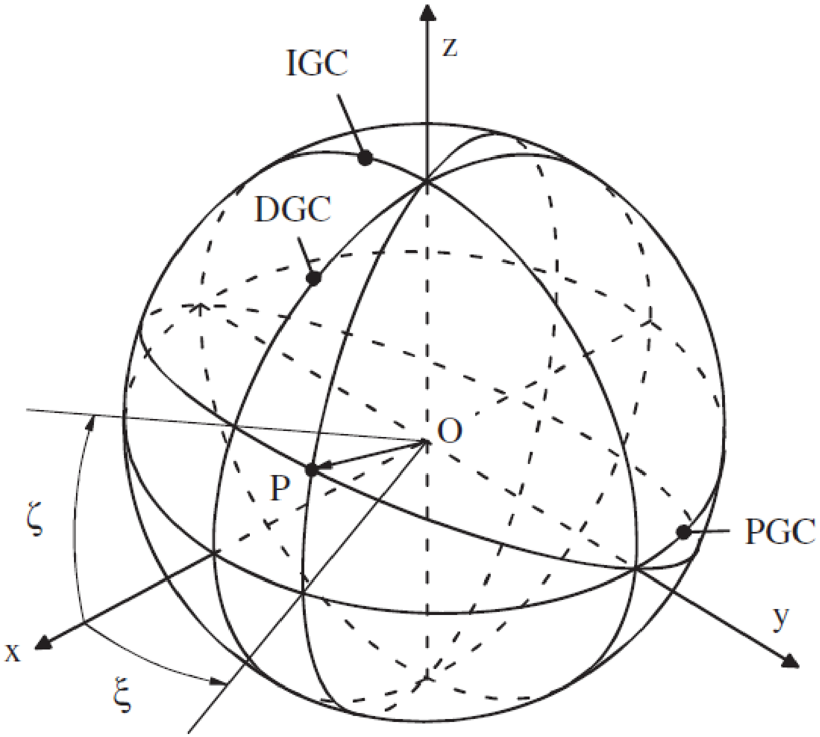
IPAs intersect the positive (negative) shell at one point (i.e., there is a one-to-one correspondence between IPAs and one shell's points), named instantaneous pole (IP)

⇓

SMs' first-order kinematics can be studied using only one (either positive or negative) shell of the RS.

Hereafter, the positive shell will be used.

From planar to spherical geometry: notations and background (contd)



Pseudo-Cartesian coordinates, (ξ, ζ), of a point, P=(x,y,z), of the positive shell

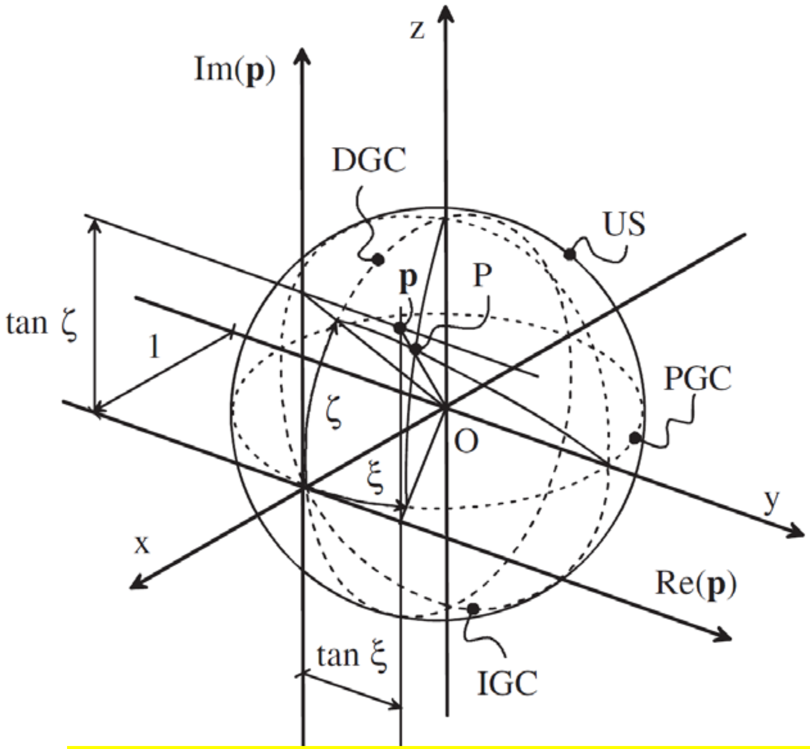
In the positive shell, the slope angle, ε, belongs to the range]π/2, π/2] (rad), *and* **it is sufficient to locate the IGC points of the shell.**
The remaining points of the shell are the points of the positive hemisphere, *and* **they need two coordinates to be located.**

Pseudo-Cartesian Coordinates (ξ, ζ):
Each RS point is located as intersection (see the figure**) of two meridians, one belonging to the pencil with ε=0, and the other belonging to the pencil with ε= π/2.**

Relationship between Cartesian and Pseudo-Cartesian coordinates (r=radius of the RS**):**

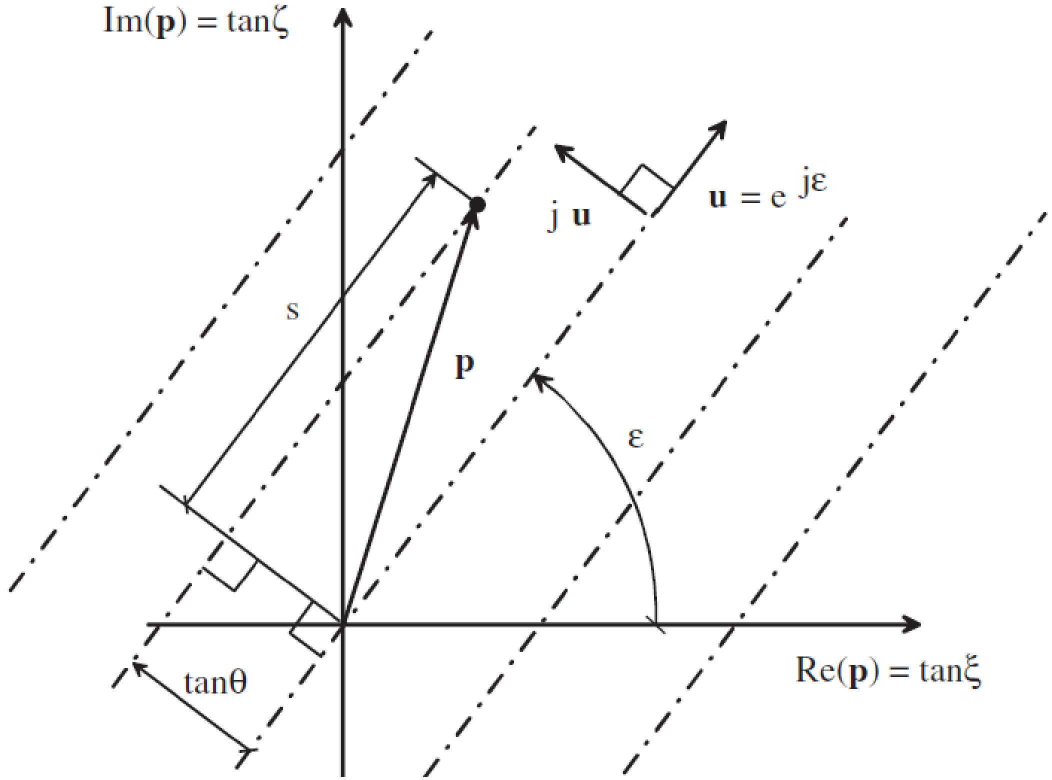
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{r}{\sqrt{1+\tan^2\xi+\tan^2\zeta}} \begin{pmatrix} 1 \\ \tan\xi \\ \tan\zeta \end{pmatrix}$$

From planar to spherical geometry: notations and background (contd)



Gnomonic projection, p, of a point, P, of the positive shell onto the reference plane (RP) tangent at $(1, 0, 0)^T$ to the unit sphere US (=RS with r=1)

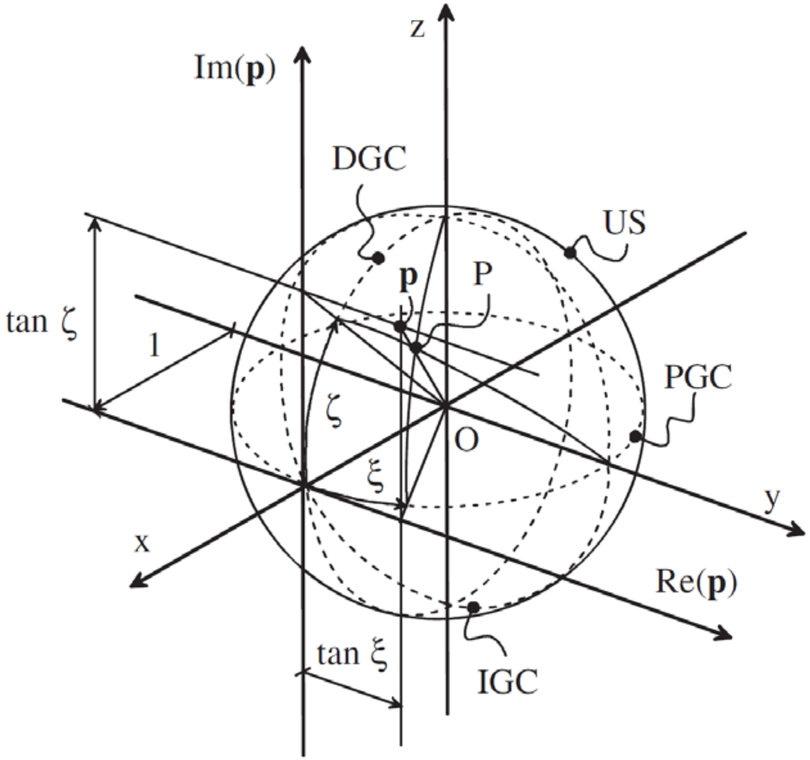
Gnomonic Projection of
a Pencil of Meridians with Slope Angle ε



Gnomonic Projection ($j=\sqrt{-1}$, u=exp(j ε)):

$$\mathbf{p} = \tan\xi + j \tan\zeta = (s + j \tan\theta)\, \mathbf{u}$$

Systematic determination of all the IPAs



The only kinematic pairs that appear in SMs are revolute pairs (R), rolling contacts (R_c), and sliding contacts (S_c)

The relative motion between two links, say links j and i, is uniquely determined by the position on the US of its instantaneous pole (IP), hereafter named P_{ij}.

