A Short Account on Leonardo Torres' Endless Spindle

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Abstract

At the end of the nineteenth century, several analog machines had been proposed for solving algebraic equations. These machines —based not only on kinematics principles but also on dynamic or hydrostatic balances, electric or electromagnetic devices, etc.— had one important drawback: lack of accuracy.

Leonardo Torres was the first to beat the challenge of designing and implementing a machine able to compute the roots of algebraic equations that, in the case of polynomials of degree eight, attained a precision down to 1/1000. The key element of Torres' machine was the *endless spindle*, an analog mechanical device designed to compute $\log(a+b)$ from $\log(a)$ and $\log(b)$. This short account gives a detailed description of this mechanism.

Keywords: Algebraic machines, history of mechanisms, endless spindle, transmissions.

1 Introduction

Leonardo Torres (1852-1936), usually known as Leonardo Torres y Quevedo in Spanish-speaking countries, was a Spanish engineer and mathematician. He was president of the Academy of Sciences of Madrid, a member of the French Academy of Sciences, and famous –mainly in Spain and France– as a prolific and successful inventor. A short biography and description of his contributions can be found in [13, 11].

Some of the earliest Torres' inventions took the form of mechanical analog devices, considered of great originality, aiming at implementing what he called the *Algebraic Machine* [3]. This machine had the ability of computing the values of arbitrary polynomial functions in one variable. Since it was an analogue machine, the variable could attain any value, not only a preestablished set of discrete values, contrarily to what happened, for example, with the celebrated Babbage's Difference Engine [18], an engine that used the method of finite differences to generate successive values of polynomial functions.

Figure 1: Leonardo Torres (1852-1936)

In Torres' Algebraic Machine, all quantities were represented by means of angular displacements in logarithmic scale. Then, adding a counter to keep track of the number of turns, it was possible to compactly represent very large variations for all quantities. When the wheel representing the variable spun round, the final result was obtained as the angular displacement of another wheel that accumulated the addition of all involved monomials. When this result was zero, a root of the function was found. Using proper modifications, it was even possible to obtain the complex roots.

The use of logarithms had two main advantages. Firstly, assuming that all absolute errors for the angular displacements were constant, all relative errors for the represented quantities were also kept constant. Secondly, the computation of monomials was greatly simplified. Actually, the logarithm of a monomial of the form $a_i x^i$ is an expression, $a_i + i \log x$, that can be easily calculated using a differential transmission. Nevertheless, the use of logarithms had a number of disadvantages. It was necessary to introduce some transformations to ensure that all quantities were positive and, what was much more challenging, it was necessary to design a sophisticated mechanical device able to compute $\log(a+b)$ from $\log(a)$ and $\log(b)$ that would permit to accumulate the sum of all involved monomials. This device was the *endless spindle*.

In 1893, Leonardo Torres presented, before the Academy of Exact, Physical and Natural Sciences, a memoir on his Algebraic Machine [19]. In his time, this was considered an extraordinary success for Spanish scientific production. In 1895 the machine was presented at a congress in Bordeaux. Later on, in 1900, he would present his calculating machine before the Paris Academy of Sciences [20].

This account is structured as follows. Next, we examine the state of the art in the analogue computation of algebraic functions using kinematic linkages at the end of the nineteenth century —against which Torres' Algebraic Machine should be analyzed. Next, we reproduce Torres' mathematical formulation for the problem of extracting the logarithm of a sum as a sum of logarithms, so that this operation could be implemented using a mechanical transmission. Then, we examine how this transmission was designed and finally implemented.

2 Historical background

Until the beginning of the nineteenth century, the diverse methods proposed for solving algebraic equations mechanically fall naturally into five types: (1) graphic and visual methods, (2) dynamic balances, (3) hydrostatic balances, (4) electric and electromagnetic methods, and (5) kinematic linkages. A survey on all these machines, written in 1905, can be found in [10]. If we focus our attention on those based on kinematic linkages, this wealth of machines is reduced to few examples, some of them little known to the English speaking audience. In all cases, quantities are represented by dimensions: the displacement of a lever or the rotation of a shaft. They consist of shafts, disks, wheels, sprockets, chains, etc. Various combinations of these parts perform by their motions the equivalent of addition, subtraction and multiplication.

