

# Approaching Dual Quaternions From Matrix Algebra

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**Abstract**—Dual quaternions give a neat and succinct way to encapsulate both translations and rotations into a unified representation that can easily be concatenated and interpolated. Unfortunately, the combination of quaternions and dual numbers seem quite abstract and somewhat arbitrary when approached for the first time. Actually, the use of quaternions or dual numbers separately are already seen as a break in mainstream robot kinematics, which is based on homogeneous transformations. This paper shows how dual quaternions arise in a natural way when approximating 3D homogeneous transformations by 4D rotation matrices. This results in a seamless presentation of rigid-body transformations based on matrices and dual quaternions which permits building intuition about the use of quaternions and their generalizations.

**Index Terms**—Spatial kinematics, quaternions, biquaternions, double quaternions, dual quaternions, Cayley factorization.

## I. INTRODUCTION

IN 1843, Hamilton defined *quaternions* as quadruples of the form  $a + bi + cj + dk$ , where  $i^2 = j^2 = k^2 = ijk = -1$ , when seeking a new kind of number that would extend the idea of complex numbers [1].

Quaternions were developed independently of their needs for any particular application. The main use of quaternions in the nineteenth century consisted in expressing physical theories in the notation of quaternions. In this context, during the end of the nineteenth century, researchers working on electromagnetic theory debated about the choice of quaternion or vector notation in their formulations. This generated a fierce dispute from about 1880 to 1900, reaching its climax in a series of letters in the journal *Nature* [2]. Then, quaternions disappeared from view, and their value discredited, having been replaced by the simpler algebra of matrices and vectors. Later on, in the mid-twentieth century, the development of computing machinery made necessary a re-examination of quaternions from the standpoint of their utility in computer simulations. The need for efficient simulations of aircraft and missile motions was responsible to a large extent for sparking the renewed interest in quaternions [3]. It was rapidly realized that quaternion algebra yields more efficient algorithms than matrix algebra for applications involving rigid-body transformations. Nowadays, quaternions play a fundamental role in the representation of spatial rotations and a chapter devoted to them can be found in nearly every advanced textbook

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on Computer Vision, Robot Kinematics and Dynamics, or Computer Graphics.

Surprisingly, despite their long life, the use of quaternions in engineering is not free from confusions which mainly concern:

- 1) The order of quaternion multiplication. Quaternions are sometimes multiplied in the opposite order than rotation matrices, as in [4]. The origin of this can be found in the way vector coordinates are represented. For example, in [5], a celebrated book on Computer Graphics, point coordinates are represented by row vectors instead of column vectors, as is the common practice in Robotics. Then, transformation matrices post-multiply a point vector to produce a new point vector. When using quaternions, instead of homogeneous transformations, the same composition rules are adopted. The result can be confusing for anyone approaching quaternions for the first time. For more details on this matter, see [6].
- 2) The way quaternions operate on vectors. Quaternions have been used to rotate vectors in three dimensions by essentially sandwiching a vector in three dimensions between a unit quaternion and its conjugate [7, Chap. 17] [8]. Nevertheless, strictly speaking, quaternions cannot operate on vectors. The word vector was introduced by Hamilton to denote the imaginary part of the quaternion which is different from today's meaning [9].
- 3) The nature of the quaternion imaginary units [6], [8]. Hamilton himself contributed to this confusion as he always identified the quaternion units with quadrantal rotations, as he called the rotations by  $\pi/2$  [10, p. 64, art. 71]. Nevertheless, they represent rotations by  $\pi$  [9].

All these confusions are seriously affecting the progress of quaternions in engineering because, as a result, they are used in recipes for manipulating sequences of rotations without a precise understanding of their meaning. The situation just worsens when working with dual quaternions, an extension of ordinary quaternions that permits encapsulating rotations and translations in a unified representation. Thus, it is not strange that many practitioners are still averse to using them despite their undeniable value.

This paper shows how quaternions do naturally emerge from 4D rotation matrices and how dual quaternions are then derived when approximating 3D homogeneous transformations by 4D rotations. As a consequence, all common misunderstandings concerning quaternions are cleared up because the derived expressions may be interpreted both as matrix expressions and as quaternions.

### A. Quaternions and rotations in $\mathbb{R}^3$ and $\mathbb{R}^4$

Soon after Hamilton introduced quaternions, he tried to use them to represent rotations in  $\mathbb{R}^3$  in the same way as complex numbers can be used to represent rotations in  $\mathbb{R}^2$ . Nevertheless, it seems that he was not aware of Rodrigues' work and his use of quaternions as a description of rotations was wrong. He believed that the expression for a rotated vector was linear in the quaternion rather than quadratic. This passage of the history of quaternions is actually a matter of controversy (see [9], [11], [12] for details). It is Cayley whom we must thank for the correct development of quaternions as a representation of rotations and for establishing the connection with the results published by Rodrigues three years before the discovery of quaternions [13]. Cayley is also credited to be the first to discover that quaternions could also be used to represent rotations in  $\mathbb{R}^4$  [14]. Cayley's results can be used to prove that any rotation in  $\mathbb{R}^4$  is a product of rotations in a pair of orthogonal two-dimensional subspaces [15]. This factorization, known as Cayley's factoring of 4D rotations, was also proved using matrix algebra by Van Elfrinkhof in 1897 in a paper [16] rescued from oblivion by Mebius in [17]. Cayley's factorization plays a central role in what follows as it provides a bridge between homogeneous transformations and quaternions that remained unnoticed in the past.

### B. Quaternions and their generalizations

In 1882, Clifford introduced the idea of a biquaternion in three papers: "*Preliminary sketch of biquaternions*", "*Notes on biquaternions*", and "*Further note on biquaternions*" [18] (see [19] for a review and summary of this work). Clifford adopted the word biquaternion, previously used by Hamilton to refer to a quaternion with complex coefficients, to denote a combination of two quaternions algebraically combined via a new symbol,  $\omega$ , defined to have the property  $\omega^2 = 0$ , so that a biquaternion has the form  $q_1 + \omega q_2$ , where  $q_1$  and  $q_2$  are both ordinary quaternions. The use of the term biquaternion is confusing. As observed in [19], even Clifford contributed to this confusion by using the symbol  $\omega$  in several different contexts. For example, in his paper "*Preliminary sketch of biquaternions paper*", it is also used with the multiplication rule  $\omega^2 = 1$ . Nowadays, in the area of robot kinematics, biquaternions of the form  $q_1 + \varepsilon q_2$ , where  $\varepsilon^2 = 0$ , are called *dual quaternions*, while those of the form  $q_1 + e q_2$ , where  $e^2 = 1$ , are called *double quaternions*. This denomination derives from the fact that the symbols  $\varepsilon$  and  $e$  designate the *dual* and the *double* units, respectively [20]. Thus, we have three *imaginary* units which can be equal either to the complex unit  $i$  ( $i^2 = -1$ ), to the dual unit  $\varepsilon$  ( $\varepsilon^2 = 0$ ), or to the double unit  $e$  ( $e^2 = 1$ ). These units define the basis of the so-called hypercomplex numbers [21].

The double quaternion  $q_1 + e q_2$  can be reformulated by introducing the symbols  $\xi = \frac{1+e}{2}$  and  $\eta = \frac{1-e}{2}$  [18], [22]. Then,  $q_1 + e q_2 = \xi(q_1 + q_2) + \eta(q_1 - q_2)$ . Since  $\xi^2 = \xi$ ,  $\eta^2 = \eta$  and  $\xi\eta = 0$ , the terms  $(q_1 + q_2)$  and  $(q_1 - q_2)$  operate independently in the double quaternion product which has been found quite convenient when manipulating kinematic equations expressed in terms of double quaternions [23]. A

third possible representation for double quaternions consists in having two quaternions expressed in different bases of imaginary units whose product is commutative. This also leads to couples of quaternions that operate independently when multiplied. Nowadays, the algebras of ordinary, double, and dual quaternions are grouped under the umbrella of Clifford algebras, also known as geometric algebras (see [24, Chap. 9] or [25] for an introduction).

While double quaternions have been found direct application to represent 4D rotations, dual quaternions found application to encapsulate both translation and rotation into a unified representation. Then, if 3D spatial displacements are approximated by 4D rotations, a beautiful connection between double and dual quaternions can be established.

Yang and Freudenstein introduced the use of dual quaternions for the analysis of spatial mechanisms [26]. Since then, dual quaternions have been used by several authors in the kinematic analysis and synthesis of mechanisms, and in computer graphics (see, for instance, the works of McCarthy [27], Angeles [28], and Perez-Gracia [29]).

### C. Quaternions and matrix algebra

Matrix algebra was developed beginning about 1858 by Cayley and Sylvester. Soon it was realized that matrices could be used to represent the imaginary units used in the definition of quaternions. Actually, a set of  $4 \times 4$  matrices, sometimes called Dirac-Eddington-Conway matrices, with real values can realize every algebraic requirement of quaternions. Alternatively, a set of  $2 \times 2$  matrices, usually called Pauli matrices, with complex values can play the same role (see [30, pp. 143-144] for details). Therefore, there are sets of matrices which all produce valid matrix representations of quaternions. The choice of one set over other has been driven by esthetic preferences, but we will show how Cayley's factorization leads to a matrix representation that attenuates this sense of arbitrariness.

While in most textbooks the matrix representation of quaternions is considered as an advanced topic, if ever mentioned, in this paper matrix algebra is used as the doorway to quaternions. This would probably be the usual practice if matrix algebra had been developed before quaternions.