IP positions are known (primary IPs) for R and R_c pairs, whereas, for S_c pairs, it lies on the GC that is normal at the contact point to both the spherical curves which slid on each other

In an SM with m links, the number of IPs that can be determined is m(m – 1)/2 (i.e., one IP for each relative motion). The positions of all the non-primary IPs (secondary IPs) must be determined.

In single-DOF SMs, IPs' positions depend only on the mechanism configuration

Aronold–Kennedy (A–K) theorem for SMs:
the instantaneous pole P_{ij} lies on the GC passing through the instantaneous poles P_{ik} and P_{jk} where link k may be any link different from links i and j.

Systematic determination of all the IPAs (contd)

Determination of Secondary IPs

From a conceptual point of view, the Procedure is the same as the one presented for ICs of planar mechanisms (PMs), that is:

DATA & TOOLS: Primary IPs + A-K Theorem + Chasles Theorem's Corollary (i.e., GC $\perp$ S$_c$)

METHODOLOGY: 1) Find two GCs the unknown IP lies on; 2) Determine (either graphically or analytically) the intersection point of those two GCs; 3) Add the found IP to the data set *and* repeat from step (1) for another unknown IP

The CENTRAL POINT for implementing the methodology is the determination of a sequence of unknown IPs to be used. Such a sequence comes out by listing,

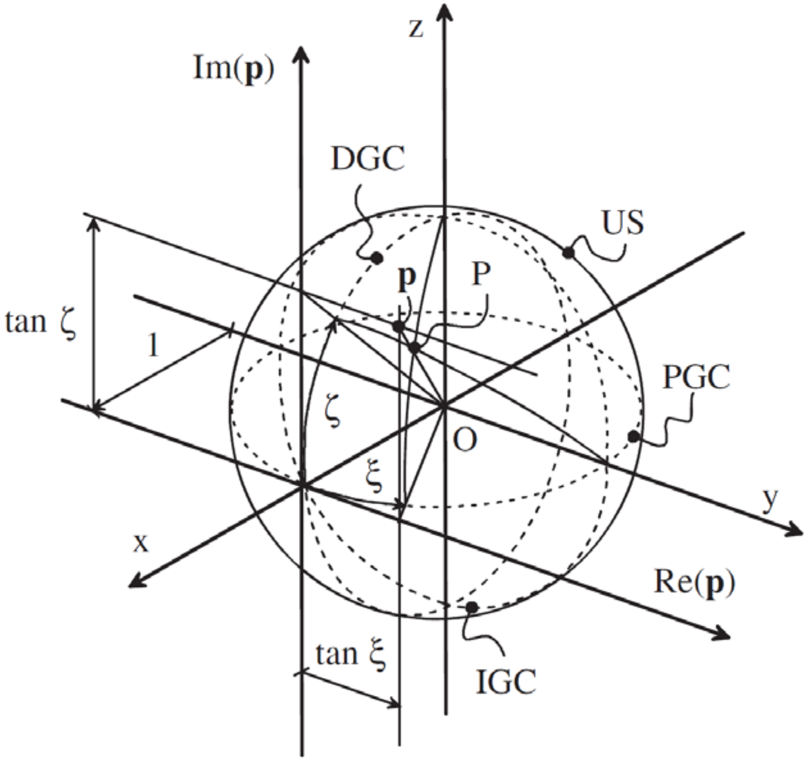
- firstly, all the IPs that can be determined by using only the primary IPs;

- secondly, all the IPs that can be determined by using the primary IPs and the IPs previously determined, and so on till all the unknown IPs are included in the list

Systematic determination of all the IPAs (contd)

Determination of Secondary IPs

Since the determination of the sequence needs only topological data (i.e., type of links' interconnections**),** *and* **topology does not change when a PM is replaced by a SM with the same types of links' interconnections, the algorithms (**circle diagram, the "Table", etc.**) ideated for PMs hold for SM, too, without any adaptation**.



Also, since the gnomonic projections of GCs are straight-lines, intersections of GCs become intersections of straight-lines in the RP

⇓

In the RP, the techniques ideated for determining the ICs of PMs can be used for determining the gnomonic projections, $\mathbf{p}_{ij}$, of the IPs, P_{ij}.

Systematic determination of all the IPAs (contd)

Determination of Secondary IPs

From an analytical point of view, by using the computed sequence, when the position of the instantaneous pole P_{ij} has to be computed, one out of the following 3 sets of geometric data has been already computed:

(a) the positions of four IPs of type P_{ik}, P_{jk}, P_{ic}, and P_{jc};

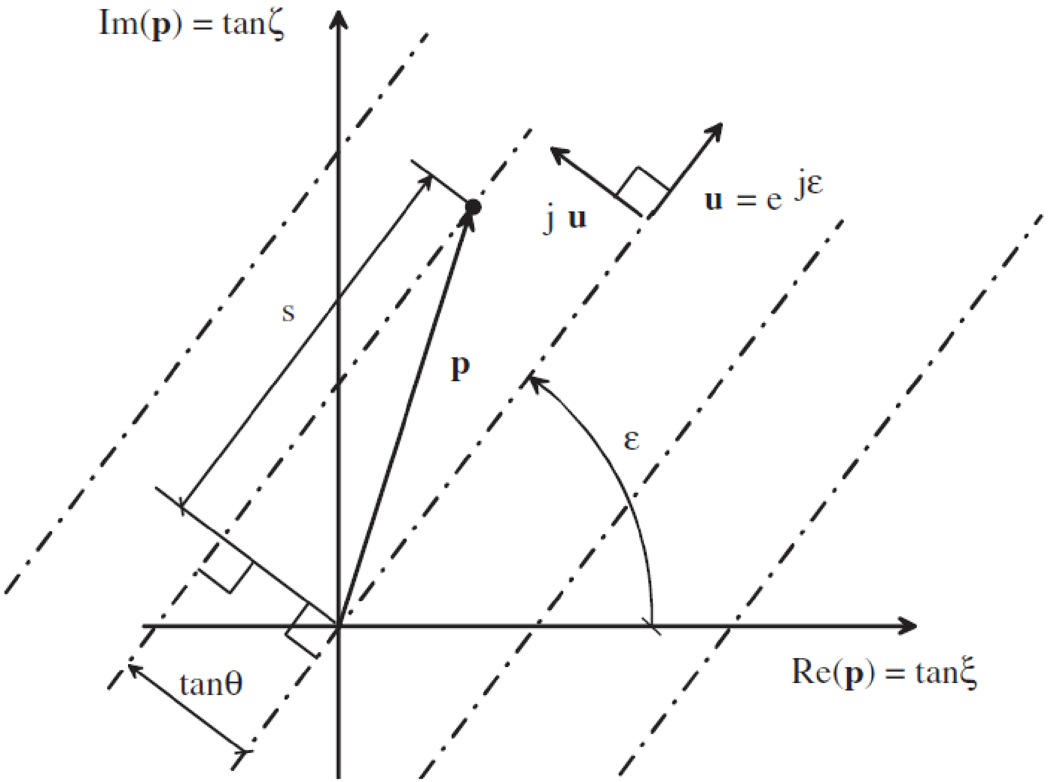
(b) the position of two IPs of type P_{ik} and P_{jk}, and one GC, P_{ij} lies on, with known slope angle, say ε_0, and passing through the contact point, say point C, of an S_c pair;

(c) 2 GCs, P_{ij} lies on, with known slope angles, say ε_1 and ε_2, and passing through 2 contact points, say points C and Q, of 2 S_c pairs.

Once the equations to solve in each of these three cases have been found, the remaining part of the general algorithm coincides with the one presented for planar mechanisms

Systematic determination of all the IPAs (contd)

Determination of Secondary IPs



Reference Plane (RP) tangent at (1, 0, 0)^T to the unit sphere US (=RS with r=1)

Gnomonic Projection ($j=\sqrt{-1}$, u=exp(j ε)):

$$\mathbf{p} = \tan\xi + j\tan\zeta = (s + j\tan\theta)\,\mathbf{u}$$

$$\Downarrow$$

Case (a): (the data are $\mathbf{p}_{ik}$, $\mathbf{p}_{jk}$, $\mathbf{p}_{ic}$, and $\mathbf{p}_{jc}$)

i) the parameter (ε,θ) of the GCs passing through 2 IPs that share one link index, say $\mathbf{p}_{ik}$ and $\mathbf{p}_{jk}$ (or $\mathbf{p}_{ic}$ and $\mathbf{p}_{jc}$), are computed as follows:

$$\varepsilon_1=\mathrm{Arg}(\mathbf{p}_{ik}-\mathbf{p}_{jk}), \quad \tan\theta_1=\mathrm{Im}[\mathbf{p}_{ik}\exp(-j\,\varepsilon_1)]$$

$$\varepsilon_2=\mathrm{Arg}(\mathbf{p}_{ic}-\mathbf{p}_{jc}), \quad \tan\theta_2=\mathrm{Im}[\mathbf{p}_{ic}\exp(-j\,\varepsilon_2)]$$

