On Cayley's Factorization of 4D Rotations and Applications

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Abstract. A 4D rotation can be decomposed into a left- and a right-isoclinic rotation. This decomposition, known as Cayley's factorization of 4D rotations, can be performed using Elfrinkhof-Rosen method. In this paper, we present a more straightforward alternative approach using the corresponding orthogonal subspaces, for which orthogonal bases can be defined. This yields easy formulations, both in the space of 4×4 real orthogonal matrices representing 4D rotations and in the Clifford algebra $\mathcal{C}_{4,0,0}$.

Cayley's factorization has many important applications. It can be used to easily transform rotations represented using matrix algebra to different Clifford algebras. As a practical application of the proposed method, it is shown how Cayley's factorization can be used to efficiently compute the screw parameters of 3D rigid-body transformations.

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1. Introduction

Any rotation in $\mathbb{R}^4$ can be seen as the composition of two rotations in a pair of orthogonal two-dimensional subspaces [1]. When the values of the rotation angles in these two subspaces are equal, the rotation is said to be isoclinic. It can be proved that any rotation in $\mathbb{R}^4$ can be factored into the commutative composition of two isoclinic rotations. Cayley realized this fact when studying the double quaternion representation of rotations in $\mathbb{R}^4$ [2]. This is why this factorization is herein named after him. It is actually Cayley whom we must thank for the correct development of quaternions as a representation of rotations, and for establishing the connection with the results published

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by Rodrigues three years before the discovery of quaternions [3]. Although Cayley's papers contain enough information to derive a practical method to perform this factorization, he wrote them before the full development of matrix algebra thus remaining somewhat cryptic to most modern readers.

The development of the first effective procedure for computing Cayley's factorization is attributed in [4] to Van Elfrinkhof [5]. Since this work, written in Dutch, remained unnoticed, other sources (see, for example, [6]) attribute to Rosen, a close collaborator of Einstein, the first procedure to obtain it [7]. The methods of Elfrinkhof and Rosen are equivalent. They are based on a clever manipulation of the 16 algebraic scalar equations resulting from imposing the factorization to an arbitrary 4D rotation matrix (see [4, 8] for a detailed explanation of this method). More recently, an alternative approach based on the computation of eigenvalues has been proposed in [16].

In this paper, it is shown how Cayley's factorization admits an explicit formula that can be expressed using matrix algebra or Clifford algebras. It is thus shown that, contrarily to what seemed unavoidable in previous formulations, there is no need to manipulate any set of algebraic equations in order to perform this factorization.

Cayley's factorization has important applications. Recently, it has been shown how it allows converting a rigid-body transformation in homogeneous coordinates to its corresponding dual quaternion representation in a very straightforward way [8]. See also [9] for a matrix representation of the algebra of double quaternions. This leads to a two-fold matrix and dual quaternion formalism for the representation of rigid-body transformations that permits a better understanding of dual quaternions and how they can be advantageously used in Kinematics. In this paper, it is shown how the application of the derived explicit formula leads to a neat way of obtaining the dual quaternion representation of rigid-body transformation, thus providing a simple alternative to the standard approach based on the computation of screw parameters [10, p. 100]. Screw parameters can actually be seen as a by-product of Cayley's factorization.

This article is organized as follows. Section 2 summarizes some basic facts about 4D rotations that are used in Sections 3 and 4 to derive a spectral decomposition of isoclinic rotations and explicit formulas in matrix and Clifford algebra for the computation of Cayley's factorization. Section 5 gives details on a mapping between general displacements in 3D and some 4D rotations to obtain the dual quaternion expression for rigid-body transformations and, as a by-product, their screw parameters. Sections 6 and 7 present an example and conclusions respectively.

2. Isoclinic Rotations

The elements of the Lie group of rotations in four-dimensional space, $SO(4)$, can be either simple or double rotations. Simple rotations have a fixed plane (a plane in which all the points are fixed under the rotation), while double rotations have a single fixed point only, the center of rotation. In addition, double rotations present at least a pair of invariant planes that are orthogonal. The double rotation has two angles of rotation, α_1 and α_2, one for each invariant plane, through which points in the planes rotate. All points not in these planes rotate through angles between α_1 and α_2. See [12] for details on the geometric interpretation of rotations in four dimensions.

Isoclinic rotations are a particular case of double rotations in which there are infinitely many invariant orthogonal planes, with same rotation angles, that is, $\alpha_1 = \pm\alpha_2$.

These rotations can be left-isoclinic, when the rotation in both planes is the same ($\alpha_1 = \alpha_2$), or right-isoclinic, when the rotations in both planes have opposite signs ($\alpha_1 = -\alpha_2$).

Isoclinic rotation matrices have several important properties:

1. The composition of two right- (left-) isoclinic rotations is a right- (left-) isoclinic rotation.
2. The composition of a right- and a left-isoclinic rotation is commutative.
3. Any 4D rotation can be decomposed into the composition of a right- and a left-isoclinic rotation.

Hence both form maximal and normal subgroups. Let us denote S^3_R the subgroups of right-isoclinic rotations, and S^3_L the subgroup of left-isoclinic rotations. The direct product $S^3_L \times S^3_R$ is a double cover of the group $SO(4)$, as four-dimensional rotations can be seen as the composition of rotations of these two subgroups, and there are two expressions for each element of the group.

3. Matrix Algebra Representation of Isoclinic Rotations

After a proper change in the orientation of the reference frame, an arbitrary 4D rotation matrix (*i.e.*, an orthogonal matrix with determinant +1) can be expressed as [11, Theorem 4]:

$$\mathbf{R} = \begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 & 0 & 0 \\ \sin\alpha_1 & \cos\alpha_1 & 0 & 0 \\ 0 & 0 & \cos\alpha_2 & -\sin\alpha_2 \\ 0 & 0 & \sin\alpha_2 & \cos\alpha_2 \end{pmatrix}. \tag{3.1}$$

This expression shows the 4D rotation as defined by the two mutually orthogonal planes of rotation with rotation angles α_1 and α_2, each of the planes being invariant in the sense that points in each plane stay within the planes.