### D. Organization of the paper

This paper is organized as follows. Section II reviews the connection between 4D rotations and double quaternions in terms of matrix algebra. Section III presents a digression, that can be skipped on a first reading, in which the expressive power of matrix algebra is explored to derive different systems of hypercomplex numbers associated with 4D rotations. In Section IV, the results presented in Section II are specialized to the 3D case. Section V deals with the problem of approximating 3D transformations in homogeneous coordinates by 4D rotations. The results obtained in Section IV and Section V constitute the basic building blocks of the proposed twofold matrix-quaternion formalism for the representation of rigid-body transformations. The reinterpretation of kinematic equations expressed as products of transformations in

homogeneous coordinates using this formalism is treated in Section VI. Section VII presents some examples and, finally, we conclude in Section VIII with a summary of the main points.

## II. 4D ROTATIONS AND DOUBLE QUATERNIONS

After a proper change in the orientation of the reference frame, an arbitrary 4D rotation matrix (*i.e.*, an orthogonal matrix with determinant +1) can be expressed as [31, Theorem 4]:

$$\begin{pmatrix} \cos \alpha_1 & -\sin \alpha_1 & 0 & 0 \\ \sin \alpha_1 & \cos \alpha_1 & 0 & 0 \\ 0 & 0 & \cos \alpha_2 & -\sin \alpha_2 \\ 0 & 0 & \sin \alpha_2 & \cos \alpha_2 \end{pmatrix}. \quad (1)$$

Thus, a 4D rotation is defined by two mutually orthogonal planes of rotation, each of which is fixed in the sense that points in each plane stay within the planes. Then, a 4D rotation has two angles of rotation,  $\alpha_1$  and  $\alpha_2$ , one for each plane of rotation, through which points in the planes rotate. All points not in the planes rotate through an angle between  $\alpha_1$  and  $\alpha_2$ . See [32] for details on the geometric interpretation of rotations in four dimensions.

If  $\alpha_1 = \pm\alpha_2$ , the rotation is called an *isoclinic rotation*. An isoclinic rotation can be *left-* or *right-isoclinic* (depending on whether  $\alpha_1 = \alpha_2$  or  $\alpha_1 = -\alpha_2$ , respectively) which can be represented by a rotation matrix of the form

$$\mathbf{R}^L = \begin{pmatrix} l_0 & -l_3 & l_2 & -l_1 \\ l_3 & l_0 & -l_1 & -l_2 \\ -l_2 & l_1 & l_0 & -l_3 \\ l_1 & l_2 & l_3 & l_0 \end{pmatrix}, \quad (2)$$

and

$$\mathbf{R}^R = \begin{pmatrix} r_0 & -r_3 & r_2 & r_1 \\ r_3 & r_0 & -r_1 & r_2 \\ -r_2 & r_1 & r_0 & r_3 \\ -r_1 & -r_2 & -r_3 & r_0 \end{pmatrix}, \quad (3)$$

respectively. Since (2) and (3) are rotation matrices, their rows and columns are unit vectors. As a consequence,

$$l_1^2 + l_2^2 + l_3^2 + l_4^2 = 1 \quad (4)$$

and

$$r_1^2 + r_2^2 + r_3^2 + r_4^2 = 1. \quad (5)$$

Without loss of generality, we have introduced some changes in the signs and indices of (2) and (3) with respect to the notation used by Cayley [14], [33] to ease the treatment given below and to provide a neat connection with the standard use of quaternions for representing rotations in three dimensions.

Isoclinic rotation matrices have three important properties:

- 1) The product of two right- (left-) isoclinic matrices is a right- (left-) isoclinic matrix.
- 2) The product of a right- and a left-isoclinic matrix is commutative.
- 3) Any 4D rotation matrix, according to Cayley's factorization, can be decomposed into the product of a right- and a left-isoclinic matrix.

Then, a 4D rotation matrix, say  $\mathbf{R}$ , can be expressed as:

$$\mathbf{R} = \mathbf{R}^L \mathbf{R}^R = \mathbf{R}^R \mathbf{R}^L \quad (6)$$

where

$$\mathbf{R}^L = l_0 \mathbf{I} + l_1 \mathbf{A}_1 + l_2 \mathbf{A}_2 + l_3 \mathbf{A}_3 \quad (7)$$

and

$$\mathbf{R}^R = r_0 \mathbf{I} + r_1 \mathbf{B}_1 + r_2 \mathbf{B}_2 + r_3 \mathbf{B}_3, \quad (8)$$

where  $\mathbf{I}$  stands for the  $4 \times 4$  identity matrix and

$$\begin{aligned} \mathbf{A}_1 &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, & \mathbf{A}_2 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\ \mathbf{A}_3 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, & \mathbf{B}_1 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\ \mathbf{B}_2 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, & \mathbf{B}_3 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \end{aligned}$$

Therefore,  $\{\mathbf{I}, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$  and  $\{\mathbf{I}, \mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3\}$  can be seen, respectively, as bases for left- and right-isoclinic rotations.

The details on how to compute Cayley's factorization (6) can be found in the appendix.

Now, it can be verified that

$$\mathbf{A}_1^2 = \mathbf{A}_2^2 = \mathbf{A}_3^2 = \mathbf{A}_1 \mathbf{A}_2 \mathbf{A}_3 = -\mathbf{I}, \quad (9)$$

and

$$\mathbf{B}_1^2 = \mathbf{B}_2^2 = \mathbf{B}_3^2 = \mathbf{B}_1 \mathbf{B}_2 \mathbf{B}_3 = -\mathbf{I}. \quad (10)$$

We can recognize in these two expressions the quaternion definition. Actually, (9) and (10) reproduce the celebrated formula that Hamilton carved into the stone of Brougham Bridge.

Expression (9) determines all the possible products of  $\mathbf{A}_1$ ,  $\mathbf{A}_2$ , and  $\mathbf{A}_3$  resulting in

$$\begin{aligned} \mathbf{A}_1 \mathbf{A}_2 &= \mathbf{A}_3, & \mathbf{A}_2 \mathbf{A}_3 &= \mathbf{A}_1, & \mathbf{A}_3 \mathbf{A}_1 &= \mathbf{A}_2, \\ \mathbf{A}_2 \mathbf{A}_1 &= -\mathbf{A}_3, & \mathbf{A}_3 \mathbf{A}_2 &= -\mathbf{A}_1, & \mathbf{A}_1 \mathbf{A}_3 &= -\mathbf{A}_2. \end{aligned} \quad (11)$$

Likewise, all the possible products of  $\mathbf{B}_1$ ,  $\mathbf{B}_2$ , and  $\mathbf{B}_3$  can be derived from expression (10). All these products can be summarized in the following product tables:

|                |                |                 |                 |                 |  |
|----------------|----------------|-----------------|-----------------|-----------------|--|
|                | $\mathbf{I}$   | $\mathbf{A}_1$  | $\mathbf{A}_2$  | $\mathbf{A}_3$  |  |
| $\mathbf{I}$   | $\mathbf{I}$   | $\mathbf{A}_1$  | $\mathbf{A}_2$  | $\mathbf{A}_3$  |  |
| $\mathbf{A}_1$ | $\mathbf{A}_1$ | $-\mathbf{I}$   | $\mathbf{A}_3$  | $-\mathbf{A}_2$ |  |
| $\mathbf{A}_2$ | $\mathbf{A}_2$ | $-\mathbf{A}_3$ | $-\mathbf{I}$   | $\mathbf{A}_1$  |  |
| $\mathbf{A}_3$ | $\mathbf{A}_3$ | $\mathbf{A}_2$  | $-\mathbf{A}_1$ | $-\mathbf{I}$   |  |

(12)

|                |                |                 |                 |                 |  |
|----------------|----------------|-----------------|-----------------|-----------------|--|
|                | $\mathbf{I}$   | $\mathbf{B}_1$  | $\mathbf{B}_2$  | $\mathbf{B}_3$  |  |
| $\mathbf{I}$   | $\mathbf{I}$   | $\mathbf{B}_1$  | $\mathbf{B}_2$  | $\mathbf{B}_3$  |  |
| $\mathbf{B}_1$ | $\mathbf{B}_1$ | $-\mathbf{I}$   | $\mathbf{B}_3$  | $-\mathbf{B}_2$ |  |
| $\mathbf{B}_2$ | $\mathbf{B}_2$ | $-\mathbf{B}_3$ | $-\mathbf{I}$   | $\mathbf{B}_1$  |  |
| $\mathbf{B}_3$ | $\mathbf{B}_3$ | $\mathbf{B}_2$  | $-\mathbf{B}_1$ | $-\mathbf{I}$   |  |

(13)

Moreover, it can be verified that

$$\mathbf{A}_i \mathbf{B}_j = \mathbf{B}_j \mathbf{A}_i. \quad (14)$$

which is actually a consequence of the commutativity of left- and right-isoclinic rotations. Then, in the composition of two 4D rotations, we have:

$$\mathbf{R}_1 \mathbf{R}_2 = (\mathbf{R}_1^L \mathbf{R}_1^R)(\mathbf{R}_2^L \mathbf{R}_2^R) = (\mathbf{R}_1^L \mathbf{R}_2^L)(\mathbf{R}_1^R \mathbf{R}_2^R). \quad (15)$$

It can be concluded that  $\mathbf{R}_i^L$  and  $\mathbf{R}_i^R$  can be seen either as  $4 \times 4$  rotation matrices or, when expressed as in (7) and (8) respectively, as unit quaternions and their product, as a double quaternion because they operate independently in the product of two 4D rotations. It is said that they are unit quaternions because their coefficients satisfy (4) and (5).