ii) once (ε_1,θ_1) and (ε_2,θ_2) are known, the linear system

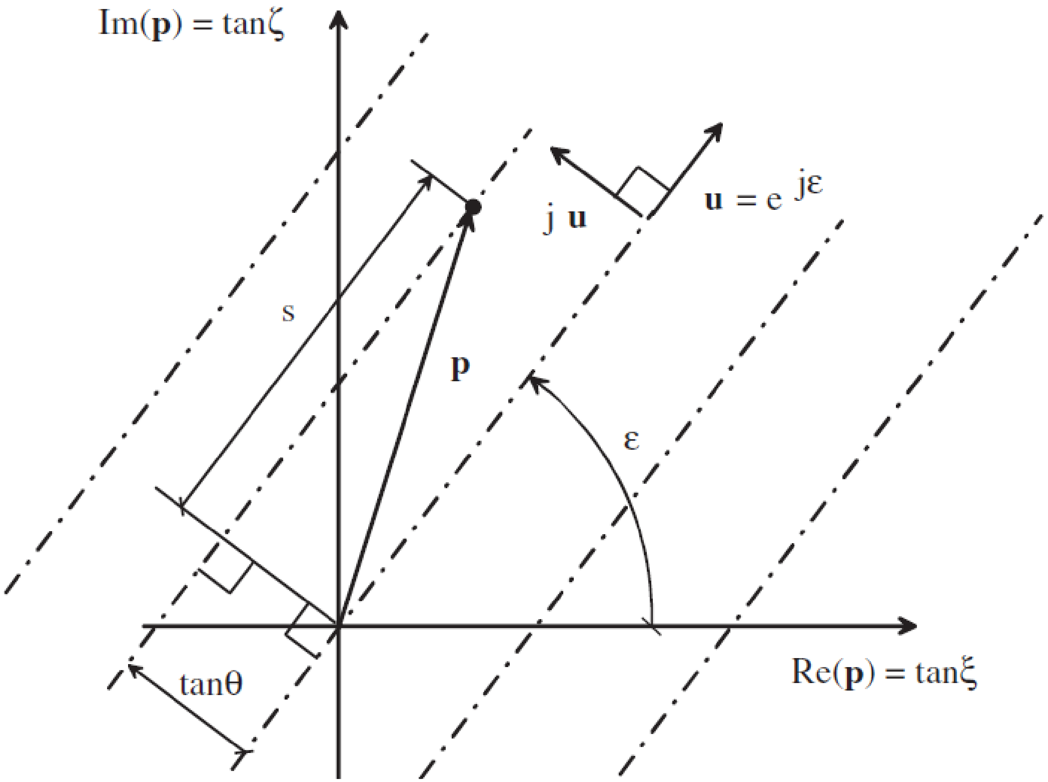
$$\mathbf{p}_{ij} = (s_1 + j\tan\theta_1)\,\mathbf{u}_1$$

$$\mathbf{p}_{ij} = (s_2 + j\tan\theta_2)\,\mathbf{u}_2$$

in the 4 unknowns, s_1, s_2,and $\mathbf{p}_{ij}$, can be solved.

Systematic determination of all the IPAs (contd)

Determination of Secondary IPs



Reference Plane (RP) tangent at (1, 0, 0)^T to the unit sphere US (=RS with r=1)

Gnomonic Projection ($j=\sqrt{-1}$, u=exp(j ε)):

$$\mathbf{p} = \tan\xi + j\tan\zeta = (s + j\tan\theta)\,\mathbf{u}$$

$$\Downarrow$$

Case (b): (the data are $\mathbf{p}_{ik}$, $\mathbf{p}_{jk}$, the point **c**, and the slope ε_c of the GC passing through C)

i) the data (ε,θ) of the GC passing through the 2 IPs that share one link index, $\mathbf{p}_{ik}$ and $\mathbf{p}_{jk}$ and the angle θ_c of the GC passing through C, are computed as follows:

$$\varepsilon_1=\mathrm{Arg}(\mathbf{p}_{ik}-\mathbf{p}_{jk}),\quad \tan\theta_1=\mathrm{Im}[\mathbf{p}_{ik}\exp(-j\,\varepsilon_1)]$$

$$\tan\theta_c=\mathrm{Im}[\mathbf{c}\,\exp(-j\,\varepsilon_c)]$$

ii) once (ε_1,θ_1) and (ε_c,θ_c) are known, the linear system

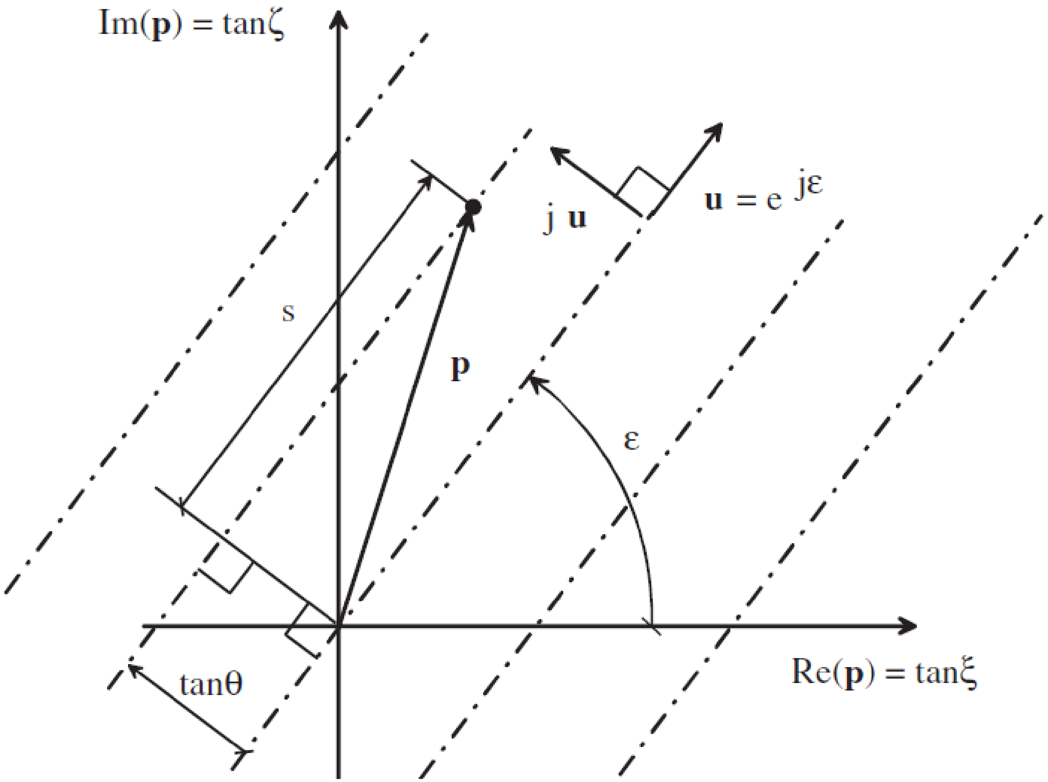
$$\mathbf{p}_{ij} = (s_1 + j\tan\theta_1)\,\mathbf{u}_1$$

$$\mathbf{p}_{ij} = (s_2 + j\tan\theta_c)\,\mathbf{u}_c$$

in the 4 unknowns, s_1, s_2,and $\mathbf{p}_{ij}$, can be solved.

Systematic determination of all the IPAs (contd)

Determination of Secondary IPs



Reference Plane (RP) tangent at (1, 0, 0)^T to the unit sphere US (=RS with r=1)

Gnomonic Projection ($j=\sqrt{-1}$, u=exp(j ε)):

$$\mathbf{p} = \tan\xi + j\tan\zeta = (s + j\tan\theta)\,\mathbf{u}$$

$$\Downarrow$$

Case (c): (the data are the points **q** and **c**, and the slopes ε_q and ε_c of the GCs passing through Q and C, respectively)

i) the angles θ_q and θ_c of the GC passing through Q and C, respectively, are computed as follows:

$$\tan\theta_q = \mathrm{Im}[\mathbf{q}\exp(-j\,\varepsilon_q)]$$

$$\tan\theta_c = \mathrm{Im}[\mathbf{c}\exp(-j\,\varepsilon_c)]$$

ii) once $(\varepsilon_q, \theta_q)$ and $(\varepsilon_c, \theta_c)$ are known, the linear system

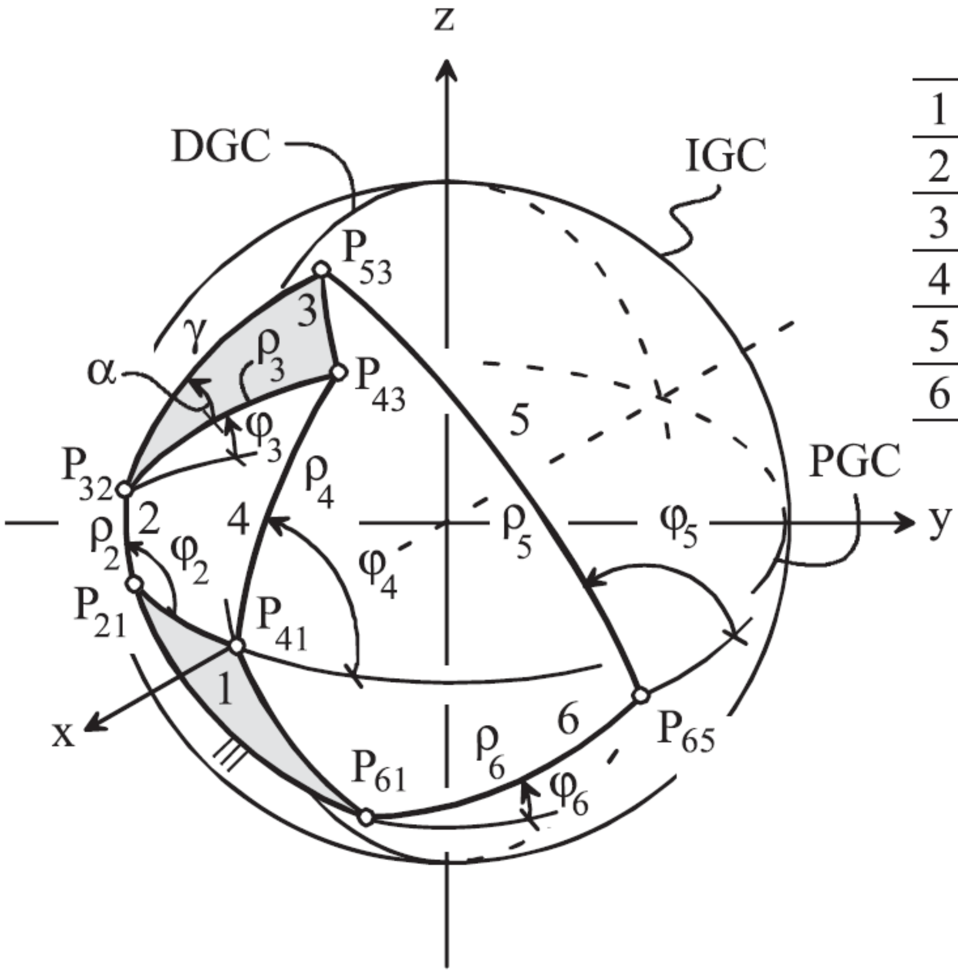
$$\mathbf{p}_{ij} = (s_1 + j\tan\theta_q)\,\mathbf{u}_q$$

$$\mathbf{p}_{ij} = (s_2 + j\tan\theta_c)\,\mathbf{u}_c$$

in the 4 unknowns, s_1, s_2,and $\mathbf{p}_{ij}$, can be solved.

