Zonotopic Fault Detection Observer with $\mathcal{H}_-$ Performance

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Abstract: This paper proposes a robust fault detection observer based on zonotopes for discrete-time uncertain systems with sensor faults and unknown but bounded uncertainties. The main advantage of this method is that the observer gain of the robust zonotopic observer is designed to be robust against bounded uncertainties while being sensitive to faults. In order to detect sensor faults with low magnitudes, the fault sensitivity is taken into account by measuring the $\mathcal{H}_-$ performance. The designed zonotopic observer gain can be obtained by solving an optimization problem including a sequence of linear matrix inequalities (LMIs). Finally, an illustrative example is provided to demonstrate the proposed method.

Key Words: Fault Detection, Zonotopes, $\mathcal{H}_-$ Fault Sensitivity, Linear Matrix Inequalities

1 Introduction

Because of the increasing demand of reliability, safety and acceptable performance of automatic systems, the fault detection problem has become an important issue and received much attention. One of the most deeply investigated approaches is the model-based fault detection approach, which relies on the mathematical model of the monitored system. However, for practical systems, model uncertainty, disturbances and measurement noise are inevitable. As a result, robust fault detection methods able to minimize all these effects while maintaining fault sensitivity have been developed [1]. Set-membership paradigm is shown to be a suitable robust fault detection approach for dealing with unknown but bounded uncertainties, disturbances and measurement noises, which are assumed to belong to compact sets (e.g. interval, polytope, ellipsoid, parallelotope and zonotopes [2]).

During the past decade, zonotopes have been widely investigated due to its geometrical flexibility, the reduced complexity and specially the efficient computation of linear transformations, such as Minkowsi sum and Pontrygin difference. In [3–8], the zonotope-based approach is introduced to compute the outer approximation of the state estimation with the unknown but bounded uncertainties and noise consideration by using two important approaches: set-membership method [5] and interval observer-based method [9]. Therefore, some research works have been developed to extend the zonotope-based approach into fault detection and isolation. In [10], an adaptive observer is designed and then a residual evaluation is performed based on zonotopes. In [11], the set-based approaches are reviewed for the application to fault diagnosis and fault tolerant control. In [12], an actuator-fault detection and isolation method is proposed based on a bank of interval observers, in which the zonotopes are used as the bounding set to propagate the effect of uncertainties. Furthermore, zonotope based method can be also applied to fault identification as in [13].

In order to avoid false alarms when no faults have occurred, only considering the robustness against disturbance is sometimes not enough. Thus, fault sensitivity is required to be taken into account in the observer design. In [14], an $\mathcal{H}_-/\mathcal{H}_\infty$ fault detection observer is firstly proposed, where the $\mathcal{H}_-$ norm is defined as the smallest nonzero singular value of the transfer function matrix from fault to residual at $\omega = 0$ and is used to evaluate the worst-case fault sensitivity. In [15], the $\mathcal{H}_-$ index definition is extended as the smallest singular value of the transfer function matrix over a given frequency range. From then on, $\mathcal{H}_-/\mathcal{H}_\infty$ fault detection observer has been attracted a lot of attention and extended to many type of systems, e.g. time-delay systems [16], linear parameter varying systems [17–19], fuzzy systems [20, 21], switched systems [22, 23], delta systems [24].

Most of the existing set-based fault detection observer methods mainly consider the robustness against disturbances. To the best of authors knowledge, no work has been done on improving the fault sensitivity based on the set-based method combined with $\mathcal{H}_-$ performance. In this paper, we aim to combine the set-based fault detection methods and $\mathcal{H}_-$ technique altogether to ensure the robustness against disturbance as well as the sensitivity to faults.

The main contribution of this paper relies on the development of a zonotopic fault detection observer designed taking into account the $\mathcal{H}_-$ fault sensitivity. In particular, the $\mathcal{H}_-$ performance is plugged in the zonotopic observer design in order to achieve the fault sensitivity performance. As a result, an optimal observer gain can be found by solving an optimization problem involving the solution of a sequence of linear matrix inequalities (LMIs). Finally, a numerical example is used to illustrate the effectiveness of the proposed method through the comparison with the standard zonotope-based method.