The left- and right-isoclinic rotations can be represented by rotation matrices of the form

$$\mathbf{R}^L = \begin{pmatrix} l_0 & -l_3 & l_2 & -l_1 \\ l_3 & l_0 & -l_1 & -l_2 \\ -l_2 & l_1 & l_0 & -l_3 \\ l_1 & l_2 & l_3 & l_0 \end{pmatrix}, \tag{3.2}$$

and

$$\mathbf{R}^R = \begin{pmatrix} r_0 & -r_3 & r_2 & r_1 \\ r_3 & r_0 & -r_1 & r_2 \\ -r_2 & r_1 & r_0 & r_3 \\ -r_1 & -r_2 & -r_3 & r_0 \end{pmatrix}, \tag{3.3}$$

respectively. Since (3.2) and (3.3) are rotation matrices, their rows and columns are unit vectors. As a consequence,

$$l_0^2 + l_1^2 + l_2^2 + l_3^2 = 1 \tag{3.4}$$

and

$$r_0^2 + r_1^2 + r_2^2 + r_3^2 = 1. \tag{3.5}$$

Without loss of generality, we have introduced some changes in the signs and indices of (3.2) and (3.3) with respect to the notation used by Cayley [2, 6] to ease the treatment given below and to provide a neat connection with the standard use of quaternions for representating rotations in three dimensions.

According to the properties in Section 2, a 4D rotation matrix, say $\mathbf{R}$, can be expressed as:

$$\mathbf{R} = \mathbf{R}^L\mathbf{R}^R = \mathbf{R}^R\mathbf{R}^L, \tag{3.6}$$

with

$$\mathbf{R}^L = l_0\mathbf{I} + l_1\mathbf{A}_1 + l_2\mathbf{A}_2 + l_3\mathbf{A}_3 \tag{3.7}$$

and

$$\mathbf{R}^R = r_0\mathbf{I} + r_1\mathbf{B}_1 + r_2\mathbf{B}_2 + r_3\mathbf{B}_3, \tag{3.8}$$

where $\mathbf{I}$ stands for the 4×4 identity matrix and

$$\mathbf{A}_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},\ \mathbf{A}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},\ \mathbf{A}_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\mathbf{B}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},\ \mathbf{B}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},\ \mathbf{B}_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Therefore, $\{\mathbf{I}, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$ and $\{\mathbf{I}, \mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3\}$ can be seen, respectively, as bases for left- and right-isoclinic rotations.

Now, it can be verified that

$$\mathbf{A}_1^2 = \mathbf{A}_2^2 = \mathbf{A}_3^2 = \mathbf{A}_1\mathbf{A}_2\mathbf{A}_3 = -\mathbf{I}, \tag{3.9}$$

and

$$\mathbf{B}_1^2 = \mathbf{B}_2^2 = \mathbf{B}_3^2 = \mathbf{B}_1\mathbf{B}_2\mathbf{B}_3 = -\mathbf{I}. \tag{3.10}$$

Expression (3.9) determines all the possible products of $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{A}_3$, resulting in

$$\begin{aligned} &\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_3, && \mathbf{A}_2\mathbf{A}_3 = \mathbf{A}_1, && \mathbf{A}_3\mathbf{A}_1 = \mathbf{A}_2, \\ &\mathbf{A}_2\mathbf{A}_1 = -\mathbf{A}_3, && \mathbf{A}_3\mathbf{A}_2 = -\mathbf{A}_1, && \mathbf{A}_1\mathbf{A}_3 = -\mathbf{A}_2. \end{aligned} \tag{3.11}$$

Likewise, all the possible products of $\mathbf{B}_1$, $\mathbf{B}_2$, and $\mathbf{B}_3$ can be derived from expression (3.10). All these products can be summarized in multiplication tables 1 and 2.

TABLE 1. Multiplication table for the basis of the left-isoclinic rotations.

	$\mathbf{I}$	$\mathbf{A}_1$	$\mathbf{A}_2$	$\mathbf{A}_3$
$\mathbf{I}$	$\mathbf{I}$	$\mathbf{A}_1$	$\mathbf{A}_2$	$\mathbf{A}_3$
$\mathbf{A}_1$	$\mathbf{A}_1$	$-\mathbf{I}$	$\mathbf{A}_3$	$-\mathbf{A}_2$
$\mathbf{A}_2$	$\mathbf{A}_2$	$-\mathbf{A}_3$	$-\mathbf{I}$	$\mathbf{A}_1$
$\mathbf{A}_3$	$\mathbf{A}_3$	$\mathbf{A}_2$	$-\mathbf{A}_1$	$-\mathbf{I}$

TABLE 2. Multiplication table for the basis of the right-isoclinic rotations.

	$\mathbf{I}$	$\mathbf{B}_1$	$\mathbf{B}_2$	$\mathbf{B}_3$
$\mathbf{I}$	$\mathbf{I}$	$\mathbf{B}_1$	$\mathbf{B}_2$	$\mathbf{B}_3$
$\mathbf{B}_1$	$\mathbf{B}_1$	$-\mathbf{I}$	$\mathbf{B}_3$	$-\mathbf{B}_2$
$\mathbf{B}_2$	$\mathbf{B}_2$	$-\mathbf{B}_3$	$-\mathbf{I}$	$\mathbf{B}_1$
$\mathbf{B}_3$	$\mathbf{B}_3$	$\mathbf{B}_2$	$-\mathbf{B}_1$	$-\mathbf{I}$

Moreover, it can be verified that

$$\mathbf{A}_i\mathbf{B}_j = \mathbf{B}_j\mathbf{A}_i, \tag{3.12}$$

which is actually a consequence of the commutativity of left- and right-isoclinic rotations. Then, in the composition of two 4D rotations, we have:

$$\mathbf{R}_1\mathbf{R}_2 = (\mathbf{R}_1^L\mathbf{R}_1^R)(\mathbf{R}_2^L\mathbf{R}_2^R) = (\mathbf{R}_1^L\mathbf{R}_2^L)(\mathbf{R}_1^R\mathbf{R}_2^R). \tag{3.13}$$