The search for mechanical means of solving algebraic equations dates back to the eighteenth century. In 1770, John Rowning (1699-1771) considered the possibility of drawing the graph of a polynomial continuously by using a number of rulers linked together so that the pencil point on the last ruler would trace the required curve [14]. This approach awaited the day of precision machinery before it could be considered practical. The equiangular linkage, described by Alfred Kempe (1849-1922) in 1873, was a theoretical device streaming from this idea [8]. Although Kempe did not attempt a description of a mechanical method for producing his equiangular linkage, this was made later in the twentieth century by Thomas Blakesley (1847-1929) [1, 2] but, unfortunately, it was just an attempt because it was found to be partially incorrect [4]. The accuracy attained by this approach, based on planar linkages, is within the limits determined by the size of the machine. One alternative to increase accuracy, without increasing the size of the machine, was to represent quantities by the endless rotation of shafts instead of planar displacements.

Léon Lalanne (1811-1892), in 1840, presented his arithmoplanimeter, a machine based on the same concept as a planimeter, that was able to compute monomials expressions [9]. This machine had an important influence in Ernest Stamm (1834-1875), the first to consider the problem of computing polynomial expression using linkages from a practical point of view. Stamm, in 1863, thanks to the combination of elements similar to Lalanne's arithmoplanimeter, was able to compute arbitrary algebraic expressions [16]. Additions and subtractions were performed by a set of gears similar to those used nowadays in the differential of an automobile. Fig. 2 shows this device. When both bevel gears turn in the same direction, the rotation of the differential shaft is proportional to the sum of their rotations. When the bevel gears rotate in opposite directions, the rotation of the differential shaft is proportional to the difference of their rotations. Mathematically this action can be expressed as $\varphi_3 = c_1\varphi_1 + c_2\varphi_2$, where the constants are determined by the gear ratios. Later on, in 1892, Stamm's ideas were improved by Federico Guarducci (1851-1931) [6]. Although Guarducci simplified Stamm's machine by introducing a mechanical element to generate a monomial of different degree and a system of pulleys and ropes to add the results, the problem was still the lack of accuracy mainly because arithmoplanimeters, and similar devices, include friction rollers so that slippage errors are unavoidable.

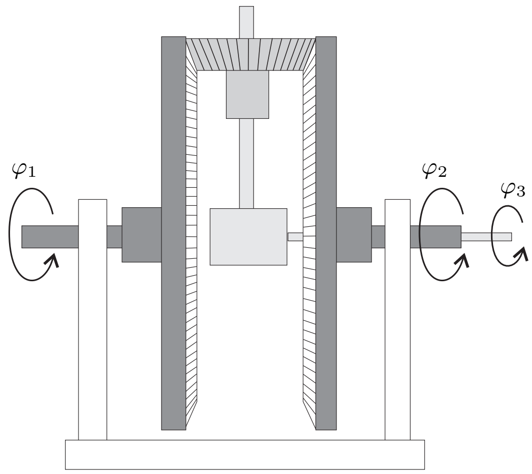

Figure 2: Differential gear train implementing the expression $\varphi_3 = a_1\varphi_1 + a_2\varphi_2$, a standard element in analogue mechanical calculators since the mid nineteenth century [17].

It seems that Torres' was not aware, at least till 1893, of all these previous developments [15]. As a consequence, his approach was fully original, the only connection with the above works being the use of a differential transmission, equivalent to that previously proposed by Stamm, for adding and subtracting quantities.