Next, in Section IV, the above twofold matrix-quaternion representation of 4D rotations is specialized to 3D rotations and, in Section V, generalized to represent 3D translations. Nevertheless, let us first explore this twofold representation a bit further.

### III. A DIGRESSION

One of the multiple advantages of the proposed matrix-quaternion formulation is that the involved imaginary units have a clear algebraic interpretation. We can operate with these units to obtain different representations of 4D rotations that would otherwise be quite abstract and difficult to derive. To see this, let us start by defining

$$\begin{aligned} \mathbf{D}_1 &= \mathbf{A}_1 \mathbf{B}_1^{-1} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \\ \mathbf{D}_2 &= -\mathbf{A}_2 \mathbf{B}_2^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \\ \mathbf{D}_3 &= \mathbf{A}_3 \mathbf{B}_3^{-1} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Then, it can be verified that

$$\mathbf{D}_1^2 = \mathbf{D}_2^2 = \mathbf{D}_3^2 = \mathbf{D}_1 \mathbf{D}_2 \mathbf{D}_3 = \mathbf{I} \quad (16)$$

which allow us to define a kind of quaternion whose imaginary units are double units. As with ordinary quaternions, (16) determines all the possible products of  $\mathbf{D}_1$ ,  $\mathbf{D}_2$ , and  $\mathbf{D}_3$  which can be summarized in the following product table:

|                |                |                |                |                |  |
|----------------|----------------|----------------|----------------|----------------|--|
|                | $\mathbf{I}$   | $\mathbf{D}_1$ | $\mathbf{D}_2$ | $\mathbf{D}_3$ |  |
| $\mathbf{I}$   | $\mathbf{I}$   | $\mathbf{D}_1$ | $\mathbf{D}_2$ | $\mathbf{D}_3$ |  |
| $\mathbf{D}_1$ | $\mathbf{D}_1$ | $\mathbf{I}$   | $\mathbf{D}_3$ | $\mathbf{D}_2$ |  |
| $\mathbf{D}_2$ | $\mathbf{D}_2$ | $\mathbf{D}_3$ | $\mathbf{I}$   | $\mathbf{D}_1$ |  |
| $\mathbf{D}_3$ | $\mathbf{D}_3$ | $\mathbf{D}_2$ | $\mathbf{D}_1$ | $\mathbf{I}$   |  |

(17)

Then, clearly  $\mathbf{D}_i = \mathbf{D}_i^{-1}$ ,  $i = 1, 2, 3$ , and, as a consequence,

$$\mathbf{B}_1 = \mathbf{D}_1 \mathbf{A}_1, \quad \mathbf{B}_2 = -\mathbf{D}_2 \mathbf{A}_2, \quad \text{and} \quad \mathbf{B}_3 = \mathbf{D}_3 \mathbf{A}_3.$$

Substituting the above expressions for  $\mathbf{B}_i$ ,  $i = 1, 2, 3$ , in (8), multiplying the result by (7), and factoring out  $\mathbf{D}_i$ ,  $i = 1, 2, 3$ , we conclude that (6) can be rewritten as:

$$\mathbf{R} = \mathbf{I} \mathbf{Q}_0 + \mathbf{D}_1 \mathbf{Q}_1 + \mathbf{D}_2 \mathbf{Q}_2 + \mathbf{D}_3 \mathbf{Q}_3 \quad (18)$$

where

$$\begin{aligned} \mathbf{Q}_0 &= r_0 (l_0 \mathbf{I} + l_1 \mathbf{A}_1 + l_2 \mathbf{A}_2 + l_3 \mathbf{A}_3), \\ \mathbf{Q}_1 &= r_1 (l_0 \mathbf{A}_1 - l_1 \mathbf{I} + l_2 \mathbf{A}_3 - l_3 \mathbf{A}_2), \\ \mathbf{Q}_2 &= r_2 (-l_0 \mathbf{A}_2 + l_1 \mathbf{A}_3 + l_2 \mathbf{I} - l_3 \mathbf{A}_1), \\ \mathbf{Q}_3 &= r_3 (l_0 \mathbf{A}_3 + l_1 \mathbf{A}_2 - l_2 \mathbf{A}_1 - l_3 \mathbf{I}). \end{aligned}$$

Now, we can shift from the basis  $\{\mathbf{I}, \mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3\}$  to the basis  $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \mathbf{E}_4\}$  defined as

$$\mathbf{E}_1 = \frac{1}{4} (\mathbf{I} - \mathbf{D}_1 + \mathbf{D}_2 - \mathbf{D}_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (19)$$

$$\mathbf{E}_2 = \frac{1}{4} (\mathbf{I} + \mathbf{D}_1 - \mathbf{D}_2 - \mathbf{D}_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (20)$$

$$\mathbf{E}_3 = \frac{1}{4} (\mathbf{I} + \mathbf{D}_1 + \mathbf{D}_2 + \mathbf{D}_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (21)$$

$$\mathbf{E}_4 = \frac{1}{4} (\mathbf{I} - \mathbf{D}_1 - \mathbf{D}_2 + \mathbf{D}_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (22)$$

The elements of this basis are distinguished by the fact that their multiplication table is the simplest possible for a basis:

|                |                |                |                |                |
|----------------|----------------|----------------|----------------|----------------|
|                | $\mathbf{E}_1$ | $\mathbf{E}_2$ | $\mathbf{E}_3$ | $\mathbf{E}_4$ |
| $\mathbf{E}_1$ | $\mathbf{E}_1$ | $\mathbf{0}$   | $\mathbf{0}$   | $\mathbf{0}$   |
| $\mathbf{E}_2$ | $\mathbf{0}$   | $\mathbf{E}_2$ | $\mathbf{0}$   | $\mathbf{0}$   |
| $\mathbf{E}_3$ | $\mathbf{0}$   | $\mathbf{0}$   | $\mathbf{E}_3$ | $\mathbf{0}$   |
| $\mathbf{E}_4$ | $\mathbf{0}$   | $\mathbf{0}$   | $\mathbf{0}$   | $\mathbf{E}_4$ |

By inverting the system of equations defined by (19)-(22), we obtain:

$$\begin{aligned} \mathbf{I} &= \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \mathbf{E}_4 \\ \mathbf{D}_1 &= -\mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 - \mathbf{E}_4 \\ \mathbf{D}_2 &= \mathbf{E}_1 - \mathbf{E}_2 + \mathbf{E}_3 - \mathbf{E}_4 \\ \mathbf{D}_3 &= -\mathbf{E}_1 - \mathbf{E}_2 + \mathbf{E}_3 + \mathbf{E}_4 \end{aligned}$$

Substituting these expressions in (18) and factoring out  $\mathbf{E}_i$ , for  $i = 1, \dots, 4$ , we obtain:

$$\mathbf{R} = \mathbf{E}_1 \mathbf{K}_1 + \mathbf{E}_2 \mathbf{K}_2 + \mathbf{E}_3 \mathbf{K}_3 + \mathbf{E}_4 \mathbf{K}_4, \quad (23)$$

where

$$\begin{aligned} \mathbf{K}_1 &= \mathbf{Q}_0 - \mathbf{Q}_1 + \mathbf{Q}_2 - \mathbf{Q}_3 \\ &= (r_0 l_0 + r_1 l_1 + r_2 l_2 + r_3 l_3) \mathbf{I} \\ &\quad + (r_0 l_1 - r_1 l_0 - r_2 l_3 + r_3 l_2) \mathbf{A}_1 \\ &\quad + (r_0 l_2 + r_1 l_3 - r_2 l_0 - r_3 l_1) \mathbf{A}_2 \\ &\quad + (r_0 l_3 - r_1 l_2 + r_2 l_1 - r_3 l_0) \mathbf{A}_3, \end{aligned}$$

$$\begin{aligned}\mathbf{K}_2 &= \mathbf{Q}_0 + \mathbf{Q}_1 - \mathbf{Q}_2 - \mathbf{Q}_3 \\ &= (r_0l_0 - r_1l_1 - r_2l_2 + r_3l_3)\mathbf{I} \\ &\quad + (r_0l_1 + r_1l_0 + r_2l_3 + r_3l_2)\mathbf{A}_1 \\ &\quad + (r_0l_2 - r_1l_3 + r_2l_0 - r_3l_1)\mathbf{A}_2 \\ &\quad + (r_0l_3 + r_1l_2 - r_2l_1 - r_3l_0)\mathbf{A}_3,\end{aligned}$$

$$\begin{aligned}\mathbf{K}_3 &= \mathbf{Q}_0 + \mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_3 \\ &= (r_0l_0 - r_1l_1 + r_2l_2 - r_3l_3)\mathbf{I} \\ &\quad + (r_0l_1 + r_1l_0 - r_2l_3 - r_3l_2)\mathbf{A}_1 \\ &\quad + (r_0l_2 - r_1l_3 - r_2l_0 + r_3l_1)\mathbf{A}_2 \\ &\quad + (r_0l_3 + r_1l_2 + r_2l_1 + r_3l_0)\mathbf{A}_3,\end{aligned}$$

and

$$\begin{aligned}\mathbf{K}_4 &= \mathbf{Q}_0 - \mathbf{Q}_1 - \mathbf{Q}_2 + \mathbf{Q}_3 \\ &= (r_0l_0 + r_1l_1 - r_2l_2 - r_3l_3)\mathbf{I} \\ &\quad + (r_0l_1 - r_1l_0 + r_2l_3 - r_3l_2)\mathbf{A}_1 \\ &\quad + (r_0l_2 + r_1l_3 + r_2l_0 + r_3l_1)\mathbf{A}_2 \\ &\quad + (r_0l_3 - r_1l_2 - r_2l_1 + r_3l_0)\mathbf{A}_3.\end{aligned}$$

It is thus concluded that a 4D rotation can be expressed as a linear combination of four quaternions. In the product of two 4D rotations, these four quaternions operate independently because  $\mathbf{E}_i^2 = \mathbf{E}_i$ , for  $i = 1, 2, 3, 4$ , and  $\mathbf{E}_i\mathbf{E}_j = \mathbf{0}$  if  $i \neq j$ . The reader interested in further exploring the connections between 4D rotations and different sets of imaginary units is referred to [33] and [34].