3.1. A spectral decomposition

The set of matrices $\{\mathbf{I}, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$ form an orthogonal basis in the sense of Hilbert-Schmidt for the real Hilbert space of 4×4 real orthogonal matrices representing left-isoclinic rotations. Then, Eq.(3.7) can be seen as a spectral decomposition. If we left-multiply it by each of the elements of the set $\{\mathbf{I}, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$, to obtain the different projection coefficients, we have that

$$l_0\mathbf{I} = -\mathbf{R}^L + l_1\mathbf{A}_1 + l_2\mathbf{A}_2 + l_3\mathbf{A}_3, \tag{3.14}$$

$$l_1\mathbf{I} = -\mathbf{A}_1\mathbf{R}^L + l_0\mathbf{A}_1 + l_2\mathbf{A}_3 - l_3\mathbf{A}_2, \tag{3.15}$$

$$l_2\mathbf{I} = -\mathbf{A}_2\mathbf{R}^L + l_0\mathbf{A}_2 - l_1\mathbf{A}_3 + l_3\mathbf{A}_1, \tag{3.16}$$

$$l_3\mathbf{I} = -\mathbf{A}_3\mathbf{R}^L + l_0\mathbf{A}_3 + l_1\mathbf{A}_2 - l_2\mathbf{A}_1. \tag{3.17}$$

By iterative substituting and rearranging terms in (3.14)-(3.17), we conclude that the coefficients of the spectral decomposition (3.7) can be expressed as:

$$l_0\mathbf{I} = -\frac{1}{4}\left(-\mathbf{R}^L + \mathbf{A}_1\mathbf{R}^L\mathbf{A}_1 + \mathbf{A}_2\mathbf{R}^L\mathbf{A}_2 + \mathbf{A}_3\mathbf{R}^L\mathbf{A}_3\right), \quad (3.18)$$

$$l_1\mathbf{I} = -\frac{1}{4}\left(\mathbf{R}^L\mathbf{A}_1 + \mathbf{A}_1\mathbf{R}^L + \mathbf{A}_3\mathbf{R}^L\mathbf{A}_2 - \mathbf{A}_2\mathbf{R}^L\mathbf{A}_3\right), \quad (3.19)$$

$$l_2\mathbf{I} = -\frac{1}{4}\left(\mathbf{R}^L\mathbf{A}_2 + \mathbf{A}_2\mathbf{R}^L + \mathbf{A}_1\mathbf{R}^L\mathbf{A}_3 - \mathbf{A}_3\mathbf{R}^L\mathbf{A}_1\right), \quad (3.20)$$

$$l_3\mathbf{I} = -\frac{1}{4}\left(\mathbf{R}^L\mathbf{A}_3 + \mathbf{A}_3\mathbf{R}^L + \mathbf{A}_2\mathbf{R}^L\mathbf{A}_1 - \mathbf{A}_1\mathbf{R}^L\mathbf{A}_2\right). \quad (3.21)$$

Likewise, we can consider the set of matrices $\{\mathbf{I}, \mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3\}$ as an orthogonal basis in the sense of Hilbert-Schmidt for right-isoclinic rotations. Then, the coefficients in (3.8) could also be obtained as above.

3.2. Matrix formulation of Cayley's factorization

Let us define the following matrix linear operators for arbitrary 4D rotation matrices:

$$\begin{aligned}
\mathcal{L}_0(\mathbf{R}) &= -\frac{1}{4}\left(-\mathbf{R} + \mathbf{A}_1\mathbf{R}\mathbf{A}_1 + \mathbf{A}_2\mathbf{R}\mathbf{A}_2 + \mathbf{A}_3\mathbf{R}\mathbf{A}_3\right), \\
\mathcal{L}_1(\mathbf{R}) &= -\frac{1}{4}\left(\mathbf{R}\mathbf{A}_1 + \mathbf{A}_1\mathbf{R} + \mathbf{A}_3\mathbf{R}\mathbf{A}_2 - \mathbf{A}_2\mathbf{R}\mathbf{A}_3\right), \\
\mathcal{L}_2(\mathbf{R}) &= -\frac{1}{4}\left(\mathbf{R}\mathbf{A}_2 + \mathbf{A}_2\mathbf{R} + \mathbf{A}_1\mathbf{R}\mathbf{A}_3 - \mathbf{A}_3\mathbf{R}\mathbf{A}_1\right), \\
\mathcal{L}_3(\mathbf{R}) &= -\frac{1}{4}\left(\mathbf{R}\mathbf{A}_3 + \mathbf{A}_3\mathbf{R} + \mathbf{A}_2\mathbf{R}\mathbf{A}_1 - \mathbf{A}_1\mathbf{R}\mathbf{A}_2\right).
\end{aligned} \quad (3.22)$$

According to (3.18)-(3.21), $\mathcal{L}_i(\mathbf{R}^L) = l_i\mathbf{I}$, $i = 0, \ldots, 3$. Using the commutativity property of left- and right-isoclinic rotations, it is straightforward to prove that

$$\mathcal{L}_i(\mathbf{R}) = \mathcal{L}_i(\mathbf{R}^L\mathbf{R}^R) = \mathcal{L}_i(\mathbf{R}^L)\mathcal{L}_i(\mathbf{R}^R) = l_i\mathbf{R}^R. \quad (3.23)$$

We arrive at an important conclusion: $\mathcal{L}_i(\mathbf{R})$ and $\mathbf{R}^R$ are equal up to a constant factor. Moreover, since $\mathbf{R}^R$ is a rotation matrix, the 2-norm of any of the rows and columns of $\mathcal{L}_i(\mathbf{R})$ is l_i. This provides a straightforward way to compute Cayley's factorization. Indeed,

$$\mathbf{R}^R = \frac{\mathcal{L}_i(\mathbf{R})}{\left[\det(\mathcal{L}_i(\mathbf{R}))\right]^{1/4}} \quad (3.24)$$

and

$$\mathbf{R}^L = \mathbf{R}(\mathbf{R}^R)^T = \frac{\mathbf{R}\mathcal{L}_i(\mathbf{R})^T}{\left[\det(\mathcal{L}_i(\mathbf{R}))\right]^{1/4}}. \quad (3.25)$$

Observe that we have two possible solutions for the factorization depending on the sign chosen for the quartic roots in (3.24)-(3.25), corresponding to the double covering of the space of rotations.