The remainder of this paper is organized as follows: In Section 2, some notations, definitions and properties are briefly introduced. In Section 3, problem formulation is addressed. In Section 4, the zonotopic fault detection observer is designed taking into account $\mathcal{H}_-$ performance. In Section 5, the proposed method is verified by an illustrative example. Finally, this work is concluded in Section 6.

This work has been partially funded by the Spanish Government and FEDER through the projects CICYT ECOCIS (ref. DPI2013-48243-C2-1-R), CICYT DEOCS (ref. DPI2016-76493-C3-3-R), CICYT HARCRICS (ref. DPI2014-58104-R) and by National Natural Science Foundation of China (Grant No. 61273162, 61403104).

2 Preliminaries

A zonotopic set $\mathcal{Z} \in \mathbb{R}^n$ is defined by a hypercube $[-1,+1]^m$ ($m \geq n$) affine projection with the center of $p \in \mathbb{R}^n$ and a generator matrix of $H \in \mathbb{R}^{n\times m}$ as $\mathcal{Z} = p \oplus H\mathbf{B}^m$. For simplicity, the zonotope $\mathcal{Z}$ is denoted as $\langle p, H\rangle$ in the following form:

$$\mathcal{Z} = \langle p, H\rangle = \left\{p + Hz, z \in \mathbf{B}^m, \|z\|_\infty \leq 1\right\}, \quad (1)$$

where $\|\cdot\|_\infty$ denotes the infinity norm.

The mathematical operators $\oplus$ and $\odot$ denote the Minkowski sum and the linear product, respectively. Therefore, the following properties hold:

$$\langle p_1, H_1\rangle \oplus \langle p_2, H_2\rangle = \langle p_1 + p_2, [H_1 \;\; H_2]\rangle, \quad (2a)$$
$$L \odot \langle p, H\rangle = \langle Lp, LH\rangle. \quad (2b)$$

The reduction operator for the zonotope firstly proposed in [25] is denoted as $\downarrow_\ell (\cdot)$, where ℓ specifies the maximum number of column of segment matrix H after reduction. Thus, $\downarrow_\ell (H)$ is computed as follows:

- Sort the column of segment matrix H on decreasing order:

$$\downarrow (H) = [h_1, h_2, \ldots, h_m], \;\; \|h_j\|^2 \geq \|h_{j+1}\|^2.$$

- Keep the first ℓ-column of $\downarrow (H)$ and enclose the set $H_<$ generated by remaining columns into a smallest box (interval hull) computed by using $rs(\cdot)$:

$$\begin{aligned}
&\textbf{If } m \leq \ell \textbf{ then } \downarrow_\ell (H) = \downarrow (H),\\
&\textbf{Else } \downarrow_\ell (H) = [H_>, rs(H_<)] \in \mathbb{R}^{n\times \ell},\\
&\quad H_> = [h_1, \ldots, h_\ell], \; H_< = [h_{\ell+1}, \ldots, h_m],\\
&\quad rs(H_<) = \sum_{j=\ell+1}^{m} |h_j|.
\end{aligned}$$

A matrix P is positive definite if the scalar $x^T Px$ is positive for arbitrary non-zero column vector x of real numbers, which is denoted by $P \succ 0$. On the other hand, the negative definite matrix is denoted by $P \prec 0$.

3 Problem Formulation

Given the discrete-time uncertain system expressed in the following form:

$$x_{k+1} = Ax_k + Bu_k + D_x w_k, \quad (3a)$$
$$y_k = Cx_k + D_y v_k + Ef_k, \quad (3b)$$

where $x_k \in \mathbb{R}^{n_x}$ and $x_{k+1} \in \mathbb{R}^{n_x}$ denote vectors of system states at time instant k and $k+1$, respectively. $u_k \in \mathbb{R}^{n_u}$ and $y_k \in \mathbb{R}^{n_y}$ denote vectors of inputs and outputs at time instant k. f_k denotes the vector of sensor faults at time instant k. $w_k \in \mathbb{R}^w$ and $v_k \in \mathbb{R}^v$ denote vectors of system disturbances and measurement noise at time instant k. Besides, A, B, C, D_x, D_y and E are system matrices of appropriate dimensions.