#### IV. 3D ROTATIONS AND ORDINARY QUATERNIONS

We have seen how double quaternions naturally emerge from Cayley's factorization of 4D rotations into isoclinic rotations. Now, we specialize this result to 3D rotations.

The homogenous matrix transformation representing a rotation by  $\phi$  about the  $x$  axis is:

$$\mathbf{R}_x(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) & 0 \\ 0 & \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (24)$$

which can be readily interpreted as a rotation in  $\mathbb{R}^4$ . Then, its Cayley's factorization into a right- and a left-isoclinic rotation, using the procedure given in the appendix, yields

$$\mathbf{R}_x^L(\phi) = \frac{1}{\sqrt{t^2+1}} \begin{pmatrix} 1 & 0 & 0 & -t \\ 0 & 1 & -t & 0 \\ 0 & t & 1 & 0 \\ t & 0 & 0 & 1 \end{pmatrix} \quad (25)$$

and

$$\mathbf{R}_x^R(\phi) = \frac{1}{\sqrt{t^2+1}} \begin{pmatrix} 1 & 0 & 0 & t \\ 0 & 1 & -t & 0 \\ 0 & t & 1 & 0 \\ -t & 0 & 0 & 1 \end{pmatrix}, \quad (26)$$

respectively, where we have introduced the change of variable  $t = \tan\left(\frac{\phi}{2}\right)$  to obtain more amenable expressions. Then,

$$\mathbf{R}_x(\phi) = \left( \frac{1}{\sqrt{t^2+1}}(\mathbf{I} + t\mathbf{A}_1) \right) \left( \frac{1}{\sqrt{t^2+1}}(\mathbf{I} + t\mathbf{B}_1) \right). \quad (27)$$

We can perform the same factorization for rotations about the  $y$  and the  $z$  axes. Table I compiles the results.

Now, let us suppose that we want to represent a general rotation using the XYZ Cardanian angles [35, p. 28]. Then,

$$\begin{aligned}\mathbf{R}_x(\phi_1)\mathbf{R}_y(\phi_2)\mathbf{R}_z(\phi_3) &= \\ &= \left( \frac{1}{\sqrt{t_1^2+1}}(\mathbf{I} + t_1\mathbf{A}_1) \right) \left( \frac{1}{\sqrt{t_1^2+1}}(\mathbf{I} + t_1\mathbf{B}_1) \right) \\ &= \left( \frac{1}{\sqrt{t_2^2+1}}(\mathbf{I} + t_2\mathbf{A}_2) \right) \left( \frac{1}{\sqrt{t_2^2+1}}(\mathbf{I} + t_2\mathbf{B}_2) \right) \\ &= \left( \frac{1}{\sqrt{t_3^2+1}}(\mathbf{I} + t_3\mathbf{A}_3) \right) \left( \frac{1}{\sqrt{t_3^2+1}}(\mathbf{I} + t_3\mathbf{B}_3) \right).\end{aligned}$$

Therefore, using the commutativity of left- and right-isoclinic rotations, and the product tables (12) and (13), we conclude that:

$$\mathbf{R}_x(\phi_1)\mathbf{R}_y(\phi_2)\mathbf{R}_z(\phi_3) = \mathbf{Q}_1\mathbf{Q}_2 = \mathbf{Q}_2\mathbf{Q}_1, \quad (28)$$

where

$$\begin{aligned}\mathbf{Q}_1 &= \frac{1}{\sqrt{(t_1^2+1)(t_2^2+1)(t_3^2+1)}} [(1 - t_1t_2t_3)\mathbf{I} \\ &\quad + (t_1 + t_2t_3)\mathbf{A}_1 + (t_2 - t_1t_3)\mathbf{A}_2 + (t_3 + t_1t_2)\mathbf{A}_3]\end{aligned}$$

and

$$\begin{aligned}\mathbf{Q}_2 &= \frac{1}{\sqrt{(t_1^2+1)(t_2^2+1)(t_3^2+1)}} [(1 - t_1t_2t_3)\mathbf{I} \\ &\quad + (t_1 + t_2t_3)\mathbf{B}_1 + (t_2 - t_1t_3)\mathbf{B}_2 + (t_3 + t_1t_2)\mathbf{B}_3].\end{aligned}$$

Similar results are obtained for other sets of Eulerian or Cardanian angles. In any case, after constraining rotations to three dimensions, the double quaternion representation becomes redundant as one quaternion can be deduced from the other by simply exchanging  $\mathbf{A}_i$  and  $\mathbf{B}_i$ . Thus, a 3D rotation can be represented either by a quaternion of the form  $(p_0\mathbf{I} + p_1\mathbf{A}_1 + p_2\mathbf{A}_2 + p_3\mathbf{A}_3)$  or  $(p_0\mathbf{I} + p_1\mathbf{B}_1 + p_2\mathbf{B}_2 + p_3\mathbf{B}_3)$  and the corresponding homogeneous matrix can be obtained by computing their commutative product. Indeed, it can be verified that

$$\begin{aligned}(p_0\mathbf{I} + p_1\mathbf{A}_1 + p_2\mathbf{A}_2 + p_3\mathbf{A}_3)(p_0\mathbf{I} + p_1\mathbf{B}_1 + p_2\mathbf{B}_2 + p_3\mathbf{B}_3) &= \\ &= \begin{pmatrix} p_0 & -p_3 & p_2 & -p_1 \\ p_3 & p_0 & -p_1 & -p_2 \\ -p_2 & p_1 & p_0 & -p_3 \\ p_1 & p_2 & p_3 & p_0 \end{pmatrix} \begin{pmatrix} p_0 & -p_3 & p_2 & p_1 \\ p_3 & p_0 & -p_1 & p_2 \\ -p_2 & p_1 & p_0 & p_3 \\ -p_1 & -p_2 & -p_3 & p_0 \end{pmatrix} \\ &= \begin{pmatrix} 1 - 2p_2^2 - 2p_3^2 & 2p_1p_2 - 2p_0p_3 & 2p_0p_2 + 2p_1p_3 & 0 \\ 2p_0p_3 + 2p_1p_2 & 1 - 2p_1^2 - 2p_3^2 & 2p_2p_3 - 2p_0p_1 & 0 \\ 2p_1p_3 - 2p_0p_2 & 2p_0p_1 + 2p_2p_3 & 1 - p_1^2 - p_2^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (29)\end{aligned}$$

Observe that this is the well-known formula that permits passing from a quaternion representation to the corresponding rotation matrix [7, p. 85].

Now, if we substitute in (29) the following values

$$\begin{aligned}p_0 &= \cos\left(\frac{\theta}{2}\right), & p_1 &= n_x \sin\left(\frac{\theta}{2}\right), \\ p_2 &= n_y \sin\left(\frac{\theta}{2}\right), & p_3 &= n_z \sin\left(\frac{\theta}{2}\right),\end{aligned}$$

TABLE I  
3D ROTATIONS IN HOMOGENEOUS COORDINATES INTERPRETED AS 4D ROTATIONS AND THEIR FACTORIZATIONS INTO LEFT- AND RIGHT- ISOCLINIC ROTATIONS ( $t = \tan\left(\frac{\phi}{2}\right)$ ).

| Transformation  | Left-Isoclinic | Right-Isoclinic | Double quaternion |
|---|----------------|-----------------|-------------------|
| $\mathbf{R}_x(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1-t^2}{t^2+1} & \frac{-2t}{t^2+1} & 0 \\ 0 & \frac{2t}{t^2+1} & \frac{1-t^2}{t^2+1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{t^2+1}} \begin{pmatrix} 1 & 0 & 0 & -t \\ 0 & 1 & -t & 0 \\ 0 & t & 1 & 0 \\ t & 0 & 0 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{t^2+1}} \begin{pmatrix} 1 & 0 & 0 & t \\ 0 & 1 & -t & 0 \\ 0 & t & 1 & 0 \\ -t & 0 & 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{t^2+1}} (\mathbf{I} + t\mathbf{A}_1) \cdot \frac{1}{\sqrt{t^2+1}} (\mathbf{I} + t\mathbf{B}_1)$  |                |                 |                   |
| $\mathbf{R}_y(\phi) = \begin{pmatrix} \frac{1-t^2}{t^2+1} & 0 & \frac{-2t}{t^2+1} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-2t}{t^2+1} & 0 & \frac{1-t^2}{t^2+1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{t^2+1}} \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & -t \\ -t & 0 & 1 & 0 \\ 0 & t & 0 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{t^2+1}} \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ -t & 0 & 1 & 0 \\ 0 & -t & 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{t^2+1}} (\mathbf{I} + t\mathbf{A}_2) \cdot \frac{1}{\sqrt{t^2+1}} (\mathbf{I} + t\mathbf{B}_2)$ |                |                 |                   |
| $\mathbf{R}_z(\phi) = \begin{pmatrix} \frac{1-t^2}{t^2+1} & \frac{-2t}{t^2+1} & 0 & 0 \\ \frac{2t}{t^2+1} & \frac{1-t^2}{t^2+1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{t^2+1}} \begin{pmatrix} 1 & -t & 0 & 0 \\ t & 1 & 0 & 0 \\ 0 & 0 & 1 & -t \\ 0 & 0 & t & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{t^2+1}} \begin{pmatrix} 1 & -t & 0 & 0 \\ t & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & -t & 1 \end{pmatrix} = \frac{1}{\sqrt{t^2+1}} (\mathbf{I} + t\mathbf{A}_3) \cdot \frac{1}{\sqrt{t^2+1}} (\mathbf{I} + t\mathbf{B}_3)$  |                |                 |                   |