4. Clifford Algebra of Isoclinic Rotations

Four-dimensional rotations can also be expressed using the Clifford algebra of the four-dimensional Euclidean space, $\mathcal{C}_{4,0,0}$. Let $\{e_1, e_2, e_3, e_4\}$ be the elements corresponding to an orthonormal basis of $\mathbb{R}^4$. The 2-blades created as the product of the elements of the basis, $e_i e_j = e_i \wedge e_j$, are denoted e_{ij} and are such that $e_{ij}^2 = -1$. The pseudoscalar satisfies $e_{1234}^2 = 1$. It is sometimes denoted as the *double unit*; we use the notation $e = e_{1234}$.

The even subalgebra, $\mathcal{C}^+_{4,0,0}$ has dimension eight and its elements are created as the linear combination of blades of grade zero, two and four. A general element of the subalgebra can be written as

$$\begin{aligned} Q =& q_0 + q_1 e_{23} + q_2 e_{31} + q_3 e_{12} + q_4 e_{41} + q_5 e_{42} + q_6 e_{43} + q_7 e_{1234} \\ =& q_0 + q_1 e_{23} + q_2 e_{31} + q_3 e_{12} + e(q_4 e_{23} + q_5 e_{31} + q_6 e_{12} + q_7). \end{aligned} \tag{4.1}$$

4.1. Bivectors and Orthogonal Components

Bivectors are linear combinations of the 2-blade basis elements, which are organized as follows: $\{e_{23}, e_{31}, e_{12}, e_{41}, e_{42}, e_{43}\}$. The geometric product of bivectors can be written as

$$B_1 B_2 = B_1 * B_2 + B_1 \times B_2 + B_1 \wedge B_2, \tag{4.2}$$

where $*$ corresponds to the scalar product and $\times$ is the commutator product. The subspace of bivectors is closed under the commutator product $B_1 \times B_2 = \frac{1}{2}(B_1 B_2 - B_2 B_1)$.

There are two types of bivectors in the algebra of the $4D$ space: simple and non-simple, or compound.

Simple bivectors can be written as the outer product of 1-vectors, and effectively define planes in $4D$. Non-simple bivectors are those that cannot be simplified as the outer product of 1-vectors; they are linear combinations of planes that not all share 1-D subspaces. The square of a bivector is, in general, the sum of a scalar plus a pseudoscalar component,

$$BB = k + k_0 e, \tag{4.3}$$

however in the case of a simple bivector, the square has only scalar part, and $k_0 = 0$. Alternatively, this can be checked using the outer product, as $B \wedge B = 0$ for simple bivectors.

The *orthogonal component* of a plane P in $4D$ space is another plane, and corresponds to the dual P^*, computed using the contraction $\rfloor$ as

$$P^* = P \rfloor e^{-1} = P \rfloor e = Pe \tag{4.4}$$

when P is a simple bivector [15]. In this fully orthogonal plane P^*, every direction is orthogonal to the plane P. The two orthogonal components intersect only at the origin. These components generate the whole $4D$ space as a direct sum.

There is a distinction to be made between orthogonal planes sharing a component of grade 1 and fully orthogonal planes. The geometric product of orthogonal planes has zero scalar, while the geometric product of fully orthogonal planes has only pseudoscalar component,

$$P_1 P_2 = P_1 \wedge P_2 = p_7 e, \tag{4.5}$$

if P_1, P_2 are fully orthogonal.

A non-simple bivector can be written as the sum of two simple bivectors. In $\mathcal{C}_{4,0,0}$, there are many ways of expressing a general bivector as the sum of two simple bivectors. However there is only one way of decomposing a bivector as the sum of two orthogonal components, $B = B_1 + B_2$, with B_1, B_2 simple and fully orthogonal bivectors.

This decomposition is calculated by solving the following conditions:

$$\begin{aligned} &B = B_1 + B_2, \\ &B_1 B_2 = ke, \\ &B_1 B_1 = k_1, \quad B_2 B_2 = k_2, \end{aligned} \tag{4.6}$$

and noticing that

$$BB = (k_1 + k_2) + 2ke. \tag{4.7}$$

The simple bivectors are obtained as

$$\begin{aligned} B_1 &= (k_1 + ke) B^{-1}, \\ B_2 &= (k_2 + ke) B^{-1}, \end{aligned} \tag{4.8}$$

with

$$k_{1,2} = \frac{1}{2}\left(\| < BB >_0 \| \pm \sqrt{\| < BB >_0 \|^2 - \| < BB >_4 \|^2}\right), \tag{4.9}$$

where the operator $<>_m$ selects the grade-m part of the element.

4.2. Versors and Rotations

A unit even versor is the geometric product of an even number of 1-vectors whose norm is one. Notice that the geometric product of an even number of 1-vectors yields an element of the even subalgebra $\mathcal{C}^+_{4,0,0}$.

A rotation R in 4D corresponds to an even unit versor of the algebra, called a *rotor*, such that $R\tilde{R} = 1$. The product of two and four unit 1-vectors v_i yields

$$\begin{aligned} &\mathsf{v}_1 \mathsf{v}_2 = k + B, \\ &\mathsf{v}_1 \mathsf{v}_2 \mathsf{v}_3 \mathsf{v}_4 = k_1 + B + k_2 e, \end{aligned} \tag{4.10}$$

where B is a bivector. Rotors can be generated as the exponentiation of bivectors.

Simple rotations are generated as the exponentiation of simple bivectors. In those, the simple bivector defines the plane in which the rotation takes place. Recalling that the square of a simple bivector yields a scalar, the exponentiation can be calculated to be

$$R = \mathrm{e}^{kB} = \cos k + \sin k B \tag{4.11}$$

for B a bivector of norm equal to one.