Assumption 1 *The unknown disturbance and noise vectors are bounded in the centered zonotopes as follows:*

$$w_k \in \mathcal{W} = \langle 0, H_w\rangle, \quad (4)$$
$$v_k \in \mathcal{V} = \langle 0, H_v\rangle, \quad (5)$$

where H_w and H_v are known segment matrices of zonotopes describing the worst-case bounds of disturbances and noise.

Assumption 2 *The initial uncertain system state x_0 is also unknown but bounded in the following zonotope:*

$$x_0 \in \hat{\mathcal{X}}_0 = \langle p_0, H_0\rangle, \quad (6)$$

where p_0 and H_0 denote the center and segment matrix of this zonotope.

A Luenberger observer is considered for monitoring (3):

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L(y_k - C\hat{x}_k), \quad (7)$$

where $L \in \mathbb{R}^{n_x \times n_y}$ denotes the observer gain. And the state estimation error at time instant k is defined as $e_k = x_k - \hat{x}_k$. Then, the error dynamics can be formulated as

$$\begin{aligned}
e_{k+1} &= x_{k+1} - \hat{x}_{k+1}\\
&= (A - LC)e_k + D_x w_k - LD_y v_k - LEf_k. \quad (8)
\end{aligned}$$

The residual signal r_k at time instant k is defined by

$$\begin{aligned}
r_k &= M\left(y_k - C\hat{x}_k\right)\\
&= MCe_k + MD_y v_k + MEf_k, \quad (9)
\end{aligned}$$

where $M \in \mathbb{R}^{n_y \times n_y}$ denotes the weighting matrix.

Considering the uncertain system state $x_k \in \hat{\mathcal{X}}_k = \langle p_k, H_k\rangle$ in (3) as prior, the uncertain system states x_{k+1} at time instant $k+1$ can also be bounded in the zonotope $\hat{\mathcal{X}}_{k+1} = \langle p_{k+1}, H_{k+1}\rangle$ that is defined as follows:

$$p_{k+1} = Ap_k + Bu_k + L(y_k - Cp_k), \quad (10a)$$
$$H_{k+1} = \left[(A - LC) \downarrow_q (H_k) \quad D_x H_w \quad -LD_y H_v\right], \quad (10b)$$

where the zonotope reduction operator $\downarrow_q (\cdot)$ is used in order to reduce the computational complexity during the state propagations.

Then, the corresponding residual zonotope $\mathcal{R}_{k+1} = \langle p^r_{k+1}, H^r_{k+1}\rangle$ can be formulated as

$$p^r_{k+1} = M\left(y_{k+1} - Cp_{k+1}\right), \quad (11a)$$
$$H^r_{k+1} = \left[MCH_{k+1} \quad MD_y H_v\right], \quad (11b)$$

Substituting p_{k+1} and H_{k+1} by (10), (11) can be reformulated

$$p^r_{k+1} = My_{k+1} - MC\left(Ap_k + Bu_k + L(y_k - Cp_k)\right), \quad (12a)$$
$$\begin{aligned}
H^r_{k+1} = \Big[&MC(A - LC) \downarrow_q (H_k) \quad MCD_x H_w\\
&- MCLD_y H_v \quad MD_y H_v\Big]. \quad (12b)
\end{aligned}$$

In order to detect the occurred faults in the uncertain system (3), this observer gain L is designed with two main objectives: (i) the zonotopic observer is robust against the uncertainties including system disturbances and measurement noise to allow distinguishing uncertainties and faults; (ii) the zonotopic fault detection observer preserves the fault sensitivity to detect faults with low magnitudes.

4 Zonotopic Fault Detection Observer Design

According to these two mentioned objectives, the observer gain L is designed by means of a procedure involving two consecutive steps. Then, the optimal observer gain L^* can be obtained by solving an optimization problem including a sequence of LMIs.