Systematic determination of all the IPAs (contd)

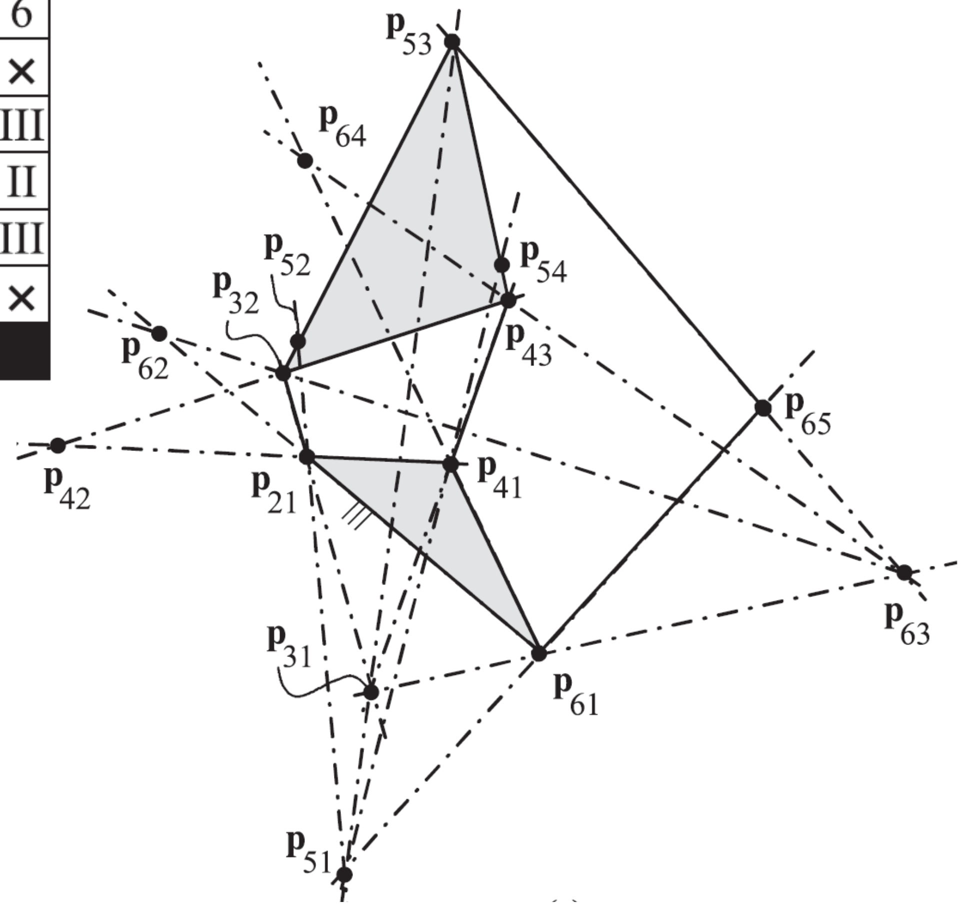
Determination of Secondary IPs: Case Study



Spherical Stephenson Mechanism

	1	2	3	4	5	6
1	■	×	I	×	II	×
2	×	■	×	I	III	III
3	I	×	■	×	×	II
4	×	I	×	■	III	III
5	II	III	×	III	■	×
6	×	III	II	III	×	■

Table



Gnomonic Projection of the SM

Singularity analysis of SMs with the IPs: Single-DOF Mechanisms

Single-DOF mechanisms have a single input.
Thus, without loosing generality, a single-input–single-output system (SISO system) **can be considered as the general case since the formulas regarding a one-dof mechanism with n output variables** (i.e. a single-input-multiple–output system (SIMO system)) **can be obtained by considering n independent SISO systems working in parallel.**

In a SISO system the general form of the instantaneous input-output relationship is [$\dot{q}$ and $\dot{y}$ are the rates of the input, q, **and the output,** y, **variables, respectively; whereas,** c **and** d **are coefficients that are functions of** q]

$$c\dot{y}=d\dot{q} \tag{2}$$

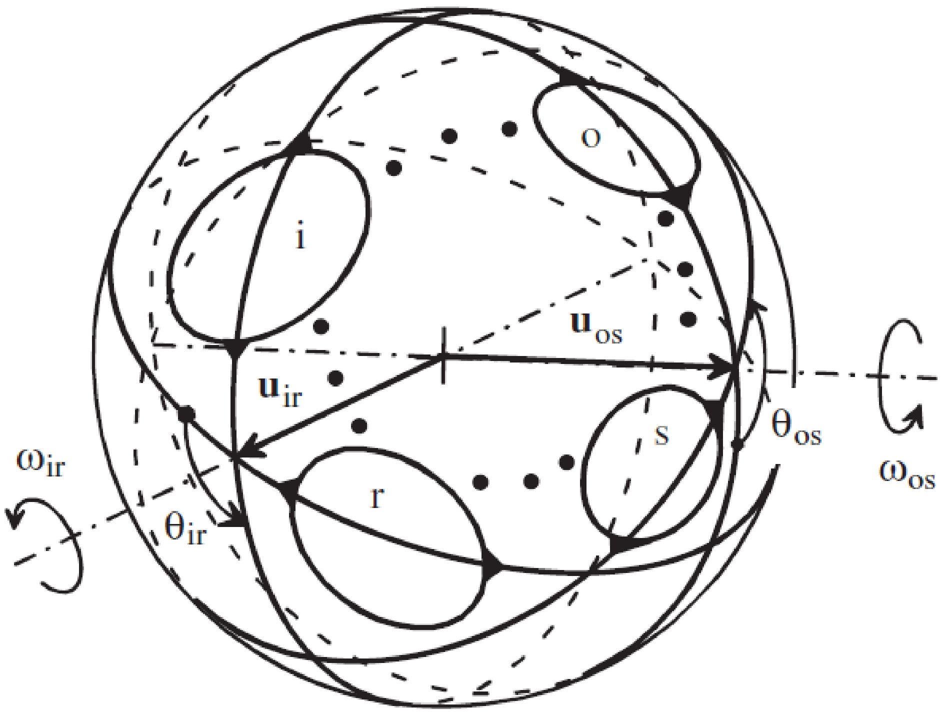
$$\Downarrow$$

d=0 $\Rightarrow$ **dead center position of the output link** (i.e., type-I singularity)

c=0 $\Rightarrow$ **dead center position of the input link** (i.e., type-II singularity)

c=d=0 $\Rightarrow$ **uncertainty configuration** (i.e., type-III singularity)

Singularity analysis of SMs with the IPs: Single-DOF Mechanisms (contd)



Single-DOF SM represented on the US as a SISO system: ω_{ir} and $\mathbf{u}_{ir}$ (ω_{os} and $\mathbf{u}_{os}$) are the input rate (output rate) and the IP position vector of the relative motion between links 'i' and 'r' ('o' and 's'), respectively.

The whole velocity analysis of a single-DOF SM can be implemented through the IPs.

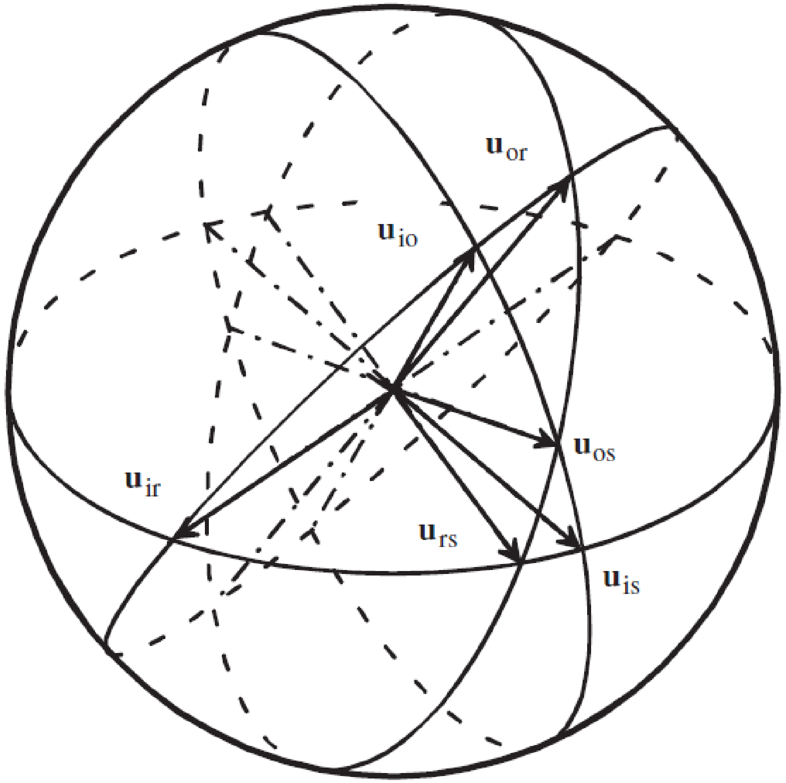
In particular [Di Gregorio, 2012]:

- the explicit expression of any input–output instantaneous relationship can be written by using the IPs;

- such relationship can be written by using only the IPs of the relative motions among 4 links: input link (i), output link (o), reference link (r) used to evaluate the rate of the input variable, and reference link (s) used to evaluate the rates of all the output variables.