The exponentiation of non-simple bivectors yields a double rotation. It is easy to distinguish between a simple and a double rotation by the fact that the pseudoscalar component in a simple rotation is zero.

For a double rotation, computed as the exponential of a non-simple bivector, we can apply the decomposition of the non-simple bivector B in simple, orthogonal bivectors,

$$R = \mathrm{e}^{B} = \mathrm{e}^{B_1+B_2} = \mathrm{e}^{B_1}\mathrm{e}^{B_2} = R_1R_2, \tag{4.12}$$

where R_1 and R_2 are the simple rotations in orthogonal planes. This derivation is possible because B_1 and B_2 commute, and make the simple rotations commutative too, $R = R_1R_2 = R_2R_1$.

The bivector and rotation angle can be recovered for simple rotations using the *log* map. Given a simple rotation R, the simple (unit) bivector B defining the rotation plane, and the rotation angle ϕ, are

$$\begin{aligned} B &= \frac{< R >_2}{\|< R >_2\|}, \\ \phi &= \arctan\left(\|< R >_2\|, \|< R >_0\|\right). \end{aligned} \tag{4.13}$$

A similar solution cannot be immediately obtained for double rotations. However it is possible to find a simpler solution if the double rotation is decomposed into two commutative double rotations, the isoclinic rotations.

4.3. Non-simple Orthogonal Bivectors

In order to decompose a four-dimensional double rotation in the product of two orthogonal double rotations, two sets of bivectors are considered that form separate ideals. A general bivector B can be written as the sum of elements B_1 and B_2 of these two ideals. Notice that the bivectors B_1 and B_2 do not have to be simple.

Assigning the bivectors to their respective ideal, decomposing $B = B_1 + B_2$ and exponentiating, we obtain

$$R = \mathrm{e}^{B} = \mathrm{e}^{B_1+B_2} = \mathrm{e}^{B_1}\mathrm{e}^{B_2} = \mathrm{e}^{B_2}\mathrm{e}^{B_1} = R_1R_2 = R_2R_1. \tag{4.14}$$

The 4D double rotation is expressed as the product of two double rotations that commute.

The ideals are created, according to [14], using two mappings called the *projectors*, defined as $P_\pm = \frac{1}{2}(1 \pm e)$. These two mappings create the elements of each of the ideals when applied to the basis blades $\{e_{23}, e_{31}, e_{12}\}$ or $\{e_{41}, e_{42}, e_{43}\}$,

$$\begin{aligned} P_+e_{23} &= \frac{1}{2}(e_{23} + e_{41}), \quad P_+e_{31} = \frac{1}{2}(e_{31} + e_{42}), \quad P_+e_{12} = \frac{1}{2}(e_{12} + e_{43}), \\ P_-e_{23} &= \frac{1}{2}(e_{23} - e_{41}), \quad P_-e_{31} = \frac{1}{2}(e_{31} - e_{42}), \quad P_-e_{12} = \frac{1}{2}(e_{12} - e_{43}). \end{aligned} \tag{4.15}$$

The geometric product of these elements yields the multiplication tables (3) and (4). It is worth noting that the elements $P_+1 = \frac{1}{2}(1 + e)$ and $P_-1 =$

$\frac{1}{2}(1-e)$ act as units in their respective ideals. The product of elements of different ideals is equal to zero, including $\frac{1}{2}(1+e)\frac{1}{2}(1-e)=0$.

TABLE 3. Multiplication table for the basis of the P_+ ideal.

	$\frac{1}{2}(e_{23}+e_{41})$	$\frac{1}{2}(e_{31}+e_{42})$	$\frac{1}{2}(e_{12}+e_{43})$	$\frac{1}{2}(1+e)$
$\frac{1}{2}(e_{23}+e_{41})$	$-\frac{1}{2}(1+e)$	$-\frac{1}{2}(e_{12}+e_{43})$	$\frac{1}{2}(e_{31}+e_{42})$	$\frac{1}{2}(e_{23}+e_{41})$
$\frac{1}{2}(e_{31}+e_{42})$	$\frac{1}{2}(e_{12}+e_{43})$	$-\frac{1}{2}(1+e)$	$-\frac{1}{2}(e_{23}+e_{41})$	$\frac{1}{2}(e_{31}+e_{42})$
$\frac{1}{2}(e_{12}+e_{43})$	$-\frac{1}{2}(e_{31}+e_{42})$	$\frac{1}{2}(e_{23}+e_{41})$	$-\frac{1}{2}(1+e)$	$\frac{1}{2}(e_{12}+e_{43})$
$\frac{1}{2}(1+e)$	$\frac{1}{2}(e_{23}+e_{41})$	$\frac{1}{2}(e_{31}+e_{42})$	$\frac{1}{2}(e_{12}+e_{43})$	$\frac{1}{2}(1+e)$

TABLE 4. Multiplication table for the basis of the P_- ideal.

	$\frac{1}{2}(e_{23}-e_{41})$	$\frac{1}{2}(e_{31}-e_{42})$	$\frac{1}{2}(e_{12}-e_{43})$	$\frac{1}{2}(1-e)$
$\frac{1}{2}(e_{23}-e_{41})$	$-\frac{1}{2}(1-e)$	$-\frac{1}{2}(e_{12}-e_{43})$	$\frac{1}{2}(e_{31}-e_{42})$	$\frac{1}{2}(e_{23}-e_{41})$
$\frac{1}{2}(e_{31}-e_{42})$	$\frac{1}{2}(e_{12}-e_{43})$	$-\frac{1}{2}(1-e)$	$-\frac{1}{2}(e_{23}-e_{41})$	$\frac{1}{2}(e_{31}-e_{42})$
$\frac{1}{2}(e_{12}-e_{43})$	$-\frac{1}{2}(e_{31}-e_{42})$	$\frac{1}{2}(e_{23}-e_{41})$	$-\frac{1}{2}(1-e)$	$\frac{1}{2}(e_{12}-e_{43})$
$\frac{1}{2}(1-e)$	$\frac{1}{2}(e_{23}-e_{41})$	$\frac{1}{2}(e_{31}-e_{42})$	$\frac{1}{2}(e_{12}-e_{43})$	$\frac{1}{2}(1-e)$