4.1 Design L against Uncertainties

In terms of the zonotope-based method, the size of the zonotope is mainly affected by the uncertainties. According to [26], the size of the zonotope (10) is measured by the P-radius of the zonotope. There exists a symmetric and positive definite matrix $P = P^T \succ 0$ such that the the P-radius of the zonotope (10) is defined as

$$\ell_k = \max_{b_1 \in \mathbf{B}^{r_1}} \|H_k b_1\|_P^2 . \quad (13)$$

This P-radius of the zonotope is converging such that the following condition is satisfied with a scalar $\gamma \in (0, 1)$:

$$\ell_{k+1} \leq \gamma \ell_k + \epsilon, \quad (14)$$

where ϵ is slack term affected by uncertainties that can be determined by

$$\epsilon = \max_{s_1 \in \mathbf{B}^{r_{s_1}}} \|D_x H_w s_1\|_2^2 + \max_{s_2 \in \mathbf{B}^{r_{s_2}}} \|D_y H_v s_2\|_2^2 . \quad (15)$$

Combining with (13) and (15), (14) can be written as

$$\max_{b_1 \in \mathbf{B}^{r_{b_1}}} \|H_{k+1} b_1\|_P^2 \leq \max_{b \in \mathbf{B}^{r_b}} \gamma \|H_k b\|_P^2 + \max_{s_1 \in \mathbf{B}^{r_{s_1}}} \|D_x H_w s_1\|_2^2 + \max_{s_2 \in \mathbf{B}^{r_{s_2}}} \|D_y H_v s_2\|_2^2 .$$

Then, $\forall b_1, b, s_1, s_2$, a sufficient condition of previous inequality can be found as

$$\|H_{k+1} b_1\|_P^2 < \gamma \|H_k b\|_P^2 + \|D_x H_w s_1\|_2^2 + \|D_y H_v s_2\|_2^2 .$$

such that

$$\begin{bmatrix} b \\ s_1 \\ s_2 \end{bmatrix}^T H_{k+1}^T P H_{k+1} \begin{bmatrix} b \\ s_1 \\ s_2 \end{bmatrix} - b^T H_k^T \gamma P H_k b - s_1^T H_w^T D_x^T D_x H_w s_1 - s_2^T H_v^T D_y^T D_y H_v s_2 < 0.$$

Considering $\theta = H_k b$, it follows

$$\begin{bmatrix} \theta \\ s_1 \\ s_2 \end{bmatrix}^T \Psi \begin{bmatrix} \theta \\ s_1 \\ s_2 \end{bmatrix} - \begin{bmatrix} \theta \\ s_1 \\ s_2 \end{bmatrix}^T \begin{bmatrix} \gamma P & 0 & 0 \\ \star & H_w^T D_x^T D_x H_w & 0 \\ \star & \star & H_v^T D_y^T D_y H_v \end{bmatrix} \begin{bmatrix} \theta \\ s_1 \\ s_2 \end{bmatrix} < 0,$$

with

$$\Psi = \begin{bmatrix} (A - LC) & D_x H_w & L D_y H_v \end{bmatrix}^T P \begin{bmatrix} (A - LC) & D_x H_w & L D_y H_v \end{bmatrix} .$$

Based on the definition of the negative definite matrix, we have

$$- \begin{bmatrix} \gamma P & 0 & 0 \\ \star & H_w^T D_x^T D_x H_w & 0 \\ \star & \star & H_v^T D_y^T D_y H_v \end{bmatrix} + \begin{bmatrix} P(A - LC) & P D_x H_w & P L D_y H_v \end{bmatrix}^T P^{-1} \begin{bmatrix} P(A - LC) & P D_x H_w & P L D_y H_v \end{bmatrix} \prec 0.$$

By using the Schur complement, the following LMI can be found:

$$\begin{bmatrix} -\gamma P & 0 & 0 & (A - LC)^T P \\ \star & -H_w^T D_x^T D_x H_w & 0 & (D_x H_w)^T P \\ \star & \star & -H_v^T D_y^T D_y H_v & (L D_y H_v)^T P \\ \star & \star & \star & -P \end{bmatrix} \prec 0. \quad (16)$$