3 Mathematical formulation

A polynomial of order n can be expressed as the addition of $n+1$ monomials, i.e.

$$f(x, a_0, \ldots, a_n) = \sum_{i=0}^{n} a_i x^i, \tag{1}$$

where x is the variable and a_i, $i = 0, \ldots, n$ the coefficients. The logarithm of each monomial has the form $\ln a_i + i \ln x$, $i = 0, \ldots, n$. Then, since both the coefficients and the roots of (1) are not necessarily positive, Torres had to introduce some transformations to guarantee positiveness. For example, to compute the negative roots of (1), he considered the polynomial

$$g(x, a_0, \ldots, a_n) = \sum_{i=0}^{n} a_i (-1)^i x^i. \tag{2}$$

It is clear that the roots of (1) correspond to the positive roots of both $f(\cdot) = 0$ and $g(\cdot) = 0$. Now, all involved monomials have constant sign for positive values of x so that we can classify them into positive or negative. Then, the positive roots of (1) are the positive roots of

$$\frac{\sum_{i \in P} a_i x^i}{\sum_{i \in N} (-a_i) x^i} = 1 \tag{3}$$

where all monomials are positive for $x > 0$. The same can be done for the roots of (2).

To accumulate the result of all monomials, as x varies, Torres had to compute $\ln(a+b)$ from $\ln(a)$ and $\ln(b)$. The problem was formulated as follows[1]:

[1]Torres used decimal logarithms. For the sake of simplicity, we use here natural logarithms. The discussion runs obviously parallel.

$$
\begin{aligned}
\ln(a+b) &= \ln\left(b\left(\frac{a}{b}\right)+b\right) \\
&= \ln(b)+\ln\left(\frac{a}{b}+1\right) \\
&= \ln(b)+\ln\left(e^{(\ln(a)-\ln(b))}+1\right) \\
&= \ln(b)-m(\ln(a)-\ln(b)) \\
&\quad +m(\ln(a)-\ln(b)) \\
&\quad +\ln\left(e^{(\ln(a)-\ln(b))}+1\right),
\end{aligned} \tag{4}
$$

where m is a design constant. The linear combinations in the above expressions can be easily implemented using the differential mechanism introduced by Stamm and described in the previous section. Thus, the problem boils down to designing a transmission mechanism able to implement the function

$$y = mx + \ln(e^x + 1) \tag{5}$$

and this is what the endless spindle was for.

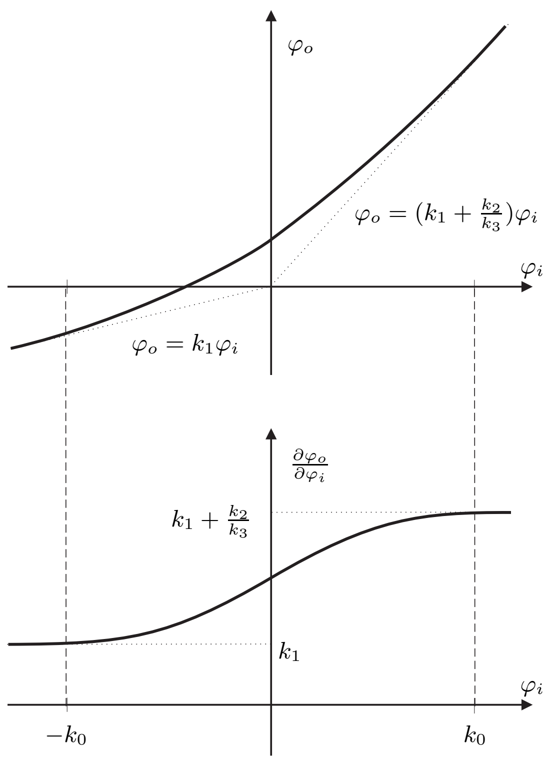

Figure 3: Plot of the function (6) (top), and its derivative (bottom). Asymptotes are shown in dotted lines.

As we said before, Torres decided to represent all quantities by angular displacements in logarithmic scale. In doing so with x and y, we can express $x = p\varphi_i$ and $y = q\varphi_o$, where p and q are constants and φ_i and φ_o angular displacements. Substituting these expressions in (5), we get [see Fig. 3(top)],

$$\varphi_o = k_1\varphi_i + k_2 \ln(e^{\frac{\varphi_i}{k_3}} + 1), \tag{6}$$

where $k_1 = m \cdot p/q$, $k_2 = 1/q$, and $k_3 = 1/p$, can be independently adjusted my properly choosing m, p, and q.