The bivectors of each ideal can be used to generate rotations. Let us call $B_\pm$ the bivectors formed as linear combination of the $P_\pm$ basis bivectors. A general bivector

$$B = b_1e_{23} + b_2e_{31} + b_3e_{12} + b_4e_{41} + b_5e_{42} + b_6e_{43} \tag{4.16}$$

can be seen as the sum or difference of the $B_\pm$ bivectors, $B = B_+ - B_-$ or $B = B_+ + B_-$. In particular, the bivectors

$$\begin{aligned} B_+ &= \frac{b_{1+}}{2}(e_{23}+e_{41}) + \frac{b_{2+}}{2}(e_{31}+e_{42}) + \frac{b_{3+}}{2}(e_{12}+e_{43}), \\ B_- &= \frac{b_{1-}}{2}(e_{23}-e_{41}) + \frac{b_{2-}}{2}(e_{31}-e_{42}) + \frac{b_{3-}}{2}(e_{12}-e_{43}), \end{aligned} \tag{4.17}$$

have coefficients

$$\begin{aligned} B = B_+ + B_- : \quad & b_{1+} = b_1 + b_4, \quad b_{2+} = b_2 + b_5, \quad b_{3+} = b_4 + b_6, \\ & b_{1-} = b_1 - b_4, \quad b_{2-} = b_2 - b_5, \quad b_{3-} = b_4 - b_6. \\ B = B_+ - B_- : \quad & b_{1+} = b_1 + b_4, \quad b_{2+} = b_2 + b_5, \quad b_{3+} = b_4 + b_6, \\ & b_{1-} = -b_1 + b_4, \quad b_{2-} = -b_2 + b_5, \quad b_{3-} = -b_4 + b_6. \end{aligned} \tag{4.18}$$

The decomposition of a non-simple bivector into bivectors of the two ideals can be easily obtained as

$$\begin{aligned} B_+ &= \frac{1}{2}(1+e)B, \\ B_- &= \frac{1}{2}(1-e)B. \end{aligned} \tag{4.19}$$

4.4. Isoclinic Rotations

Using the decomposition in Eq.(4.19), the general 4D rotation can be written as

$$\mathrm{e}^{B} = \mathrm{e}^{B_+}\mathrm{e}^{B_-}. \tag{4.20}$$

To prove this, construct the exponential of a bivector of each ideal, with

$$\begin{aligned}
B_+^2 &= -(b_{1+}^2 + b_{2+}^2 + b_{3+}^2)\frac{1}{2}(1+e),\\
B_+^3 &= -(b_{1+}^2 + b_{2+}^2 + b_{3+}^2)B_+,\\
B_+^4 &= (b_{1+}^2 + b_{2+}^2 + b_{3+}^2)^2\frac{1}{2}(1+e),\\
B_+^5 &= (b_{1+}^2 + b_{2+}^2 + b_{3+}^2)^2 B_+,\\
&\dots
\end{aligned} \tag{4.21}$$

and likewise for B_-.

Consider bivectors $B_{u+} = \frac{1}{k_+}B_+$, with $k_+ = \sqrt{b_{1+}^2 + b_{2+}^2 + b_{3+}^2}$, so that $B_{u+}^2 = -\frac{1}{2}(1+e)$. Then,

$$\begin{aligned}
R_+ &= \mathrm{e}^{k_+B_{u+}} = \frac{1}{2}(1-e) + \cos k_+\frac{1}{2}(1+e) + \sin k_+ B_{u+},\\
R_- &= \mathrm{e}^{k_-B_{u-}} = \frac{1}{2}(1+e) + \cos k_-\frac{1}{2}(1-e) + \sin k_- B_{u-},
\end{aligned} \tag{4.22}$$

and their product yields the general double rotation,

$$\begin{aligned}
R =& \mathrm{e}^{k_+B_{u+}+k_-B_{u-}} = \mathrm{e}^{k_+B_{u+}}\mathrm{e}^{k_-B_{u-}}\\
=& \cos k_-\frac{1}{2}(1-e) + \cos k_+\frac{1}{2}(1+e) + \sin k_- B_{u-} + \sin k_+ B_{u+}.
\end{aligned} \tag{4.23}$$

This decomposition is commutative.

4.5. Invariant Planes for Isoclinic Rotations

Isoclinic rotations can be decomposed as the product of simple rotations for which the angles of rotation about the fully orthogonal planes have the same value, with same or opposite sign. In Eq.(4.9) we can see that, in order to obtain the same or opposite value for the rotation angles, $\|< BB >_0\| = \pm\|< BB >_4\|$. For the non-simple bivectors in each ideal, we have

$$\begin{aligned}
B_+^2 &= -(b_{1+}^2 + b_{2+}^2 + b_{3+}^2)\frac{1}{2}(1+e),\\
B_-^2 &= -(b_{1-}^2 + b_{2-}^2 + b_{3-}^2)\frac{1}{2}(1-e),
\end{aligned} \tag{4.24}$$

which shows that $k_1 = k_2$ in Eq.(4.9) for both B_+ and B_-.

Each of the non-simple, orthogonal bivectors B_+ and B_- can be easily decomposed into the sum of two simple, orthogonal planes. Let B_+ be as in

Eq.(4.17), then simple, orthogonal planes can be computed as

$$B_{1+} = \frac{b_{1+}}{2}e_{23} + \frac{b_{2+}}{2}e_{31} + \frac{b_{3+}}{2}e_{12},$$
$$B_{2+} = \frac{b_{1+}}{2}e_{41} + \frac{b_{2+}}{2}e_{42} + \frac{b_{3+}}{2}e_{43} \tag{4.25}$$

for B_+, and

$$B_{1-} = \frac{b_{1-}}{2}e_{23} + \frac{b_{2-}}{2}e_{31} + \frac{b_{3-}}{2}e_{12},$$
$$B_{2-} = -\frac{b_{1-}}{2}e_{41} - \frac{b_{2-}}{2}e_{42} - \frac{b_{3-}}{2}e_{43}, \tag{4.26}$$

for B_-.