Let $W = PL$, the previous LMI can be reformulated by

$$\begin{bmatrix} -\gamma P & 0 & 0 & A^T P - C^T W^T \\ \star & -H_w^T D_x^T D_x H_w & 0 & H_w^T D_x^T P \\ \star & \star & -H_v^T D_y^T D_y H_v & H_v^T D_y^T W^T \\ \star & \star & \star & -P \end{bmatrix} \prec 0. \quad (17)$$

4.2 Design L with $\mathcal{H}_-$ Fault Sensitivity

Considering that there are no disturbances and noise $w_k = 0$ and $v_k = 0$ for $k \in \mathbb{N}_+$ in the system, the state estimation error e_k^f is only affected by the possible occurred faults at time instant k that is defined as

$$e_k^f = x_k - p_k . \quad (18)$$

Then, this error dynamics can be formulated by

$$\begin{aligned} e_{k+1}^f &= x_{k+1} - p_{k+1} \\ &= (A - LC) e_k^f - L E f_k . \end{aligned} \quad (19)$$

Ignoring disturbances and measurement noise, the residual zonotope $\mathcal{R}_k$ reduces to its center $p_k^{r_f}$ at time instant k. Therefore, this residual $p_k^{r_f}$ only affected by the fault vector f_k can be formulated as

$$\begin{aligned} p_k^{r_f} &= M(y_k - C p_k) \\ &= M(C x_k + E f_k - C p_k) \\ &= M C e_k^f + M E f_k \end{aligned} \quad (20)$$

As introduced in [20], the $\mathcal{H}_-$ performance is used for forcing that the residual $p_k^{r_f}$ satisfy the condition $\|p_k^{r_f}\|_2 > \beta^2 \|f_k\|_2$ under zero initial condition $e_0^f = 0$ and for any non-zero $f_k \in \ell_2[0, \infty)$ with a positive scalar β.

Then, the criterion function of the $\mathcal{H}_-$ performance can be defined as

$$\mathcal{J}_-^N = \sum_{k=0}^{N-1} \left({p_k^{r_f}}^T p_k^{r_f} - \beta^2 f_k^T f_k \right), \quad (21)$$

where N is an arbitrary time instant.

Considering that there exists a positive definite matrix Q such that a Lyapunov function can be chosen as

$$V_k = {e_k^f}^T Q e_k^f . \quad (22)$$

Then, it follows

$$\begin{aligned} \Delta V_k &= V_{k+1} - V_k \\ &= {e_k^f}^T \left((A - LC)^T Q (A - LC) - Q \right) e_k^f \\ &\quad + {e_k^f}^T (A - LC)^T Q (-LE) f_k \\ &\quad + f_k^T (-LE)^T Q (A - LC) e_k^f \\ &\quad + f_k^T (LE)^T Q (LE) f_k . \end{aligned}$$

For any nonzero $f_k \in \mathcal{L}_2[0,\infty)$ and zero initial condition $e_0^f = 0$, (21) is equivalent to

$$\mathcal{J}_-^N = \sum_{k=0}^{N-1} \left({p_k^{r_f}}^T p_k^{r_f} - \beta^2 f_k^T f_k - \Delta V_k \right) + V_N. \quad (23)$$

The condition $\left\| p_k^{r_f} \right\|_2 > \beta^2 \left\| f_k \right\|_2$ is satisfied by guaranteeing $\mathcal{J}_-^N > 0$. Therefore, it can be derived:

$$ {p_k^{r_f}}^T p_k^{r_f} - \beta^2 f_k^T f_k - \Delta V_k \succ 0 \quad (24)$$

By substituting $p_k^{r_f}$ in (24) by (20), (24) is equivalent to:

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \star & \Phi_{22} \end{bmatrix} \prec 0, \quad (25)$$

with

$$\Phi_{11} = -C^T M^T M C + (A - LC)^T Q (A - LC) - Q,$$
$$\Phi_{12} = (A - LC)^T Q(-LE) - C^T M^T M E,$$
$$\Phi_{22} = \beta^2 I - E^T M^T M E + (LE)^T Q (LE),$$