The algebraic manipulations in (4) might seem unnecessary involved to the reader. Torres knew that a transmission mechanism cannot represent a function with constant asymptotic behavior because this would imply null transmission velocities. This is why he finally came up with the idea of implementing a transmission mechanism for function (5). Some accounts overcome this fact and give a simplified presentation which does not correspond to what Torres finally implemented. To better understand this fact, we can differentiate equation (6) to obtain:

$$\frac{\partial \varphi_o}{\partial \varphi_i} = k_1 + \frac{k_2}{k_3} \frac{1}{1 + e^{-\frac{\varphi_i}{k_3}}}. \tag{7}$$

Thus, if $k_1 > 0$, the transmission velocity law is always positive [see Fig. 3(bottom)].

Function (6) and its derivative (7) can be approximated for large and small values of φ_i by their asymptotes. That is,

$$\varphi_o = \begin{cases} k_1 \varphi_i, & x < -k_0 \\ (k_1 + \frac{k_2}{k_3})\varphi_i, & x > k_0 \end{cases} \tag{8}$$

$$\frac{\partial \varphi_o}{\partial \varphi_i} = \begin{cases} k_1, & x < -k_0 \\ k_1 + \frac{k_2}{k_3}, & x > k_0 \end{cases} \tag{9}$$

Once arrived at this point, the challenge for Torres was to design a transmission implementing the function given by function (6).

4 Designing the transmission

Let us consider two curves, $\mathcal{C}_1$ and $\mathcal{C}_2$ that can rotate around O_1 and O_2, respectively, so that they always keep in contact. Let φ_i and φ_o represent the angular displacements of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, as shown in Fig. 4.

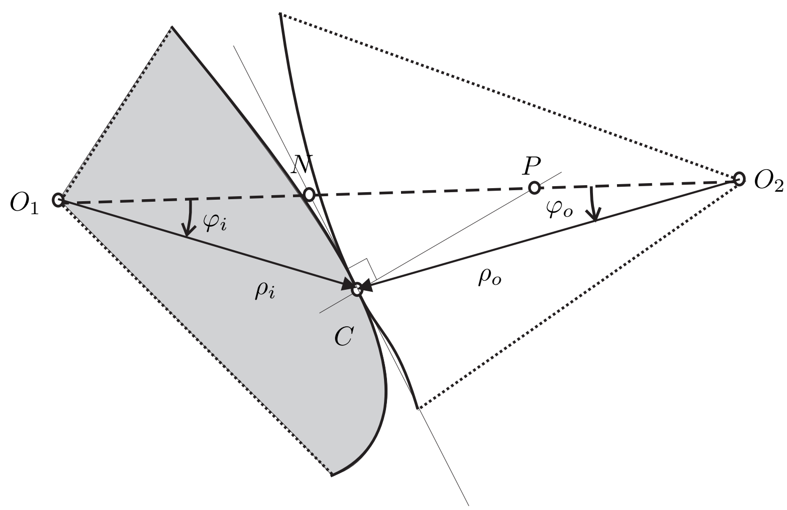

Figure 4: Two matting curves that rotate so that they always keep in contact at a point without slippage.

Although the two curves have different velocities at the contact point, say C, their velocities orthogonal to the tangent to both curves at the contact point are equal in both magnitude and direction. Otherwise, the two curves would separate from each other. Therefore, it can be proved that the angular velocities of the driving profile to the driven profile, or velocity ratio, of the matting curves is given by:

$$\frac{\partial \varphi_o}{\partial \varphi_i} = \frac{\rho_i}{\rho_o}, \tag{10}$$

where $\rho_i = O_1P$, $\rho_o = O_2P$, and P –known as the pitch point– is the intersection of the normal at the contact point between the two curves with the line of centers O_1O_2 (see, for example, [7]). Then, substituting in (7), we get

$$\frac{\rho_o}{\rho_i} = \frac{1}{k_1 + \frac{k_2}{k_3}\frac{1}{1+e^{-\frac{\varphi_i}{k_3}}}} \tag{11}$$