The decomposition into simple, orthogonal planes

$$k_+B_{u+} = k_{1+}B_{1u+} + k_{2+}B_{2u+},$$
$$k_+B_{u-} = k_{1-}B_{1u-} + k_{2-}B_{2u-}, \tag{4.27}$$

where

$$k_{1+} = k_{2+} = \frac{k_+}{2},$$
$$k_{1-} = -k_{2-} = \frac{k_+}{2}, \tag{4.28}$$

are used to calculate the exponentials

$$\mathrm{e}^{k_+B_+} = \mathrm{e}^{k_{1+}B_{1u+}+k_{2+}B_{2u+}} = (\cos k_{1+} + \sin k_{1+}B_{1u+})(\cos k_{2+} + \sin k_{2+}B_{2u+}),$$
$$\mathrm{e}^{k_-B_-} = \mathrm{e}^{k_{1-}B_{1u-}+k_{2-}B_{2u-}} = (\cos k_{1-} + \sin k_{1-}B_{1u-})(\cos k_{2-} + \sin k_{2-}B_{2u-}), \tag{4.29}$$

which yield Eq.(4.22) if we notice that $B_{1u+} + B_{2u+} = 2B_{u+}$, $B_{1u-} - B_{2u-} = 2B_{u-}$ and $B_{1u+}B_{2u+} = B_{1u-}B_{2u-} = -e$.

4.6. Isoclinic Decomposition of 4D Rotations

Given a double rotation obtained as the product of rotations on planes in the ideal, it is easy to recover these rotations. Let R be a double rotation that can be expressed as in Eq.(4.23). The products

$$\frac{1}{2}(1+e)R + \frac{1}{2}(1-e) = R_+,$$
$$\frac{1}{2}(1-e)R + \frac{1}{2}(1+e) = R_- \tag{4.30}$$

can be used to find the corresponding left- and right- isoclinic rotations.

5. A useful mapping

Chasles' theorem states that the general spatial motion of a rigid body can be expressed as a rotation about an axis and a translation along the direction given by the same axis. Such a combination of translation and rotation is called a general screw motion [13]. In the definition of screw motion, a positive rotation corresponds to a positive translation along the screw axis according to the right-hand rule.

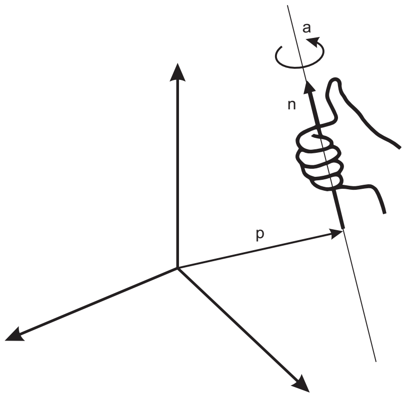

FIGURE 1. Geometric parameters used to describe a general screw motion.

In Fig. 1, a screw axis is defined by $\mathbf{n} = (n_x, n_y, n_z)^T$, a unit vector defining its direction, and $q\mathbf{p}$, the position vector of a point lying on it, where $\mathbf{p} = (p_x, p_y, p_z)^T$ is also a unit vector. The angle of rotation θ and the translational distance d are called the screw parameters. These screw parameters together with the screw axis completely define the general displacement of a rigid body.

In [8], the following mapping between 3D transformations in homogeneous coordinates and a subset of 4D rotation matrices was proposed:

$$\mathbf{T} = \begin{pmatrix} \mathbf{R}_{3\times3} & \mathbf{t} \\ 0^T & 1 \end{pmatrix} \rightleftarrows \widetilde{\mathbf{T}} = \begin{pmatrix} \mathbf{R}_{3\times3} & \varepsilon\mathbf{t} \\ -\varepsilon\mathbf{t}^T\mathbf{R}_{3\times3} & 1 \end{pmatrix}, \quad (5.1)$$

where ε is the standard dual unit ($\varepsilon^2 = 0$). The interesting thing about this mapping is that Cayley's factorization of $\widetilde{\mathbf{T}}$ can be expressed as $\widetilde{\mathbf{T}}^L\widetilde{\mathbf{T}}^R$ where

$$\widetilde{\mathbf{T}}^R = \cos\left(\frac{\hat{\theta}}{2}\right)\mathbf{I} + \sin\left(\frac{\hat{\theta}}{2}\right)(\hat{n}_x\mathbf{B}_1 + \hat{n}_y\mathbf{B}_2 + \hat{n}_z\mathbf{B}_3) \quad (5.2)$$

where $\hat{\mathbf{n}} = (\hat{n}_x, \hat{n}_y, \hat{n}_z)^T = \mathbf{n} + \varepsilon\, q\,(\mathbf{p}\times\mathbf{n})$ and $\hat{\theta} = \theta + \varepsilon\, d$ (see [8] for details). Thus, the coefficients of the Cayley's factorization of $\widetilde{\mathbf{T}}$ give us the screw parameters of $\mathbf{T}$. This is exemplified in the next section.