By transforming (25) and using the Schur complement, (25) can be reformulated as

$$\begin{bmatrix} -Q & QA - QLC & -QLE \\ \star & -Q - C^T M^T M C & -C^T M^T M E \\ \star & \star & -E^T M^T M E + \beta^2 I \end{bmatrix} \prec 0. \quad (26)$$

Set $Y = M^T M$, there exists a positive scalar α such that the matrix Q is linked with the positive matrix P in the P-radius of the zonotope by $Q = \alpha P$. Then, (26) is equivalent to

$$\begin{bmatrix} -\alpha P & \alpha P A - \alpha W C & -\alpha W E \\ \star & -\alpha P - C^T Y C & -C^T Y E \\ \star & \star & -E^T Y E + \beta^2 I \end{bmatrix} \prec 0. \quad (27)$$

4.3 Optimal Observer Gain L^*

Considering the above two aspects, the optimal observer gain L^* can be found by solving an optimization problem. On one hand, the size of the state zonotope affected by all the uncertainties is required to be minimized. On the other hand, the center of residual zonotope affected by the occurred faults is expected to be maximized in order to detect the faults with low magnitudes.

In general, there exists positive scalars α, β and $\gamma \in (0, 1)$ such that the optimal observer gain L^* can be found by solving the following optimization problem:

$$\min_{\beta,P,W,Y} -\mathrm{tr}(P) - \beta^2 \quad (28a)$$

subject to

$$\begin{bmatrix} -\gamma P & 0 & 0 & A^T P - C^T W^T \\ \star & -H_w^T D_x^T D_x H_w & 0 & H_w^T D_x^T P \\ \star & \star & -H_v^T D_y^T D_y H_v & H_v^T D_y^T W^T \\ \star & \star & \star & -P \end{bmatrix} \prec 0, \quad (28b)$$

$$\begin{bmatrix} -\alpha P & \alpha P A - \alpha W C & -\alpha W E \\ \star & -\alpha P - C^T Y C & -C^T Y E \\ \star & \star & -E^T Y E + \beta^2 I \end{bmatrix} \prec 0. \quad (28c)$$

The feasible solutions of the optimization problem (28) are denoted as β^*, P^*, W^* and Y^*. Therefore, the optimal observer gain L^* and matrix M^* can be computed by

$$L^* = {P^*}^{-1} W^*, \quad (29)$$
$$M^* = {Y^*}^{\frac{1}{2}}. \quad (30)$$

Remark 1 *In order to obtain $\mathcal{H}_-$ fault sensitivity, the optimal result of β is expected to be as large as possible. Therefore, those faults with low magnitudes can be easier to detect. Meanwhile, the matrix M is expect to be not too big in case that the effects by uncertainties in the residuals are enlarged.*

4.4 Zonotopic Fault Detection Procedure

As explained in [11], the set-membership approach is useful to implement robust fault detection. The procedure of the zonotope-based fault detection method can be summarized in the following. When some faults occur, the center of zonotope is moved. Hence, faults can be detected by testing

$$\begin{cases} \mathbf{0} \in \mathcal{R}_k, & f_s = 0 \text{ (No Fault)} \\ \mathbf{0} \notin \mathcal{R}_k, & f_s = 1 \text{ (Fault Detected)} \end{cases} \quad (31)$$

for $k \in \mathbb{N}_+$ and f_s denotes the auxiliary variable for representing the fault detection.

After solving the optimization problem, the optimal observer gain L^* can be obtained. Therefore, the residual zonotope (11) can be subsequently found. In order to implement this test, the residual zonotope can be characterized in a half-space representation. Therefore, the zonotopic set contains a sequence of linear constraints, which can be formulated as follows:

$$\mathcal{R}_k = \left\{ r_k \in \mathbb{R}^{n_y} \mid \Sigma r_k \leq \vartheta \right\}, \quad (32)$$

where Σ and ϑ denote a matrix and a vector from the half-space representation of the residual zonotope.