Now, the problem was to compute the equation of both curves, $\rho_i(\varphi_i)$ and $\rho_o(\varphi_o)$, so that the above relation holds. It seems that Torres tried to obtain the exact solution to this problem but he found no way to integrate the resulting differential equations. He had to introduce a simplification. He though that, if ρ_i varies very slowly with respect to φ_i, then $NP \to 0$, and O_1C can be approximated by O_1P. It is clear that we can always bound the error committed by this approximation by properly choosing k_1, k_2, and k_3. Then, under this assumption, if the distance between the two centers is d,

$$\rho_o \simeq d - \rho_i. \tag{12}$$

Finally, the sought equations in polar coordinates are

$$\rho_i(\varphi_i) = d\left(1 - \frac{1}{1 + k_1 + \frac{k_2}{k_3}\frac{1}{1+e^{-\frac{\varphi_i}{k_3}}}}\right) \tag{13}$$

and

$$\rho_o(\varphi_o) = d\left(\frac{1}{1 + k_1 + \frac{k_2}{k_3}\frac{1}{1+e^{-\frac{\varphi_i}{k_3}}}}\right). \tag{14}$$

As an example, let us take $k_1 = 0.5$, $k_2 = 20$, $k_3 = 4$, and $d = 1$. The resulting spiral-like curves are shown in Fig. 5.

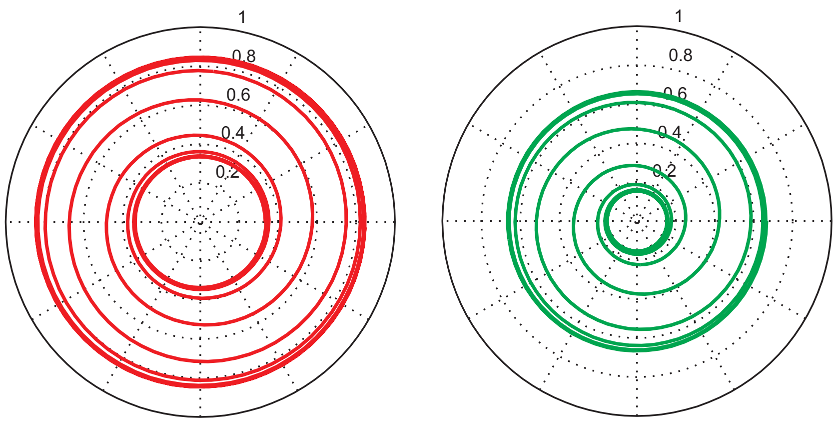

Figure 5: Input profile, $\rho_i(\varphi_i)$ (left), and output profile, $\rho_o(\varphi_o)$ (right), for $k_1 = 0.5$, $k_2 = 20$, $k_3 = 4$, and $d = 1$.

5 The implementation

Since the obtained spiral-like profiles overlap when in contact at their matting points, the only possibility to implement the corresponding transmission in practice was to stretch both profiles, perpendicularly to their plane, into helices so that the matting points correspond to points at the same height in the resulting helices. Nevertheless, it was still necessary to introduce an idle wheel, a wheel interposed between both profiles to convey motion to one to the other, to guarantee that both helices would not collide. Finally, in order to ensure no slippage errors, contacts were performed through gears. The result can be seen in Fig. 6.

Now, it was necessary to set all parameters (k_0, k_1, k_2, and k_3). This setting began by deciding the scale factor, i.e. the number of revolutions a shaft must spin round to represent one unit of the quantity under consideration, and a bound for the error committed due to the approximation introduced by equation (12) [19].

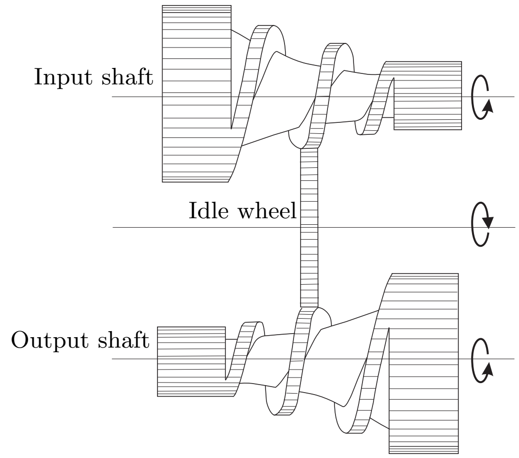

Figure 6: Schematic representation of the endless spindle as presented in [20]: Two helical geared shafts connected by an idle gear.