with  $n_x^2 + n_y^2 + n_z^2 = 1$ , the result can be recognized as the rotation through an angle  $\theta$  about an axis that passes through the origin and has direction given by the unit vector  $\mathbf{n} = (n_x, n_y, n_z)$  (see, for example, [36, p. 47]). In sum,

$$\mathbf{R}_n(\theta) = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \mathbf{I} + \sin\left(\frac{\theta}{2}\right) (n_x \mathbf{A}_1 + n_y \mathbf{A}_2 + n_z \mathbf{A}_3) \\ \cos\left(\frac{\theta}{2}\right) \mathbf{I} + \sin\left(\frac{\theta}{2}\right) (n_x \mathbf{B}_1 + n_y \mathbf{B}_2 + n_z \mathbf{B}_3) \end{bmatrix}. \quad (30)$$

This alternative to Rodrigues' formula provides a seamless connection between homogeneous transformations and quaternions that can easily be grasped by anyone approaching quaternions for the first time. This formula also demonstrates two important concepts. First, a 3D rotation can be seen as the composition of two 4D isoclinic rotations. Second, contrary to popular belief, a unit quaternion actually represents a 4D rotation (see [37] for an alternative proof of these two facts using heavier mathematical machinery).

Equation (30) is also interesting because it can be concluded from it that Cayley's factorization gives the axis-angle representation of a 3D rotation.

Notice that we can operate with  $\mathbf{A}_i$  and  $\mathbf{B}_i$ ,  $i = 1, \dots, 3$ , as symbols whose products commute and satisfy the product tables (12) and (13). We do not need to substitute for them by their corresponding matrices unless we explicitly need the matrix representation. Keeping them as symbols has benefits in three important practical applications:

- 1) to generate more compact expressions involving rotations and, in general, to increase speed and reduce storage for calculations involving long sequences of rotations,
- 2) to avoid distortions arising from numerical inaccuracies caused by floating point computations with rotations, and

- 3) to interpolate between two rotations for generating trajectories. The interpolation of two quaternions still represents a valid rotation contrarily to what happens when interpolating two rotation matrices.

Actually, these are the well-known advantages of using quaternions. For their detailed analysis see [30, Chap. IX].

## V. 3D TRANSLATIONS AND DUAL QUATERNIONS

Let us suppose that we want to obtain the nearest rotation matrix, under the Frobenius norm, to a given homogeneous transformation matrix  $\mathbf{M}$ . To find this rotation matrix, say  $\mathbf{R}$ , we can use the singular value decomposition  $\mathbf{M} = \mathbf{U} \Sigma \mathbf{V}^T$  to write  $\mathbf{R} = \mathbf{U} \mathbf{V}^T$  [38], [39].

If we compute the singular value decomposition of the homogeneous transformation matrix representing an infinitesimal translation along the x-axis, we obtain

$$\begin{pmatrix} 1 & 0 & 0 & d/\delta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big|_{\delta \rightarrow \infty} \simeq \begin{pmatrix} \frac{+\sqrt{2+d/\delta}}{2} & 0 & 0 & \frac{-\sqrt{2-d/\delta}}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{+\sqrt{2-d/\delta}}{2} & 0 & 0 & \frac{+\sqrt{2+d/\delta}}{2} \end{pmatrix} \\ \begin{pmatrix} 1 - \frac{d}{2\delta} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 + \frac{d}{2\delta} \end{pmatrix} \begin{pmatrix} \frac{+\sqrt{2-d/\delta}}{2} & 0 & 0 & \frac{+\sqrt{2+d/\delta}}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-\sqrt{2+d/\delta}}{2} & 0 & 0 & \frac{+\sqrt{2-d/\delta}}{2} \end{pmatrix}.$$

Then, the nearest (in Frobenius norm) 4D rotation matrix approximating an infinitesimal translation along the x-axis in homogeneous coordinates is given by:

$$\begin{pmatrix} 1 & 0 & 0 & d/\delta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big|_{\delta \rightarrow \infty} \simeq \begin{pmatrix} 1 & 0 & 0 & d/2\delta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -d/2\delta & 0 & 0 & 1 \end{pmatrix},$$

which can be factored into the product of a right- and left-isoclinic rotation. The same can be performed for infinitesimal

TABLE II

INFINITESIMAL 3D TRANSLATIONS IN HOMOGENEOUS COORDINATES APPROXIMATED BY 4D ROTATIONS AND THEIR FACTORIZATIONS INTO LEFT- AND RIGHT-ISOCLINIC ROTATIONS

| Transformation   | Left-Isoclinic | Right-Isoclinic | Double quaternion |
|--|----------------|-----------------|-------------------|
| $\mathbf{T}_x\left(\frac{d}{\delta}\right)\Big _{\delta\rightarrow\infty} = \begin{pmatrix} 1 & 0 & 0 & \frac{d}{\delta} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\Big _{\delta\rightarrow\infty} \simeq \begin{pmatrix} 1 & 0 & 0 & \frac{d}{2\delta} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{d}{2\delta} & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \frac{d}{4\delta} \\ 0 & 1 & \frac{d}{4\delta} & 0 \\ 0 & -\frac{d}{4\delta} & 1 & 0 \\ -\frac{d}{4\delta} & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & \frac{d}{4\delta} \\ 0 & 1 & -\frac{d}{4\delta} & 0 \\ 0 & \frac{d}{4\delta} & 1 & 0 \\ -\frac{d}{4\delta} & 0 & 0 & 1 \end{pmatrix} = \left(\mathbf{I} - \frac{d}{4\delta}\mathbf{A}_1\right) \cdot \left(\mathbf{I} + \frac{d}{4\delta}\mathbf{B}_1\right)$ |                |                 |                   |
| $\mathbf{T}_y\left(\frac{d}{\delta}\right)\Big _{\delta\rightarrow\infty} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{d}{\delta} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\Big _{\delta\rightarrow\infty} \simeq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{d}{2\delta} \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{d}{2\delta} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{d}{4\delta} & 0 \\ 0 & 1 & 0 & \frac{d}{4\delta} \\ \frac{d}{4\delta} & 0 & 1 & 0 \\ 0 & -\frac{d}{4\delta} & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \frac{d}{4\delta} & 0 \\ 0 & 1 & 0 & \frac{d}{4\delta} \\ -\frac{d}{4\delta} & 0 & 1 & 0 \\ 0 & -\frac{d}{4\delta} & 0 & 1 \end{pmatrix} = \left(\mathbf{I} - \frac{d}{4\delta}\mathbf{A}_2\right) \cdot \left(\mathbf{I} + \frac{d}{4\delta}\mathbf{B}_2\right)$ |                |                 |                   |
| $\mathbf{T}_z\left(\frac{d}{\delta}\right)\Big _{\delta\rightarrow\infty} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{d}{\delta} \\ 0 & 0 & 0 & 1 \end{pmatrix}\Big _{d\rightarrow 0} \simeq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{d}{2\delta} \\ 0 & 0 & -\frac{d}{2\delta} & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{d}{4\delta} & 0 & 0 \\ -\frac{d}{4\delta} & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{d}{4\delta} \\ 0 & 0 & -\frac{d}{4\delta} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -\frac{d}{4\delta} & 0 & 0 \\ \frac{d}{4\delta} & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{d}{4\delta} \\ 0 & 0 & -\frac{d}{4\delta} & 1 \end{pmatrix} = \left(\mathbf{I} - \frac{d}{4\delta}\mathbf{A}_3\right) \cdot \left(\mathbf{I} + \frac{d}{4\delta}\mathbf{B}_3\right)$          |                |                 |                   |

translations along the other coordinate axes. Table II compiles the results.