Therefore, this test involves solving a constraint satisfaction problem. If the problem is feasible, then the origin of the coordinate is included in the residual zonotope. Otherwise, it is not included.

5 Illustrative Example

In order to illustrate the proposed method, a numerical example is provided. The uncertain system is defined in (3) with the following system matrices:

$$A = \begin{bmatrix} 0.413 & 0 & -0.02 \\ 0.033 & 0.521 & -0.042 \\ -0.01 & 0 & 0.257 \end{bmatrix},$$

$$B = \begin{bmatrix} -1.773 & 0.07 & 0.074 \\ 0.093 & 0.466 & 0.105 \\ -0.042 & -0.09 & 2.075 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, D_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$D_y = E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The initial uncertain state x_0 is bounded in the zonotope $\hat{\mathcal{X}}_0 = \langle p_0, H_0 \rangle$ with

$$p_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, H_0 = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}.$$

and the uncertainty zonotopes are given by

$$H_w = \begin{bmatrix} 0.085 & 0 & 0 \\ 0 & 0.085 & 0 \\ 0 & 0 & 0.085 \end{bmatrix}, H_v = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}.$$

The additive sensor faults are added into the simulation as follows:

$$f_k = \begin{cases} \begin{bmatrix} 0.3 \\ -0.25 \end{bmatrix} & 20 < k \leq 40, \\ \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{Otherwise.} \end{cases}$$

By solving the optimization problem (28), the optimal feasible solutions can be obtained with $\alpha = 10$ and $\gamma = 0.6$ as follows:

$$L^* = \begin{bmatrix} 0.2681 & 0.0011 \\ 0.0228 & 0.3463 \\ -0.0085 & -0.0032 \end{bmatrix},$$
$$M^* = \begin{bmatrix} 44.1097 & 21.1622 \\ 21.1622 & 173.3860 \end{bmatrix},$$

and the maximum value of β is 2.47. The uncertain states are bounded in the state zonotopes $\hat{\mathcal{X}}_k$ for $k \in \mathbb{N}_+$ that are plotted in Fig. 1. From this plot, it is clear that the uncertainties are propagated into system states and the sizes of uncertain state zonotopes are convergent satisfying the first objective of the observer gain design.

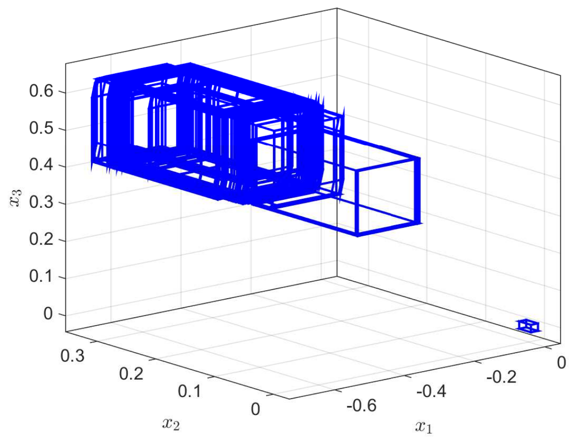


Fig. 1: Uncertain state zonotopes

The residual zonotopes are obtained applying the proposed method as shown in Fig. 2. These zonotopes change because of the occurrence of faults. Following the procedure mentioned in Section 4.4, the faults can be detected by checking whether the origin of the coordinate is inside the

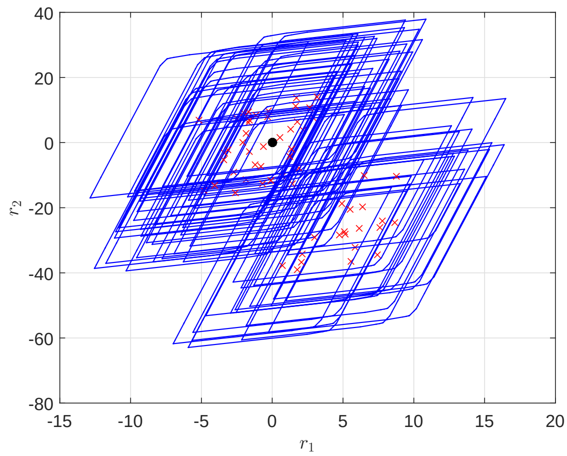


Fig. 2: Residual zonotopes by using the proposed method: blue lines are for zonotopes; red cross points are zonotope centers; black point is the origin of the coordinate