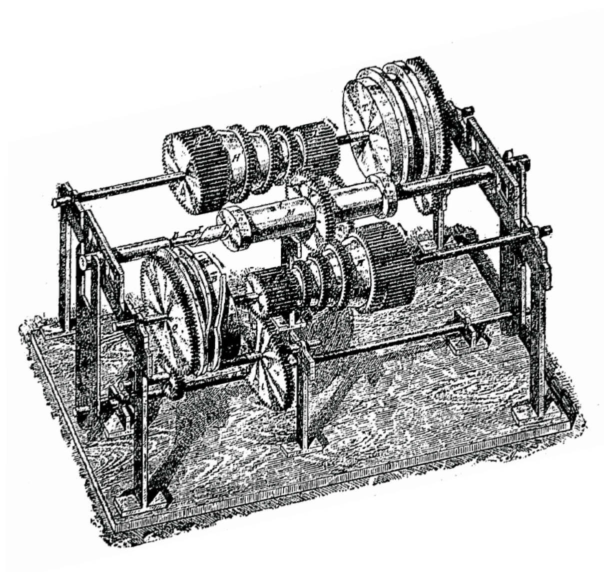

Figure 7: Artistic representation of the endless spindle. It appeared in [19] in an incorrect perspective projection which has been modified here to give a correct appearance.

Torres skills were not limited to the theoretical side. In order to construct the different parts of this transmission, he also proposed new tools and manufacturing techniques. Using several units of the described endless spindle, he constructed several machines that could solve different algebraic equations, even one with eight terms (Fig. 8). He began to build this latter machine in 1910 and concluded it in 1920, at least thirty five years after it was theoretically conceived. The result was remarkable, the machine was able to compute the roots of arbitrary polynomials of order eight, including the complex ones, with a precision down to thousandths. As many other Torres' inventions, some still in working order, this machine is stored at the Civil Engineering School of the Technical University of Madrid. As recently pointed out in [5], it would be not only convenient but also necessary that these machines would be put in working order and on exhibition.

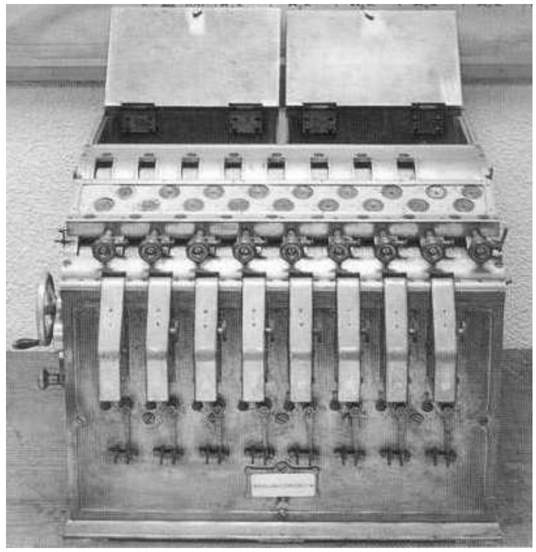

Figure 8: An instance of Torres' Algebraic Machine. This particular one was able to compute the roots of arbitrary polynomials of order eight.

6 Conclusion

Leonardo Torres was probably the most eminent Spanish engineer in the first half of the 20th century. He had a great influence, truncated by the Spanish Civil War, in the development of automatic control in Spain. As Randell points out in [13], we can only speculate on what might happened if Torres' writings had become better known to the English-speaking world. For instance, he qualifies Torres' paper *Essays on Automatics* [21] as "a fascinating work which well repays reading even today." The paper contains what Randell believes to be the first proposal of the idea of floating-point arithmetic. It seems clear that Torres' contributions deserve much wider appreciation outside Spain. We hope that this short account will contribute to this end.

7 Acknowledgment

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