6. Example

Let us consider, as an example, the transformation in homogeneous coordinates

$$\mathbf{T} = \begin{pmatrix} 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.1}$$

Then, according to (5.1),

$$\widetilde{\mathbf{T}} = \begin{pmatrix} 0 & 0 & 1 & 4\varepsilon \\ 1 & 0 & 0 & -3\varepsilon \\ 0 & 1 & 0 & 7\varepsilon \\ 3\varepsilon & -7\varepsilon & -4\varepsilon & 1 \end{pmatrix}, \tag{6.2}$$

and, according to (3.22),

$$\begin{aligned} \mathcal{L}_0(\widetilde{\mathbf{T}}) &= -\frac{1}{4}\left(-\widetilde{\mathbf{T}} + \mathbf{A}_1\widetilde{\mathbf{T}}\mathbf{A}_1 + \mathbf{A}_2\widetilde{\mathbf{T}}\mathbf{A}_2 + \mathbf{A}_3\widetilde{\mathbf{T}}\mathbf{A}_3\right) \\ &= -\frac{1}{4}\begin{pmatrix} 1 & -1-11\varepsilon & 1+4\varepsilon & 1+\varepsilon \\ 1+11\varepsilon & 1 & -1-\varepsilon & 1+4\varepsilon \\ -1-4\varepsilon & 1-\varepsilon & 1 & 1+11\varepsilon \\ -1-\varepsilon & -1-4\varepsilon & -1-11\varepsilon & 1 \end{pmatrix} \\ &= -\frac{1}{4}[\mathbf{I} + (1+\varepsilon)\mathbf{B}_1 + (1+4\varepsilon)\mathbf{B}_2 + (1+11\varepsilon)\mathbf{B}_3] \end{aligned} \tag{6.3}$$

Therefore,

$$\widetilde{\mathbf{T}}^R = -\frac{1}{4l_0}[\mathbf{I} + (1+\varepsilon)\mathbf{B}_1 + (1+4\varepsilon)\mathbf{B}_2 + (1+11\varepsilon)\mathbf{B}_3]. \tag{6.4}$$

Since, according to (3.5), $r_0^2 + r_1^2 + r_2^2 + r_3^2 = 1$, we have that

$$\frac{1}{16l_0^2}\left[1 + (1+\varepsilon)^2 + (1+4\varepsilon)^2 + (1+11\varepsilon)^2\right] = \frac{4+32\varepsilon}{16l_0^2} = 1. \tag{6.5}$$

Thus,

$$l_0 = \pm\left(\tfrac{1}{2} + 2\varepsilon\right). \tag{6.6}$$

If we take the negative sign (remember that the solution is unique up to a sign change), we conclude that

$$r_0 = \frac{1}{2+8\varepsilon} = \tfrac{1}{2} - 2\varepsilon, \quad r_1 = \frac{1+\varepsilon}{2+8\varepsilon} = \tfrac{1}{2} - \tfrac{3}{2}\varepsilon,$$

$$r_2 = \frac{1+4\varepsilon}{2+8\varepsilon} = \tfrac{1}{2}, \qquad r_3 = \frac{1+11\varepsilon}{2+8\varepsilon} = \tfrac{1}{2} + \tfrac{7}{2}\varepsilon.$$

That is, the unit dual quaternion representing the transformation in homogenous coordinates given by $\mathbf{T}$ can be expressed as:

$$\widetilde{\mathbf{T}}^R = \left(\tfrac{1}{2} - 2\varepsilon\right)\mathbf{I} + \left(\tfrac{1}{2} - \tfrac{3}{2}\varepsilon\right)\mathbf{B}_1 + \left(\tfrac{1}{2}\right)\mathbf{B}_2 + \left(\tfrac{1}{2} + \tfrac{7}{2}\varepsilon\right)\mathbf{B}_3. \tag{6.7}$$

To obtain the corresponding screw parameters for this rigid-body transformation, we can simply identify (6.7) with (5.2). This identification yields:

$$\cos\left(\frac{\hat{\theta}}{2}\right) = 0.5 - 2\varepsilon, \tag{6.8}$$

$$\hat{n}_x \sin\left(\frac{\hat{\theta}}{2}\right) = 0.5 - 1.5\varepsilon, \tag{6.9}$$

$$\hat{n}_y \sin\left(\frac{\hat{\theta}}{2}\right) = 0.5, \tag{6.10}$$

$$\hat{n}_z \sin\left(\frac{\hat{\theta}}{2}\right) = 0.5 + 3.5\varepsilon. \tag{6.11}$$

Solving (6.8) for $\hat{\theta} = \theta + \varepsilon d$ we get

$$\theta = \tfrac{2}{3}\pi \text{ and } d = \tfrac{8}{\sqrt{3}}. \tag{6.12}$$

Then, substituting $\hat{\theta} = \frac{2}{3}\pi + \varepsilon\frac{8}{\sqrt{3}}$ in (6.9)-(6.11), we conclude that

$$\mathbf{n} = \left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right)^T, \tag{6.13}$$

and

$$q(\mathbf{p} \times \mathbf{n}) = \left(-\tfrac{6\sqrt{3}-1}{6}, -\tfrac{1}{6}, \tfrac{14\sqrt{3}-1}{6}\right)^T. \tag{6.14}$$

If $\mathbf{p}$ and $\mathbf{n}$ are assumed to be orthogonal, it is concluded from (6.14) that $q = \sqrt{\frac{699-40\sqrt{3}}{36}}$. As a consequence,

$$\mathbf{p} \times \mathbf{n} = (-0.3742, -0.03984, 0.9264)^T. \tag{6.15}$$

Finally, using (6.13) and (6.15), we have that

$$\mathbf{p} = \mathbf{n} \times (\mathbf{p} \times \mathbf{n}) = (-0.5579, 0.7509, -0.1930)^T. \tag{6.16}$$

7. Conclusions

Rotations in three dimensions are determined by a rotation axis and the rotation angle about it, where the rotation axis is perpendicular to the plane in which points are being rotated. The situation in four dimensions is more complicated. In this case, rotations are determined by two orthogonal planes and two angles, one for each plane. Cayley proved that a general 4D rotation can always be decomposed into two 4D rotations, each of them being determined by two equal rotation angles up to a sign change.

In this paper, we have presented explicit formulas for this decomposition using both matrix algebra and the Clifford algebra $\mathcal{C}_{4,0,0}$. The use of so different formalisms to solve the same problem has allowed us to provide a more insightful view of this decomposition. The presented results reveal of practical interest in kinematics as they provide, for example, a neat connection between the algebra of homogeneous transformations in three dimensions and the algebra of dual quaternions. As an example, this connection has been exploited to compute the screw parameters of a homogeneous transformation.

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