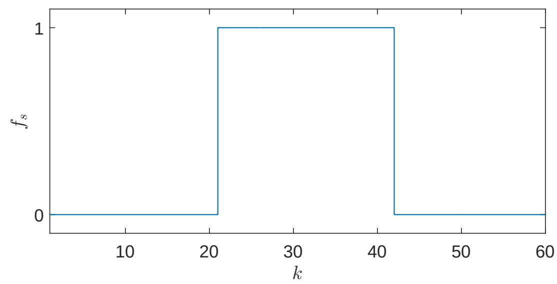
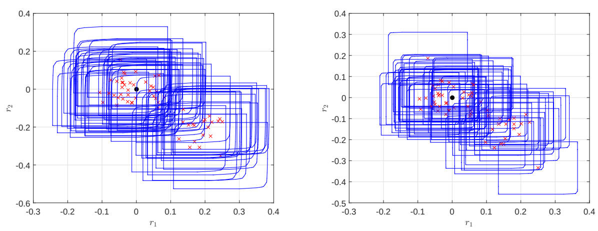
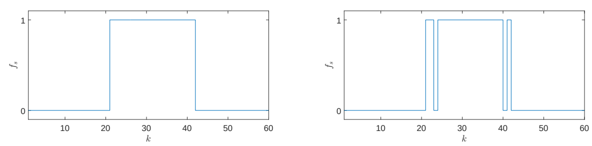


Fig. 3: Fault detection result by using the proposed method



(a) With $\mathcal{H}_-$ and $M = I$ (b) Without $\mathcal{H}_-$

Fig. 4: Comparison of residual zonotopes: blue lines are for zonotopes; red cross points are zonotope centers; black point is the origin of the coordinate



(a) With $\mathcal{H}_-$ and $M = I$ (b) Without $\mathcal{H}_-$

Fig. 5: Comparison of fault detection results

residual zonotope and the fault detection result is shown in Fig. 3. Thus, the additive sensor faults can be detected by using the proposed method with $\mathcal{H}_-$ performance.

In order to test the effectiveness of the proposed method with $\mathcal{H}_-$ performance, the standard zonotope-based method is used. In this case, the observer gain can also be designed by solving the following optimization problem without taking into account $\mathcal{H}_-$ performance:

$$\min_{P,W} -\mathrm{tr}(P) \tag{33a}$$

subject to

$$\begin{bmatrix} -\gamma P & 0 & 0 & A^T P - C^T W^T \\ \star & -H_w^T D_x^T D_x H_w & 0 & H_w^T D_x^T P \\ \star & \star & -H_v^T D_y^T D_y H_v & H_v^T D_y^T W^T \\ \star & \star & \star & -P \end{bmatrix} \prec 0. \tag{33b}$$

The comparison results of the residual zonotopes are shown in Fig. 4. Fig. 4(a) is obtained by applying the proposed method considering that M is chosen as an identity matrix. Note that the residual zonotopes for faulty and no-faulty cases by using the standard zonotope-based method overlap. If the residual zonotopes are placed too close in the different system situations, then the faults are difficult to be detected. In Fig. 5, the additive faults cannot be detected all the time by using the standard zonotope-based method while they all can be detected by using the proposed method.

6 Conclusion

In this paper, a zonotopic fault detection observer is designed considering $\mathcal{H}_-$ performance. The $\mathcal{H}_-$ performance is used to achieve the $\mathcal{H}_-$ fault sensitivity. The optimal observer gain can be found by solving an optimization problem including a sequence of LMIs. By means of the comparison with a zonotope-based method without $\mathcal{H}_-$ performance, the proposed method is more sensitive to faults and achieves the better performance than the standard approach.

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