[The input variable is a geometric parameter that defines the pose (position and orientation) of link "i" with respect to link "r "; whereas, the output variables are geometric parameters which define the pose of link "o" with respect to link "s"]

Singularity analysis of SMs with the IPs: Single-DOF Mechanisms (contd)



Kinematic relationships:

$$\omega_{os}\mathbf{u}_{os} = \omega_{ir}\mathbf{u}_{ir} + \omega_{rs}\mathbf{u}_{rs} - \omega_{io}\mathbf{u}_{io}$$
$$\omega_{or}\mathbf{u}_{or} = \omega_{ir}\mathbf{u}_{ir} + \omega_{os}\mathbf{u}_{os} - \omega_{is}\mathbf{u}_{is}$$
$$\Downarrow$$

Input-Output Instantaneous Relationships ($c\dot{y}=d\dot{q}$):

i) $\dot{q}=\omega_{ir}$, $\dot{y}=\omega_{os}$:

$$[\mathbf{u}_{rs}\cdot(\mathbf{u}_{os}\times\mathbf{u}_{io})]\omega_{os} = [\mathbf{u}_{rs}\cdot(\mathbf{u}_{ir}\times\mathbf{u}_{io})]\omega_{ir}$$

ii) $\dot{q}=\omega_{ir}$, $\dot{y}=\omega_{or}$:

$$[\mathbf{u}_{or}\cdot(\mathbf{u}_{os}\times\mathbf{u}_{is})]\omega_{or} = [\mathbf{u}_{ir}\cdot(\mathbf{u}_{os}\times\mathbf{u}_{is})]\omega_{ir}$$

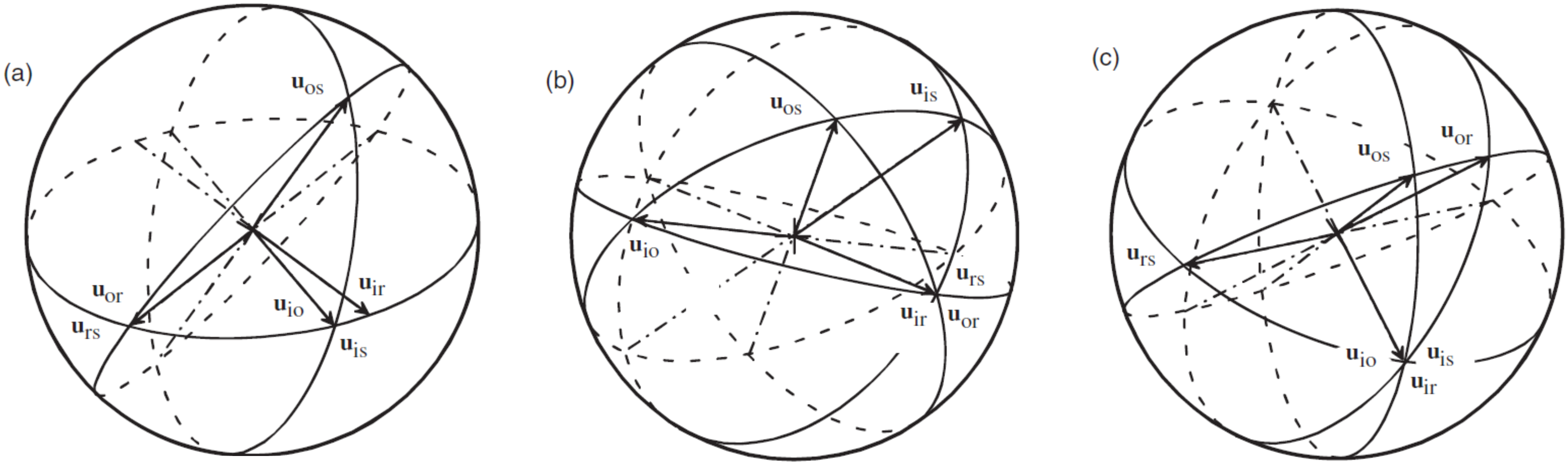
The above-deduced relationships are general, but they can be expressed in many alternative ways by exploiting the geometric relationships among the six IP position vectors {$\mathbf{u}_{or}$, $\mathbf{u}_{ir}$, $\mathbf{u}_{io}$, $\mathbf{u}_{os}$, $\mathbf{u}_{is}$, $\mathbf{u}_{rs}$}

With reference to the Figure, such relationships simply express the coplanarity of the four triplets of IP position vectors, {$\mathbf{u}_{or}$, $\mathbf{u}_{ir}$, $\mathbf{u}_{io}$}, {$\mathbf{u}_{io}$, $\mathbf{u}_{os}$, $\mathbf{u}_{is}$}, {$\mathbf{u}_{ir}$, $\mathbf{u}_{is}$, $\mathbf{u}_{rs}$}, {$\mathbf{u}_{or}$, $\mathbf{u}_{os}$, $\mathbf{u}_{rs}$}.

Singularity analysis of SMs with the IPs: Single-DOF Mechanisms (contd)

Type-I singularity conditions (d=0):

i) $\dot{q}=\omega_{ir}$, $\dot{y}=\omega_{os}$: $$[\mathbf{u}_{rs}\cdot(\mathbf{u}_{os}\times\mathbf{u}_{io})]\omega_{os}=[\mathbf{u}_{rs}\cdot(\mathbf{u}_{ir}\times\mathbf{u}_{io})]\omega_{ir}$$



The planes of the two triplets $\{\mathbf{u}_{ir}, \mathbf{u}_{is}, \mathbf{u}_{rs}\}$ and $\{\mathbf{u}_{or}, \mathbf{u}_{ir}, \mathbf{u}_{io}\}$ coincide

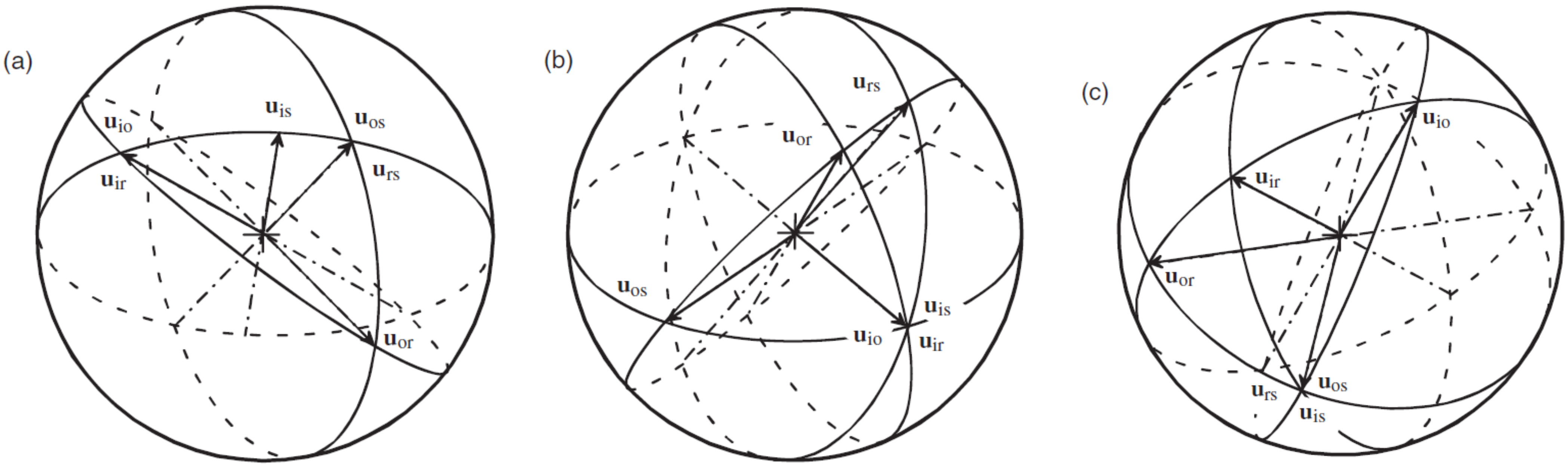
$\mathbf{u}_{ir}$ is parallel to $\mathbf{u}_{rs}$

$\mathbf{u}_{ir}$ is parallel to $\mathbf{u}_{io}$

Singularity analysis of SMs with the IPs: Single-DOF Mechanisms (contd)

Type-I singularity conditions (d=0):

ii) $\dot{q}=\omega_{ir}$, $\dot{y}=\omega_{or}$: $$[\mathbf{u}_{or} \cdot (\mathbf{u}_{os} \times \mathbf{u}_{is})]\omega_{or} = [\mathbf{u}_{ir} \cdot (\mathbf{u}_{os} \times \mathbf{u}_{is})]\omega_{ir}$$



The planes of the two triplets $\{\mathbf{u}_{io}, \mathbf{u}_{os}, \mathbf{u}_{is}\}$ and $\{\mathbf{u}_{ir}, \mathbf{u}_{is}, \mathbf{u}_{rs}\}$ coincide

$\mathbf{u}_{is}$ is parallel to $\mathbf{u}_{ir}$

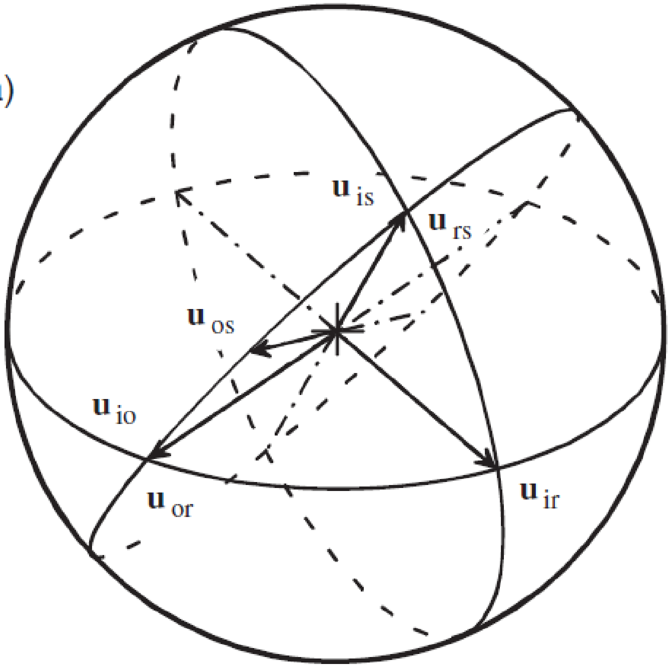
$\mathbf{u}_{is}$ is parallel to $\mathbf{u}_{os}$