Now, let us suppose that we perform an infinitesimal translation in the direction given by the vector  $\mathbf{t} = (t_x, t_y, t_z)^T$ . Then,

$$\begin{aligned} \mathbf{T}_x\left(\frac{t_x}{\delta}\right) \mathbf{T}_y\left(\frac{t_y}{\delta}\right) \mathbf{T}_z\left(\frac{t_z}{\delta}\right)\Big|_{\delta\rightarrow\infty} \\ \simeq \left(\mathbf{I} - \frac{t_x}{4\delta}\mathbf{A}_1\right) \left(\mathbf{I} + \frac{t_x}{4\delta}\mathbf{B}_1\right) \left(\mathbf{I} - \frac{t_y}{4\delta}\mathbf{A}_2\right) \\ \left(\mathbf{I} + \frac{t_y}{4\delta}\mathbf{B}_2\right) \left(\mathbf{I} - \frac{t_z}{4\delta}\mathbf{A}_3\right) \left(\mathbf{I} + \frac{t_z}{4\delta}\mathbf{B}_3\right). \end{aligned}$$

Using the product tables (12) and (13) and neglecting higher-order infinitesimals, we conclude that

$$\begin{aligned} \mathbf{T}_x\left(\frac{t_x}{\delta}\right) \mathbf{T}_y\left(\frac{t_y}{\delta}\right) \mathbf{T}_z\left(\frac{t_z}{\delta}\right)\Big|_{\delta\rightarrow\infty} \\ \simeq \begin{bmatrix} \mathbf{I} - \frac{1}{4\delta}(t_x\mathbf{A}_1 + t_y\mathbf{A}_2 + t_z\mathbf{A}_3) \\ \mathbf{I} + \frac{1}{4\delta}(t_x\mathbf{B}_1 + t_y\mathbf{B}_2 + t_z\mathbf{B}_3) \end{bmatrix} \\ = \begin{pmatrix} 1 & 0 & 0 & t_x/2\delta \\ 0 & 1 & 0 & t_y/2\delta \\ 0 & 0 & 1 & t_z/2\delta \\ -t_x/2\delta & -t_y/2\delta & -t_z/2\delta & 1 \end{pmatrix}. \end{aligned}$$

An alternative way to perform the above operation in a more elegant way is by introducing the symbol  $\varepsilon$ ,  $\varepsilon^2 = 0$  (*i.e.*, the dual unit). It can actually be shown that

$$\begin{aligned} \mathbf{T}_x(\varepsilon t_x) \mathbf{T}_y(\varepsilon t_y) \mathbf{T}_z(\varepsilon t_z) \\ = \begin{bmatrix} \mathbf{I} - \varepsilon \frac{1}{4}(t_x\mathbf{A}_1 + t_y\mathbf{A}_2 + t_z\mathbf{A}_3) \\ \mathbf{I} + \varepsilon \frac{1}{4}(t_x\mathbf{B}_1 + t_y\mathbf{B}_2 + t_z\mathbf{B}_3) \end{bmatrix} \\ = \begin{pmatrix} 1 & 0 & 0 & \varepsilon t_x/2 \\ 0 & 1 & 0 & \varepsilon t_y/2 \\ 0 & 0 & 1 & \varepsilon t_z/2 \\ -\varepsilon t_x/2 & -\varepsilon t_y/2 & -\varepsilon t_z/2 & 1 \end{pmatrix}. \end{aligned} \quad (31)$$

In other words,  $\varepsilon d$  permits working as if  $d$  were infinitely small without explicitly having to compute limits. Therefore, based on (31), we can establish the following one-to-one correspondence between general 3D translations expressed as homogeneous transformation matrices and 4D rotation matrices

$$\begin{pmatrix} \mathbf{I}_{3\times 3} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix} \iff \begin{pmatrix} \mathbf{I}_{3\times 3} & \varepsilon \mathbf{t} \\ -\varepsilon \mathbf{t}^T & 1 \end{pmatrix} \quad (32)$$

Observe that we have dropped the  $1/2$  term as it is just a constant scaling factor.

Since any arbitrary rigid motion can be expressed as a translation followed by a rotation, the above correspondence induces the following correspondence between general 3D rigid motions expressed as homogeneous transformation matrices and 4D rotation matrices:

$$\begin{aligned} \begin{pmatrix} \mathbf{I}_{3\times 3} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R}_{3\times 3} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix} & \iff \begin{pmatrix} \mathbf{I}_{3\times 3} & \varepsilon \mathbf{t} \\ -\varepsilon \mathbf{t}^T & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R}_{3\times 3} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix} \\ & \parallel \\ & \begin{pmatrix} \mathbf{R}_{3\times 3} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix} \xrightarrow[\mathcal{F}^{-1}]{\mathcal{F}} \begin{pmatrix} \mathbf{R}_{3\times 3} & \varepsilon \mathbf{t} \\ -\varepsilon \mathbf{t}^T \mathbf{R}_{3\times 3} & 1 \end{pmatrix} \end{aligned} \quad (33)$$

From now on, we will denote  $\widetilde{\mathbf{M}} = \mathcal{F}(\mathbf{M})$ , where  $\mathbf{M}$  is an arbitrary transformation matrix. Observe that  $\mathcal{F}(\mathbf{M}_1\mathbf{M}_2) = \mathcal{F}(\mathbf{M}_1)\mathcal{F}(\mathbf{M}_2)$ . From this property, one can deduce that  $\mathcal{F}$  maps the identity element into the identity element, and it also maps inverses to inverses in the sense that  $\mathcal{F}(\mathbf{M}^{-1}) = (\mathcal{F}(\mathbf{M}))^{-1}$ . Technically speaking,  $\mathcal{F}$  is a group isomorphism [40, Section 2.2].

At this point, it is important to realize that the 4D rotation matrix corresponding to a 3D homogeneous transformation matrix through  $\mathcal{F}$  is not an approximation, it is an exact representation, and that the use of dual numbers to represent general 3D translations in terms of 4D rotation matrices is completely different from the standard approach based on 3D rotation matrices [41], [42], [43]. The use given here, while allowing a clear-cut distinction between translations and

rotations, provides at the same time a neat connection with standard homogeneous transformations.

Now, the 4D rotation matrix corresponding to the translation in the direction given by the unit vector  $\mathbf{n} = (n_x, n_y, n_z)^T$  a distance  $d$ , after factoring it using Cayley's factorization, can be expressed as:

$$\tilde{\mathbf{T}}_{\mathbf{n}}(d) = \begin{bmatrix} \mathbf{I} - \varepsilon \frac{d}{2} (n_x \mathbf{A}_1 + n_y \mathbf{A}_2 + n_z \mathbf{A}_3) \\ \mathbf{I} + \varepsilon \frac{d}{2} (n_x \mathbf{B}_1 + n_y \mathbf{B}_2 + n_z \mathbf{B}_3) \end{bmatrix}. \quad (34)$$

As with 3D rotations, the double quaternion representation of 3D translations is also redundant because one quaternion can be deduced from the other by exchanging  $\mathbf{A}_i$  and  $\mathbf{B}_i$  and changing the sign of the dual part.

## VI. KINEMATIC EQUATIONS

Consider the kinematic equation:

$$\mathbf{M}_0 = \mathbf{M}_1 \mathbf{M}_2 \cdots \mathbf{M}_n, \quad (35)$$

where  $\mathbf{M}_i$ ,  $i = 0, \dots, n$ , is an arbitrary transformation in homogeneous coordinates. This kinematic equation can be translated, through the mapping in (33), into a kinematic equation fully expressed in terms of 4D rotation matrices. That is,

$$\tilde{\mathbf{M}}_0 = \tilde{\mathbf{M}}_1 \tilde{\mathbf{M}}_2 \cdots \tilde{\mathbf{M}}_n. \quad (36)$$

Then, using Cayley's factorization, we obtain

$$\tilde{\mathbf{M}}_0^L \tilde{\mathbf{M}}_0^R = \tilde{\mathbf{M}}_1^L \tilde{\mathbf{M}}_1^R \tilde{\mathbf{M}}_2^L \tilde{\mathbf{M}}_2^R \cdots \tilde{\mathbf{M}}_n^L \tilde{\mathbf{M}}_n^R. \quad (37)$$

Therefore, using the properties of left- and right-isoclinic rotations, we conclude that

$$\tilde{\mathbf{M}}_0^L = \tilde{\mathbf{M}}_1^L \tilde{\mathbf{M}}_2^L \cdots \tilde{\mathbf{M}}_n^L, \quad (38)$$

$$\tilde{\mathbf{M}}_0^R = \tilde{\mathbf{M}}_1^R \tilde{\mathbf{M}}_2^R \cdots \tilde{\mathbf{M}}_n^R. \quad (39)$$

Each rotation matrix in (38), or (39), can readily be interpreted as a quaternion if expressed in the basis  $\{\mathbf{I}, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$ , or  $\{\mathbf{I}, \mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3\}$ . Moreover, as we have already seen, any of these two equations can be obtained from the other by exchanging  $\mathbf{A}_i$  and  $\mathbf{B}_i$  and changing the sign of the dual symbol.

Although translations and rotations provide the basic building blocks of any kinematic equation, in many applications it is interesting to have a more compact expression for motions combining a rotation about an axis and a translation in the direction given by the same axis (*i.e.*, screw motions). Indeed, if we define

$$\mathbf{S} = \mathbf{R}_{\mathbf{n}}(\theta) \mathbf{T}_{\mathbf{n}}(d), \quad (40)$$

then

$$\begin{aligned} \tilde{\mathbf{S}}^R = & \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) - \varepsilon \frac{d}{2} \sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) + \varepsilon \frac{d}{2} \cos\left(\frac{\theta}{2}\right) \end{bmatrix} \mathbf{I} \\ & + \begin{bmatrix} \sin\left(\frac{\theta}{2}\right) + \varepsilon \frac{d}{2} \cos\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) - \varepsilon \frac{d}{2} \sin\left(\frac{\theta}{2}\right) \end{bmatrix} (n_x \mathbf{B}_1 + n_y \mathbf{B}_2 + n_z \mathbf{B}_3). \end{aligned} \quad (41)$$

Since all powers greater or equal to two of  $\varepsilon$  vanish, the Taylor expansion of  $f(a + \varepsilon b)$  about  $a$  yields  $f(a) + \varepsilon b f'(a)$ .

As a consequence,  $\sin(a + \varepsilon b) = \sin a + \varepsilon b \cos a$  and  $\cos(a + \varepsilon b) = \cos a - \varepsilon b \sin a$  [43, p. 3]. Therefore, (41) can be more compactly expressed as:

$$\tilde{\mathbf{S}}^R = \cos\left(\frac{\hat{\theta}}{2}\right) \mathbf{I} + \sin\left(\frac{\hat{\theta}}{2}\right) (n_x \mathbf{B}_1 + n_y \mathbf{B}_2 + n_z \mathbf{B}_3), \quad (42)$$

where  $\hat{\theta} = \theta + \varepsilon d$ . The dual quaternion (42) is undoubtedly a much more compact representation of a screw motion than the expansion of (40). Thus, it is not surprising that dual quaternions defining successive screw displacements are introduced to simplify the structure of the design equations in the synthesis of mechanisms [29].

The screw displacement given by (42) is not general as the rotation axis passes through the origin. The general form is derived as an example in the next section.

## VII. EXAMPLES

The following examples illustrate different aspects on how to operate with the presented twofold matrix-quaternion formalism.

### A. Derivation of the sandwich formula

It is straightforward to prove that, if  $\mathbf{q} = \mathbf{C} \mathbf{p}$ , where  $\mathbf{C}$  is an arbitrary rotation matrix in homogeneous coordinates and  $\mathbf{p}$  a unit vector, then

$$\mathbf{R}_{\mathbf{q}}(\theta) = \mathbf{C} \mathbf{R}_{\mathbf{p}}(\theta) \mathbf{C}^T. \quad (43)$$

If we substitute (30) in (43), set  $\theta = \pi$ , and separate left- and right-isoclinic rotations, we obtain

$$(q_x \mathbf{A}_1 + q_y \mathbf{A}_2 + q_z \mathbf{A}_3) = \mathbf{C}^L (p_x \mathbf{A}_1 + p_y \mathbf{A}_2 + p_z \mathbf{A}_3) (\mathbf{C}^L)^T \quad (44)$$

and

$$(q_x \mathbf{B}_1 + q_y \mathbf{B}_2 + q_z \mathbf{B}_3) = \mathbf{C}^R (p_x \mathbf{B}_1 + p_y \mathbf{B}_2 + p_z \mathbf{B}_3) (\mathbf{C}^R)^T \quad (45)$$

respectively.

Observe that either (44) or (45), after expressing  $\mathbf{C}^R$  and  $\mathbf{C}^L$  in quaternion form according to (2) and (3), respectively, is the well-known quaternion sandwich formula used to rotate vectors. It is interesting to observe the number of pages devoted to the derivation of this formula in current textbooks [4, pp. 127-134], [7], and the existence of recent papers essentially devoted to its justification [8], while its derivation using the proposed twofold matrix-quaternion formulation seems much easier, at least for those used to work with homogenous transformations.

### B. Approximating dual quaternions by double quaternions

Let us suppose that we need to obtain the quaternion representation of the following transformation in homogeneous coordinates

$$\begin{aligned} \mathbf{M} = & \mathbf{T}_{\mathbf{x}}(4) \mathbf{T}_{\mathbf{y}}(-3) \mathbf{T}_{\mathbf{z}}(7) \mathbf{R}_{\mathbf{y}}\left(\frac{\pi}{2}\right) \mathbf{R}_{\mathbf{z}}\left(\frac{\pi}{2}\right) \\ = & \begin{pmatrix} 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \quad (46)$$



The 4D rotation matrix resulting from the mapping in (33) is:

$$\widetilde{\mathbf{M}} = \begin{pmatrix} 0 & 0 & 1 & 4\varepsilon \\ 1 & 0 & 0 & -3\varepsilon \\ 0 & 1 & 0 & 7\varepsilon \\ 3\varepsilon & -7\varepsilon & -4\varepsilon & 1 \end{pmatrix}. \quad (47)$$

Cayley's factoring of this matrix into a left- and right-isoclinic rotation matrix, using the procedure given in the appendix, yields:

$$\begin{aligned} \widetilde{\mathbf{M}}^L &= \begin{pmatrix} 0.5 + 2\varepsilon & -0.5 + 3.5\varepsilon & 0.5 & -0.5 - 1.5\varepsilon \\ 0.5 - 3.5\varepsilon & 0.5 + 2\varepsilon & -0.5 - 1.5\varepsilon & -0.5 \\ -0.5 & 0.5 + 1.5\varepsilon & 0.5 + 2\varepsilon & -0.5 + 3.5\varepsilon \\ 0.5 + 1.5\varepsilon & 0.5 & 0.5 - 3.5\varepsilon & 0.5 + 2\varepsilon \end{pmatrix} \\ &= (0.5 + 2\varepsilon)\mathbf{I} + (0.5 + 1.5\varepsilon)\mathbf{A}_1 + 0.5\mathbf{A}_2 + (0.5 - 3.5\varepsilon)\mathbf{A}_3 \end{aligned} \quad (48)$$

and

$$\begin{aligned} \widetilde{\mathbf{M}}^R &= \begin{pmatrix} 0.5 - 2\varepsilon & -0.5 - 3.5\varepsilon & 0.5 & 0.5 - 1.5\varepsilon \\ 0.5 + 3.5\varepsilon & 0.5 - 2\varepsilon & -0.5 + 1.5\varepsilon & 0.5 \\ -0.5 & 0.5 - 1.5\varepsilon & 0.5 - 2\varepsilon & 0.5 + 3.5\varepsilon \\ -0.5 + 1.5\varepsilon & -0.5 & -0.5 - 3.5\varepsilon & 0.5 - 2\varepsilon \end{pmatrix} \\ &= (0.5 - 2\varepsilon)\mathbf{I} + (0.5 - 1.5\varepsilon)\mathbf{B}_1 + 0.5\mathbf{B}_2 + (0.5 + 3.5\varepsilon)\mathbf{B}_3, \end{aligned} \quad (49)$$

respectively. Hence,

$$\begin{aligned} \widetilde{\mathbf{M}} &= \widetilde{\mathbf{M}}^L \widetilde{\mathbf{M}}^R = [(0.5 + 2\varepsilon)\mathbf{I} + (0.5 + 1.5\varepsilon)\mathbf{A}_1 \\ &\quad + 0.5\mathbf{A}_2 + (0.5 - 3.5\varepsilon)\mathbf{A}_3] \\ &\quad [(0.5 - 2\varepsilon)\mathbf{I} + (0.5 - 1.5\varepsilon)\mathbf{B}_1 \\ &\quad + 0.5\mathbf{B}_2 + (0.5 + 3.5\varepsilon)\mathbf{B}_3]. \end{aligned}$$

This double quaternion representation is redundant as one quaternion can be deduced from the other by exchanging  $\mathbf{A}_i$  and  $\mathbf{B}_i$  and changing the sign of the dual part. Thus, either the dual quaternion (48) or (49) unambiguously represents the transformation (46).

The rotation matrix  $\widetilde{\mathbf{M}}$  is an exact representation of  $\mathbf{M}$ , it is not an approximation. Alternatively, if we are not interested in using dual numbers, we can approximate  $\mathbf{M}$  by

$$\widetilde{\widetilde{\mathbf{M}}} = \begin{pmatrix} 0 & 0 & 1 & 4/\delta \\ 1 & 0 & 0 & -3/\delta \\ 0 & 1 & 0 & 7/\delta \\ 3/\delta & -7/\delta & -4/\delta & 1 \end{pmatrix}. \quad (50)$$

where  $\delta$  is a scaling factor.  $\widetilde{\widetilde{\mathbf{M}}}$  is closer to a 4D rotation matrix as  $\delta$  tends to infinity (it can be verified that  $\det(\widetilde{\widetilde{\mathbf{M}}}) = 1 - 5/\delta^2$ ). This is the approach pioneered in [44] and [45] to approximate 3D homogeneous transformations by 4D rotation matrices and used, for example, in [23] in dimensional synthesis, or in [46] to solve the inverse kinematics of a 6R robot. This kind of approximation introduces a tradeoff between numerical stability and accuracy of the approximation (see [45] for details). If we factor  $\widetilde{\widetilde{\mathbf{M}}}$  into a left- and a right-isoclinic rotation, with  $\delta = 100$ , we obtain:

$$\begin{aligned} \widetilde{\widetilde{\mathbf{M}}}^L &= \begin{pmatrix} 0.5204 & -0.4632 & 0.4996 & -0.5152 \\ 0.4632 & 0.5204 & -0.5152 & -0.4996 \\ -0.4996 & 0.5152 & 0.5204 & -0.4632 \\ 0.5152 & 0.4996 & 0.4632 & 0.5204 \end{pmatrix} \\ &= 0.5204\mathbf{I} + 0.5152\mathbf{A}_1 + 0.4996\mathbf{A}_2 + 0.4632\mathbf{A}_3 \end{aligned} \quad (51)$$

and

$$\begin{aligned} \widetilde{\widetilde{\mathbf{M}}}^R &= \begin{pmatrix} 0.4804 & -0.5332 & 0.4996 & 0.4852 \\ 0.5332 & 0.4804 & -0.4852 & 0.4996 \\ -0.4996 & 0.4852 & 0.4804 & 0.5332 \\ -0.4852 & -0.4996 & -0.5332 & 0.4804 \end{pmatrix} \\ &= 0.4804\mathbf{I} + 0.4852\mathbf{B}_1 + 0.4996\mathbf{B}_2 + 0.5332\mathbf{B}_3. \end{aligned} \quad (52)$$

Hence,  $\widetilde{\widetilde{\mathbf{M}}}^L \widetilde{\widetilde{\mathbf{M}}}^R$  can be used as an approximation of  $\mathbf{M}$  by properly scaling the translations. This approximate double quaternion representation of  $\mathbf{M}$  is not redundant as one quaternion cannot be deduced from the other.

### C. Computation of screw parameters

Chasles' theorem states that the general spatial motion of a rigid body can be produced a rotation about an axis and a translation along the direction given by the same axis. Such a combination of translation and rotation is called a general screw motion [47]. In the definition of screw motion, a positive rotation corresponds to a positive translation along the screw axis by the right-hand rule.