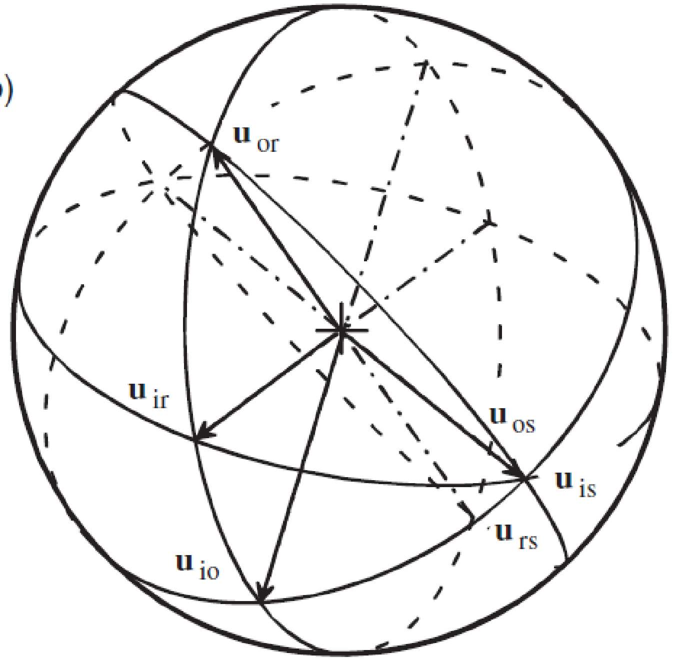
Singularity analysis of SMs with the IPs: Single-DOF Mechanisms (contd)

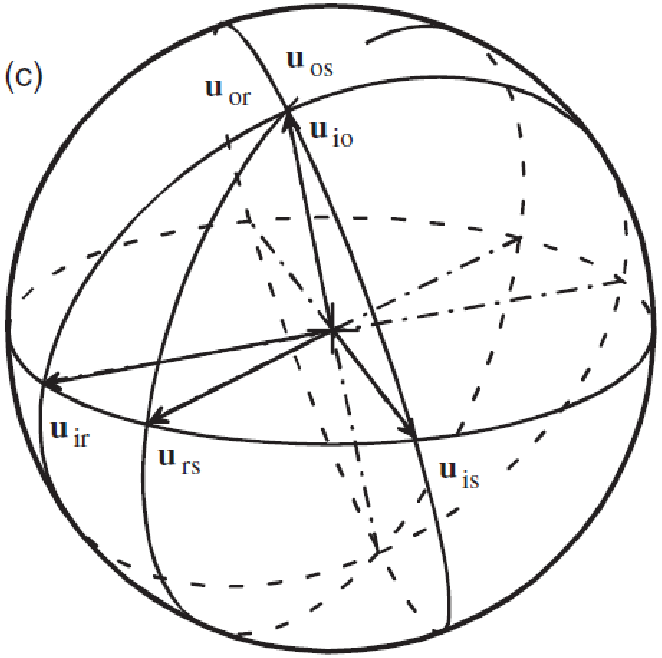
Type-II singularity conditions (c=0):

i) $\dot{q}=\omega_{ir}$, $\dot{y}=\omega_{os}$: $$[\mathbf{u}_{rs}\cdot(\mathbf{u}_{os}\times\mathbf{u}_{io})]\omega_{os}=[\mathbf{u}_{rs}\cdot(\mathbf{u}_{ir}\times\mathbf{u}_{io})]\omega_{ir}$$

ii) $\dot{q}=\omega_{ir}$, $\dot{y}=\omega_{or}$: $$[\mathbf{u}_{or}\cdot(\mathbf{u}_{os}\times\mathbf{u}_{is})]\omega_{or}=[\mathbf{u}_{ir}\cdot(\mathbf{u}_{os}\times\mathbf{u}_{is})]\omega_{ir}$$







The planes of the two triplets {$\mathbf{u}_{or}$, $\mathbf{u}_{os}$, $\mathbf{u}_{rs}$} and {$\mathbf{u}_{io}$, $\mathbf{u}_{os}$, $\mathbf{u}_{is}$} coincide

$\mathbf{u}_{os}$ is parallel to $\mathbf{u}_{rs}$
or
$\mathbf{u}_{os}$ is parallel to $\mathbf{u}_{is}$

$\mathbf{u}_{os}$ is parallel to $\mathbf{u}_{io}$
or
$\mathbf{u}_{os}$ is parallel to $\mathbf{u}_{or}$

Singularity analysis of SMs with the IPs: Single-DOF Mechanisms (contd)

Type-III singularity conditions (c=d=0):

i) $\dot{q}=\omega_{ir}$, $\dot{y}=\omega_{os}$:

$$[\mathbf{u}_{rs} \cdot (\mathbf{u}_{os} \times \mathbf{u}_{io})]\omega_{os} = [\mathbf{u}_{rs} \cdot (\mathbf{u}_{ir} \times \mathbf{u}_{io})]\omega_{ir}$$

(III.1) The six IP position vectors of set $\{\mathbf{u}_{or}, \mathbf{u}_{ir}, \mathbf{u}_{io}, \mathbf{u}_{os}, \mathbf{u}_{is}, \mathbf{u}_{rs}\}$ are all coplanar (Figure 6)

(III.2) $\mathbf{u}_{rs}$, $\mathbf{u}_{os}$ and $\mathbf{u}_{ir}$ are all parallel

(III.3) $\mathbf{u}_{io}$, $\mathbf{u}_{os}$ and $\mathbf{u}_{ir}$ are all parallel

(III.4) $\mathbf{u}_{ir}$ is parallel to $\mathbf{u}_{rs}$ and, simultaneously, $\mathbf{u}_{os}$ is parallel to $\mathbf{u}_{io}$

(III.5) $\mathbf{u}_{ir}$ is parallel to $\mathbf{u}_{io}$ and, simultaneously, $\mathbf{u}_{os}$ is parallel to $\mathbf{u}_{rs}$

ii) $\dot{q}=\omega_{ir}$, $\dot{y}=\omega_{or}$:

$$[\mathbf{u}_{or} \cdot (\mathbf{u}_{os} \times \mathbf{u}_{is})]\omega_{or} = [\mathbf{u}_{ir} \cdot (\mathbf{u}_{os} \times \mathbf{u}_{is})]\omega_{ir}$$

(III.a) The six IP position vectors of set $\{\mathbf{u}_{or}, \mathbf{u}_{ir}, \mathbf{u}_{io}, \mathbf{u}_{os}, \mathbf{u}_{is}, \mathbf{u}_{rs}\}$ are all coplanar (Figure 6)

(III.b) The planes of the two triplets $\{\mathbf{u}_{io}, \mathbf{u}_{os}, \mathbf{u}_{is}\}$ and $\{\mathbf{u}_{ir}, \mathbf{u}_{is}, \mathbf{u}_{rs}\}$ coincide and, simultaneously, $\mathbf{u}_{os}$ is parallel to $\mathbf{u}_{is}$

(III.c) the planes of the two triplets $\{\mathbf{u}_{or}, \mathbf{u}_{os}, \mathbf{u}_{rs}\}$ and $\{\mathbf{u}_{io}, \mathbf{u}_{os}, \mathbf{u}_{is}\}$ coincide and, simultaneously, $\mathbf{u}_{is}$ is parallel to $\mathbf{u}_{os}$

(III.d) $\mathbf{u}_{is}$ is parallel to $\mathbf{u}_{ir}$ and, simultaneously, $\mathbf{u}_{os}$ is parallel to $\mathbf{u}_{or}$

(III.e) $\mathbf{u}_{is}$, $\mathbf{u}_{ir}$ and $\mathbf{u}_{os}$ are all parallel

(III.f) $\mathbf{u}_{is}$, $\mathbf{u}_{os}$ and $\mathbf{u}_{or}$ are all parallel

(III.g) $\mathbf{u}_{is}$ is parallel to $\mathbf{u}_{os}$ (Figure 4(c))

Singularity analysis of SMs with the IPs: Single-DOF Mechanisms (contd)

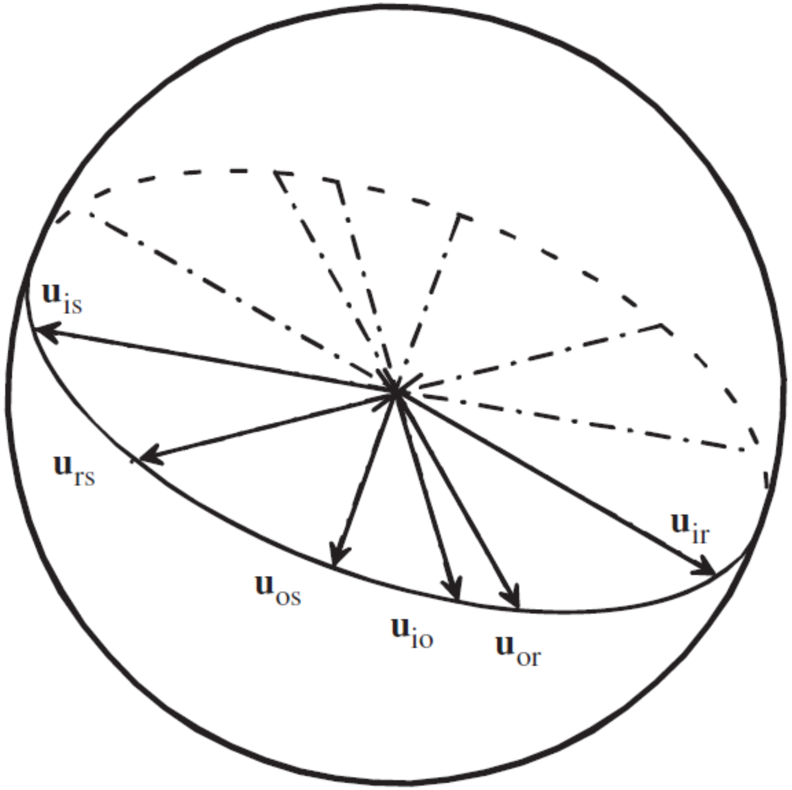
Type-III singularity conditions (c=d=0):

i) $\dot{q}=\omega_{ir}$, $\dot{y}=\omega_{os}$:

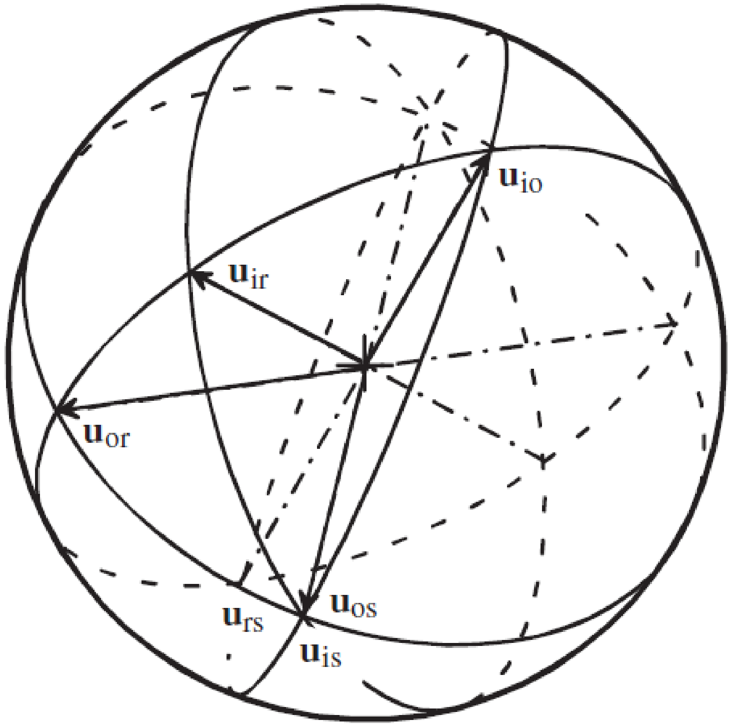
$$[\mathbf{u}_{rs}\cdot(\mathbf{u}_{os}\times\mathbf{u}_{io})]\omega_{os}=[\mathbf{u}_{rs}\cdot(\mathbf{u}_{ir}\times\mathbf{u}_{io})]\omega_{ir}$$

ii) $\dot{q}=\omega_{ir}$, $\dot{y}=\omega_{or}$:

$$[\mathbf{u}_{or}\cdot(\mathbf{u}_{os}\times\mathbf{u}_{is})]\omega_{or}=[\mathbf{u}_{ir}\cdot(\mathbf{u}_{os}\times\mathbf{u}_{is})]\omega_{ir}$$



(c)



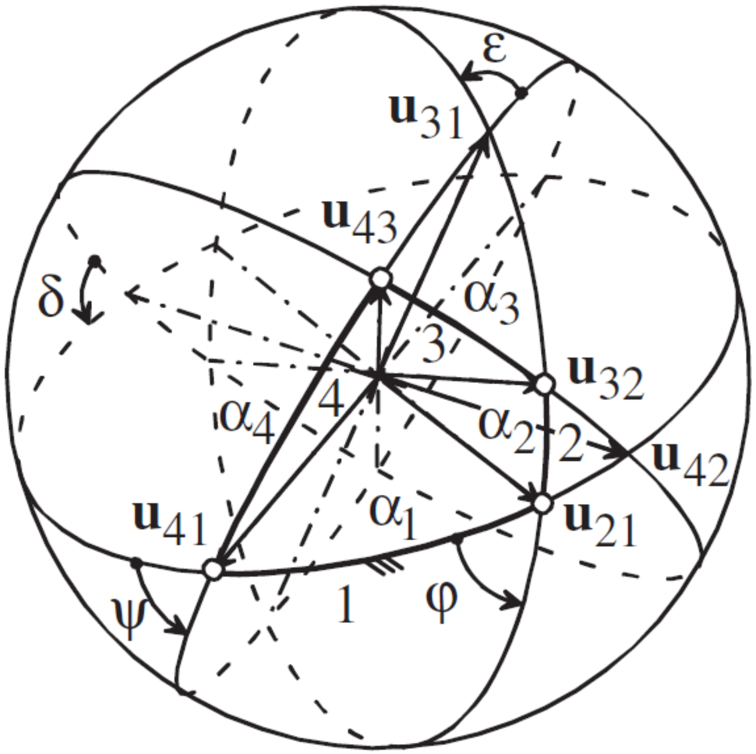
The six IP position vectors of the set $\{\mathbf{u}_{or}, \mathbf{u}_{ir}, \mathbf{u}_{io}, \mathbf{u}_{os}, \mathbf{u}_{is}, \mathbf{u}_{rs}\}$ are all coplanar

$\mathbf{u}_{is}$ is parallel to $\mathbf{u}_{os}$

Singularity analysis of SMs with the IPs: Single-DOF Mechanisms (contd)

From an analytic point of view,

- the above-deduced geometric conditions (parallelism and/or coplanarity of particular IPs) become equations that, once the position analysis has been solved and the IPs have been determined, contain only the input variable as unknown
- the solutions of these equations analytically determine the singular configurations of the SM



Spherical Four-Bar Linkage

Spherical Four-Bar Linkage

Input-Output Instantaneous Relationship:

$$[\mathbf{u}_{41} \cdot (\mathbf{u}_{43} \times \mathbf{u}_{32})]\omega_{41} = [\mathbf{u}_{21} \cdot (\mathbf{u}_{43} \times \mathbf{u}_{32})]\omega_{21}$$

$$\Downarrow$$

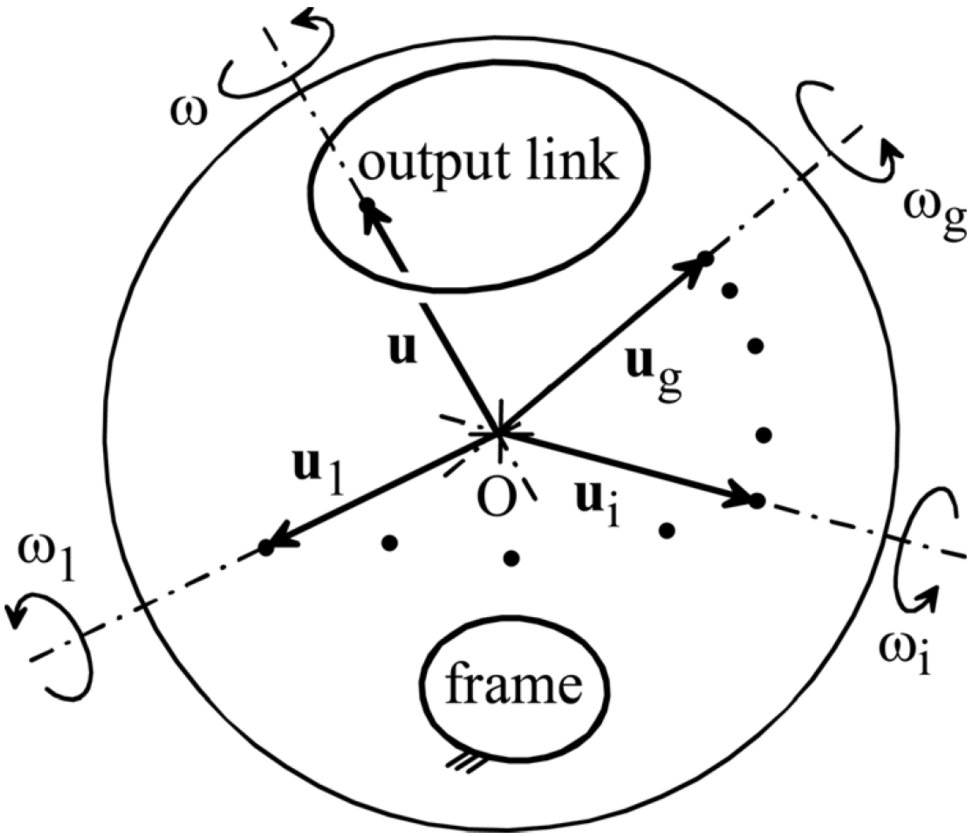
Type-I singularities' analytic condition:

$$\mathbf{u}_{21} \cdot (\mathbf{u}_{43} \times \mathbf{u}_{32}) = 0$$

Type-II singularities' analytic condition:

$$\mathbf{u}_{41} \cdot (\mathbf{u}_{43} \times \mathbf{u}_{32}) = 0$$

Singularity analysis of SMs with the IPs: Multi-DOF Mechanisms



General scheme of a g-DOF SM:
O = center of the spherical motion

g = DOF number

q_i = i-th input variable

u and ω are unit vector (IP on the US) **and signed magnitude**, respectively, **of output-link's angular velocity**,

u_i and ω_i for i=1,…,g **are unit vector** (IP on the US) **and signed magnitude**, respectively, **of output-link's angular velocity**, **when all the inputs are locked but the i-th one**

Superposition Principle (superposition of g single-DOF SMs)**:**

$$\boldsymbol{\omega} = \sum_{i=1,g} \boldsymbol{\omega}_i \quad \Rightarrow \quad \omega\ \mathbf{u} = \sum_{i=1,g} \omega_i\ \mathbf{u}_i$$

Input-Output Instantaneous relationship of the i-th single-DOF SM generated from the g-DOF SM by locking all the inputs but the i-th one:

$$\mathrm{c}_i\ \omega_i = \mathrm{d}_i\ \dot{\mathrm{q}}_i \qquad \mathrm{i} = 1, \ldots, \mathrm{g}$$

Singularity analysis of SMs with the IPs: Multi-DOF Mechanisms (contd)

$$\omega\,\mathbf{u} = \sum_{i=1,g} \omega_i\,\mathbf{u}_i \quad \& \quad \mathrm{c}_i\,\omega_i = \mathrm{d}_i\,\dot{\mathrm{q}}_i \quad \mathrm{i} = 1,\ldots,\mathrm{g}$$

$$\Downarrow$$

Input-Output Instantaneous relationship of the g-DOF SM:

$$(\omega\mathbf{u}) \prod_{i=1,g} c_i = \sum_{i=1,g} \left[\mathbf{u}_i \left(d_i \prod_{\substack{j=1,g \\ j\neq i}} c_j \right) \right] \dot{q}_i$$

In this expression:

- output link's angular velocity $\{\omega\mathbf{u}\}^T$ is the output

- the coefficients both of the input rates and of the output involve only the IPA unit vectors of the g single-DOF SMs generated from the g-DOF SM, which depend only on the SM configuration

[i.e., the coefficients c_i and d_i for i=1,…,g, and the IPA unit vectors, $\mathbf{u}_i$ for I = 1,…,g]

Singularity analysis of SMs with the IPs: Multi-DOF Mechanisms (contd)

$$(\omega\mathbf{u})\prod_{i=1,g} c_i = \sum_{i=1,g}\left[\mathbf{u}_i\left(d_i \prod_{\substack{j=1,g\\ j\neq i}} c_j\right)\right]\dot{q}_i \qquad (7)$$

1) at a non-singular configuration,

1a) for g≥3, system (7) consists of 3 independent scalar equations; whereas,

1b) for g = 2, it reduces itself to only two independent scalar equations, [in this case, **u is a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ and must lie on the plane located by $\mathbf{u}_1$ and $\mathbf{u}_2$**]

2) type-I singularities occur if system (7) cannot provide a unique solution for a number of input rates equal to the minimum between g **and** 3 **when output-link's angular velocity** ω**u**, **and**, for g ≥ 3, **the remaining input rates are assigned**.

3) type-II singularities occur if system (7) cannot provide a unique solution for output-link's angular velocity ω**u**, **when the input rates are all assigned**

Singularity analysis of SMs with the IPs: Multi-DOF Mechanisms (contd)

$$(\omega \mathbf{u}) \prod_{i=1,g} c_i = \sum_{i=1,g} \left[\mathbf{u}_i (d_i \prod_{\substack{j=1,g \\ j \neq i}} c_j) \right] \dot{q}_i \tag{7}$$

Type-I Singularities

Equation (7) reveals that,

at a given configuration, **if $d_r = 0$**, $r \in \{1, 2, \ldots, g\}$, **then the r-th input rate** $\dot{q}_r$ **is undetermined**.

$\Downarrow$

Statement 1: **the union of the g sets of type-I singularities of the single-DOF SMs, generated from a g-dof SM, is a subset of the set of all the type-I singularities of that g-DOF SM**

Theorem 1: **For g≥3** *and provided that* **all the $\mathbf{c}_i$ and $\mathbf{d}_i$ coefficients are different from zero, a type-I singularity occurs if and only if 3 IPAs unit vectors**, $\mathbf{u}_i$, **are coplanar**

Singularity analysis of SMs with the IPs: Multi-DOF Mechanisms (contd)

$$(\omega \mathbf{u}) \prod_{i=1,g} c_i = \sum_{i=1,g} \left[\mathbf{u}_i (d_i \prod_{\substack{j=1,g \\ j \neq i}} c_j) \right] \dot{q}_i \tag{7}$$

Type-II Singularities

Equation (7) reveals that,

at a given configuration, **the motion of the output link is undetermined even though the input rates are all assigned, if and only if at least one** c_i **coefficient is equal to zero**

$\Downarrow$

Statement 2: **the set of all the type-II singularities of a g-DOF SM is the union of the g sets of type-II singularities of the single-DOF SMs generated from that g-dof SM**

Theorem 2: **the coincidence of all the $\mathbf{u}_i$ unit vectors for i = 1,…, g, identifies a particular type-II singularity where all the c_i coefficients vanish**

Singularity analysis of SMs with the IPs: Multi-DOF Mechanisms (contd)

$$(\omega \mathbf{u}) \prod_{i=1,g} c_i = \sum_{i=1,g} \left[\mathbf{u}_i (d_i \prod_{\substack{j=1,g \\ j \neq i}} c_j) \right] \dot{q}_i \qquad (7)$$

Type-III Singularities

Equation (7) reveals that,

at a given configuration, **if at least one out of the c_i coefficients together with at least one out of the d_i coefficients are equal to zero, a type-III singularity will occur**

$\Downarrow$

Statement 3: **In a g-DOF SM, a type-(III) singularity occurs if and only if at least one c_j and one d_k, with $j,k \in \{1,\ldots,g\}$, are simultaneously equal to zero**

$\Downarrow$

Corollary 1: **the union of the g sets of type-III singularities of the single-DOF SMs, generated from a g-dof SM, is a subset of the set of all the type-III singularities of that g-DOF SM**

Singularity analysis of SMs with the IPs: Multi-DOF Mechanisms (contd)

$$(\omega \mathbf{u}) \prod_{i=1,g} c_i = \sum_{i=1,g} \left[\mathbf{u}_i (d_i \prod_{\substack{j=1,g \\ j \neq i}} c_j) \right] \dot{q}_i \quad (7)$$

Discussion

The above results proved that there, in g-DOF SMs, there are two groups of singularities:

1st group : singularities that are also singularities of at least one single-DOF SM generated from the g-DOF SM,

For these singularities the equations to solve are:

$$c_i(\mathbf{q}) = 0 \quad i = 1, \dots, g$$

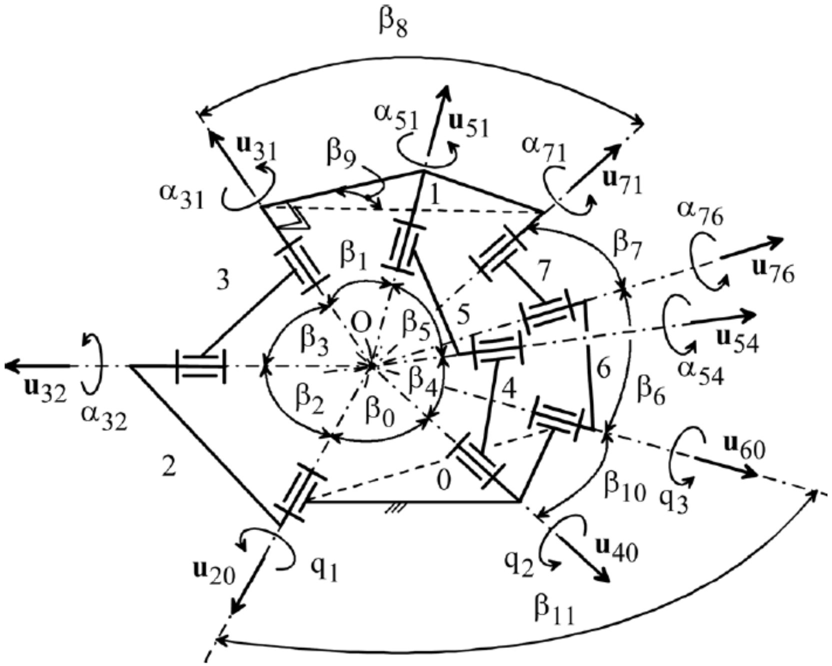
$$d_i(\mathbf{q}) = 0 \quad i = 1, \dots, g$$

2nd group: singularities that are NOT singularities of any single-DOF SM generated from the g-DOF PM,

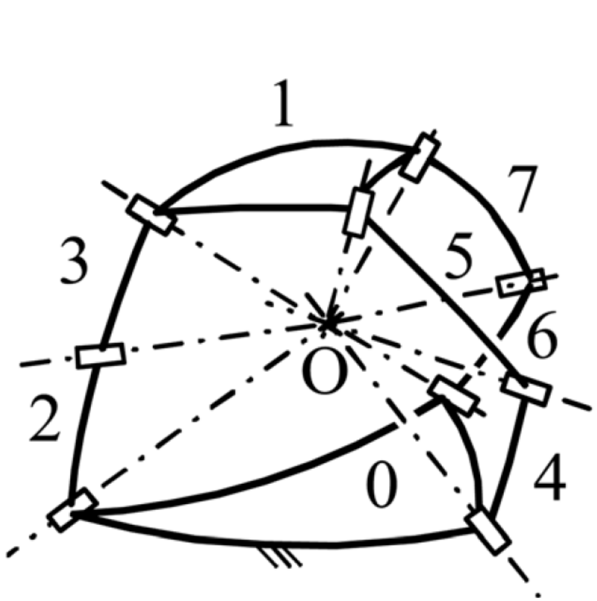
These singularities are only those identified by theorem 1 $\Rightarrow$ the equations to solve are obtained by imposing the coplanarity of u_i triplets [the components of each u_i can be easily computed with the above-presented general method for determining IPAs' unit vectors in single-DOF SMs]

Singularity analysis of SMs with the IPs: Multi-DOF Mechanisms (contd)

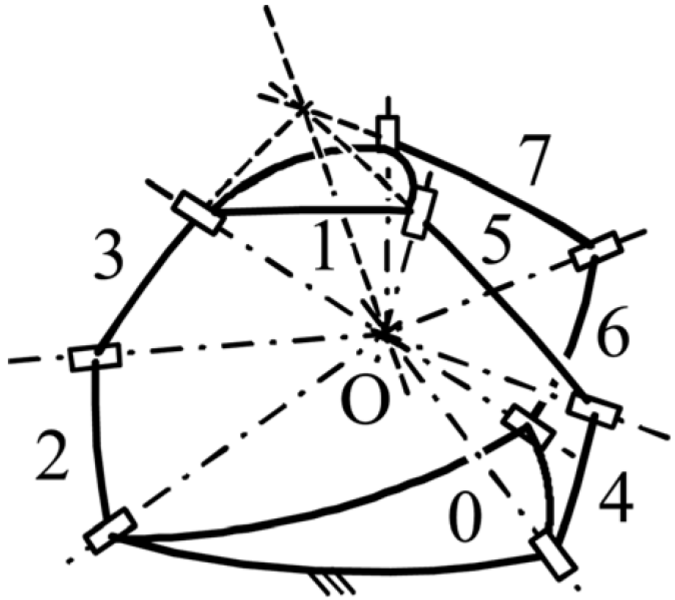
Case Study: 3-RRR SM



3-RRR SM



Example of type-I singularity: $\mathbf{u}_{20}$, $\mathbf{u}_{32}$, and $\mathbf{u}_{31}$ are coplanar



Example of type-II singularity: $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ coincide

Further details (proofs of the theorems, computation algorithm, etc.) on Multi-DOF SMs are reported in

- Di Gregorio R., 2015, "Analytic and Geometric Technique for the Singularity Analysis of Multi-Degree-of-Freedom Spherical Mechanisms," *ASME J. Mechanisms and Robotics*, 7(3): 031008 (9 pages)