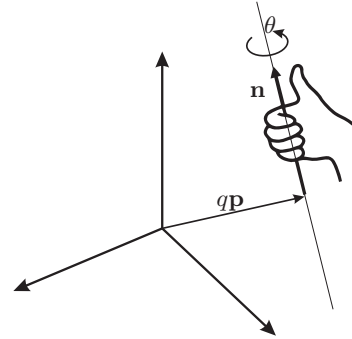


Fig. 1. Geometrix parameters used to describe a general screw motion.

In Fig. 1, a screw axis is defined by  $\mathbf{n} = (n_x, n_y, n_z)^T$ , a unit vector defining its direction, and  $q\mathbf{p}$ , the position vector of a point lying on it, where  $\mathbf{p} = (p_x, p_y, p_z)^T$  is also a unit vector. The angle of rotation  $\theta$  and the translational distance  $d$  are called the screw parameters. These screw parameters together with the screw axis completely define the general displacement of a rigid body. In terms of homogeneous transformations, in a way similar to (43), this can be expressed as:

$$\mathbf{S} = \mathbf{T}_p(q)\mathbf{R}_n(\theta)\mathbf{T}_n(d)\mathbf{T}_p(-q) \quad (53)$$

Then, using (34) and (42),

$$\tilde{\mathbf{S}}^R = \begin{bmatrix} \mathbf{I} + \varepsilon \frac{q}{2} (p_x \mathbf{B}_1 + p_y \mathbf{B}_2 + p_z \mathbf{B}_3) \\ \cos\left(\frac{\hat{\theta}}{2}\right) \mathbf{I} + \sin\left(\frac{\hat{\theta}}{2}\right) (n_x \mathbf{B}_1 + n_y \mathbf{B}_2 + n_z \mathbf{B}_3) \\ \mathbf{I} - \varepsilon \frac{q}{2} (p_x \mathbf{B}_1 + p_y \mathbf{B}_2 + p_z \mathbf{B}_3) \end{bmatrix} \quad (54)$$

where  $\hat{\theta} = \theta + \varepsilon d$ . This can be rewritten, after simplification, as:

$$\tilde{\mathbf{S}}^R = \cos\left(\frac{\hat{\theta}}{2}\right) \mathbf{I} + \sin\left(\frac{\hat{\theta}}{2}\right) (\hat{n}_x \mathbf{B}_1 + \hat{n}_y \mathbf{B}_2 + \hat{n}_z \mathbf{B}_3) \quad (55)$$

where  $\hat{\mathbf{n}} = (\hat{n}_x, \hat{n}_y, \hat{n}_z)^T = \mathbf{n} + \varepsilon q \mathbf{p} \times \mathbf{n}$ .

Thus, using the presented formalism, the derivation of the screw parameters of an arbitrary 3D transformation in homogenous coordinates entails finding the corresponding 4D rotation matrix through the mapping (33), obtaining its Cayley's factorization and, finally, identifying the resulting right-isoclinic rotation with (55). For example, to obtain the screw parameters of (46), we have to identify (49) with (55). This identification yields:

$$\cos\left(\frac{\hat{\theta}}{2}\right) = 0.5 - 2\varepsilon \quad (56)$$

$$\hat{n}_x \sin\left(\frac{\hat{\theta}}{2}\right) = 0.5 - 1.5\varepsilon \quad (57)$$

$$\hat{n}_y \sin\left(\frac{\hat{\theta}}{2}\right) = 0.5 \quad (58)$$

$$\hat{n}_z \sin\left(\frac{\hat{\theta}}{2}\right) = 0.5 + 3.5\varepsilon \quad (59)$$

From (56), we have that  $\hat{\theta} = \theta + \varepsilon d = \frac{2}{3}\pi + \varepsilon \frac{8}{\sqrt{3}}$ . Then, substituting this value in (57)-(59), we conclude that  $\mathbf{n} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)^T$  and  $q\mathbf{p} \times \mathbf{n} = \left(-\frac{6\sqrt{3}-1}{6}, -\frac{1}{6}, \frac{14\sqrt{3}-1}{6}\right)^T$ .

The conversion of a transformation in homogeneous coordinates to its corresponding dual quaternion counterpart has traditionally been performed by computing its screw parameters [48, p. 100]. We have shown how Cayley's factorization performs this task in a more straightforward way. Actually, the screw parameters can be seen as a by-product of this factorization.

## VIII. CONCLUSIONS

We have presented a two-fold matrix-quaternion formalism for the representation of rigid-body transformations that permits a better understanding of what dual quaternions are and how they can be manipulated. This formalism stems from Cayley's factorization of 4D rotation matrices whose use has been crucial for at least the following three reasons:

- Cayley's factorization leads to a matrix representation of quaternions that alleviates the sense of arbitrariness that has dominated the representation of quaternions using matrices.

- Cayley's factorization, together with a new one-to-one correspondence between 3D homogeneous transformation matrices and 4D rotation matrices, permits deriving dual quaternions from homogeneous transformations in a way that a deeper understanding of dual quaternions can be attained.
- Cayley's factorization permits converting a transformation in homogeneous coordinates to its corresponding dual quaternion without having to compute screw parameters. It is not even necessary to know the existence of Chasles' theorem to perform this conversion.

Thus, Cayley's factorization certainly deserves a more prominent place in the arsenal of the applied kinematician. This work can ultimately be seen as a vindication of its importance.

## APPENDIX. CAYLEY'S FACTORIZATION OF 4D ROTATION MATRICES

The problem of factoring a 4D rotation matrix, say  $\mathbf{S}$ , into the product of a right- and a left- isoclinic rotation matrix consists in finding the values of  $l_0, \dots, l_3$  and  $r_0, \dots, r_3$  that satisfy the following matrix equation:

$$\mathbf{S} = \begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{pmatrix} = \begin{pmatrix} l_0 & -l_3 & l_2 & -l_1 \\ l_3 & l_0 & -l_1 & -l_2 \\ -l_2 & l_1 & l_0 & -l_3 \\ l_1 & l_2 & l_3 & l_0 \end{pmatrix} \begin{pmatrix} r_0 & -r_3 & r_2 & r_1 \\ r_3 & r_0 & -r_1 & r_2 \\ -r_2 & r_1 & r_0 & r_3 \\ -r_1 & -r_2 & -r_3 & r_0 \end{pmatrix}. \quad (60)$$

According to [33], this problem was first solved by Rosen, a close collaborator of Einstein, in [49].

Equation (60) can be rewritten as:

$$\begin{pmatrix} l_0 r_0 & l_0 r_1 & l_0 r_2 & l_0 r_3 \\ l_1 r_0 & l_1 r_1 & l_1 r_2 & l_1 r_3 \\ l_2 r_0 & l_2 r_1 & l_2 r_2 & l_2 r_3 \\ l_3 r_0 & l_3 r_1 & l_3 r_2 & l_3 r_3 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} s_{11} + s_{22} + s_{33} + s_{44} & s_{31} + s_{42} - s_{13} - s_{24} \\ s_{41} + s_{32} - s_{23} - s_{14} & -s_{21} - s_{12} - s_{43} - s_{34} \\ -s_{31} + s_{42} + s_{13} + s_{24} & s_{11} - s_{22} + s_{33} - s_{44} \\ s_{21} - s_{12} + s_{43} - s_{34} & s_{41} - s_{32} - s_{23} + s_{14} \\ -s_{41} + s_{32} - s_{23} + s_{14} & s_{21} - s_{12} - s_{43} + s_{34} \\ s_{21} + s_{12} - s_{43} - s_{34} & s_{41} + s_{32} + s_{23} + s_{14} \\ s_{11} - s_{22} - s_{33} + s_{44} & s_{31} - s_{42} + s_{13} - s_{24} \\ s_{31} + s_{42} + s_{13} + s_{24} & -s_{11} - s_{22} + s_{33} + s_{44} \end{pmatrix}. \quad (61)$$

Then, if we square and add all the entries in row  $i$  of the above matrix equation, we obtain

$$l_{i-1}^2 (r_0^2 + r_1^2 + r_2^2 + r_3^2) = \frac{1}{16} \sum_{j=1}^4 w_{i,j}^2, \quad (62)$$

where  $w_{i,j}$  denotes the entry  $(i, j)$  of the matrix on the right-hand side of (61). Hence, according to (5),

$$l_{i-1} = \pm \sqrt{\frac{1}{16} \sum_{j=1}^4 w_{i,j}^2}. \quad (63)$$

Therefore, assuming that  $l_{i-1} \neq 0$ , which is always true for at least one value of  $i$  according to (4), the entries of the right-isoclinic matrix can be obtained as follows:

$$r_{j-1} = \frac{w_{i,j}}{l_{i-1}}. \quad (64)$$

Now, if we take a value of  $j$  for which  $r_{j-1} \neq 0$ , all other entries of the left-isoclinic matrix, besides that obtained in (63), can be obtained as follows:

$$l_{k-1} = \frac{w_{k,j}}{r_{j-1}}. \quad (65)$$

Observe that we have two possible solutions for the factorization depending on the sign chosen for the square root in (63). This simply says that the factorization of  $\mathbf{S}$  into isoclinic rotations can either be expressed as  $\mathbf{S}^L \mathbf{S}^R$  or  $(-\mathbf{S}^L)(-\mathbf{S}^R)$ . In other words, Cayley's factorization is unique up to a sign change. The consequence of this fact is that quaternions provide a double covering of the space of rotations.

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