Zonotopic Fault Estimation Filter Design for Discrete-time Descriptor Systems ⋆

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Abstract: This paper considers actuator-fault estimation for discrete-time descriptor systems with unknown but bounded system disturbance and measurement noise. A zonotopic fault estimation filter is designed based on the analysis of fault detectability indexes. To ensure estimation accuracy, the filter gain in the zonotopic fault estimation filter is optimized through the zonotope minimization. The designed zonotopic filter not only can estimate fault magnitudes, but it also provides fault estimation results in an interval, i.e. the upper and lower bounds of fault magnitudes. Moreover, the proposed fault estimation filter has a non-singular structure and hence is easy to implement. Finally, simulation results are provided to illustrate the effectiveness of the proposed method.

Keywords: Actuator fault, set-based fault estimation, descriptor systems, discrete-time model, zonotopes.

1. INTRODUCTION

Descriptor systems, which are also known as singular systems, implicit systems or differential-algebraic systems, are more general than regular dynamic systems (only including ordinary differential equations). Descriptor systems may contain static relations between system variables. Due to their powerful modeling capability, descriptor systems are encountered in diverse applications such as electrical circuits (Dai, 1989), water networks (Wang et al., 2016) and chemical processes (Kumar and Daoutidis, 1995). Therefore, control analysis and synthesis problems for descriptor systems have attracted considerable attention during the past decades (Duan, 2010).

Fault diagnosis plays an important role in increasing applications allowing to detect, isolate and identify/estimate the occurred faults appearing in actuators, sensors or the plant itself. Fault diagnosis is also a necessary part of active fault-tolerant control (Blanke et al., 2016). In recent years, significant research effort have been dedicated to fault diagnosis for descriptor systems, such as H_-/H_∞ fault detection methods for nonlinear descriptor systems have been studied in López Estrada et al. (2015); Boulkroune et al. (2013), fault estimation of nonlinear descriptor systems (Koenig, 2005; Gao and Ding, 2007).

In an uncertain control system, system disturbances and measurement noise have an important effect on the fault diagnosis result. In order to attenuate the effect of those uncertainties and guarantee the fault diagnosis performance, the fault diagnosis methods are expected to be robust with respect to uncertainties. In terms of fault estimation, the main idea behind most robust methods is to attenuate the effect of uncertainties on fault estimation as much as possible. However, it does not provide any information on the accuracy of the fault estimation. Since fault estimation is a key issue for using an active fault-tolerant control scheme, it is very useful not only to provide the fault estimation but also the confidence intervals of estimated faults. In the last decades, interval state estimation for systems without fault has been extensively studied within a set-theoretic framework, see e.g. in Gouzé et al. (2000); Puig et al. (2008); Puig (2010); Efimov and Raïssi (2016); Chambon et al. (2016) and the references therein.

From the literature, few works have been carried out on interval estimation of fault or unknown input for descriptor systems. Until now, Gucik-Derigny et al. (2016) have studied interval estimation of unknown inputs in continuous-time systems via interval observer design. However, the interval estimation method in (Gucik-Derigny et al., 2016) requires an evaluation of the measurement derivatives which must be computed using a numerical differentiator. As is known, numerical differentiation is an ill-posed problem and may enlarge measurement noise. Moreover, the interval estimation method needs the unknown in-

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put to satisfy the matching condition, which restricts its application scope. Most recently, Wang et al. (2015a) proposed a novel fault estimation filter design method without constant fault assumption for discrete-time linear descriptor systems. On the other hand, Combastel (2015) presented a zonotopic Kalman observer which can provide interval estimation with optimal robust convergence. In this work, a novel zonotopic fault estimation method for discrete-time descriptor systems is developed by means of the interval observer-based approach within a set-theoretic framework. The proposed method can provide both the optimal fault estimation and its upper and lower bounds with the desired performance.

The main contribution of this paper is to design a zonotopic fault estimation filter for uncertain discrete-time descriptor systems. The advantages of the proposed fault estimation filter are three-fold. Firstly, the proposed filter can provide an interval fault estimation in the form of a zonotopic residual set. Secondly, an optimal filter gain is designed by minimizing the zonotope size. Finally, this filter has a non-singular structure, which is convenient for implementation.

The remainder of this paper is organized as follows. After background in Section 2, the problem statement is presented in Section 3. The zonotopic fault estimation filter design is described in Section 4. The simulation results of applying the proposed fault estimation algorithm to an illustrative example are shown in Section 5. Finally, the paper is concluded in Section 6.

2. BACKGROUND: SOME DEFINITIONS AND PROPERTIES

2.1 Zonotopes and Set Operations

A m-order zonotope $\mathcal{Z} \in \mathbb{R}^n$ $(m \geq n)$ is defined by a hypercube $\mathbf{B}^m = [-1,+1]^m$ affine projection with the center $p \in \mathbb{R}^n$ and a generator matrix $H \in \mathbb{R}^{n\times m}$ as $\mathcal{Z} = p \oplus H\mathbf{B}^m$. For simplicity, the zonotope $\mathcal{Z}$ is simply denoted as $\langle p, H\rangle$ with the following form:

$$\mathcal{Z} = \langle p, H\rangle = \{p + Hz, z \in \mathbf{B}^m, \|z\|_\infty \leq 1\}, \tag{1}$$

where $\|\cdot\|_\infty$ denotes the infinity norm.

The operators $\oplus$ and $\odot$ denote the Minkowski sum and the linear image, respectively. Meanwhile, the following properties hold:

$$\langle p_1, H_1\rangle \oplus \langle p_2, H_2\rangle = \langle p_1 + p_2, [H_1 \;\; H_2]\rangle, \tag{2a}$$

$$L \odot \langle p, H\rangle = \langle Lp, LH\rangle, \tag{2b}$$

with L is an arbitrary matrix of appropriate dimension.

The interval hull $rs(H) \in \mathbb{R}^{n\times n}$ of the zonotope $\langle p, H\rangle$ can be regarded as an aligned minimum box. Therefore, the following condition holds:

$$\langle p, H\rangle \subset \langle p, rs(H)\rangle, \tag{3}$$

where $rs(H)$ is a diagonal matrix such that $rs(H)_{i,i} = \sum_{j=1}^{m} |H_{i,j}|$ for $i = 1, 2, \ldots, n$.

The reduction operator for the zonotope firstly proposed in (Combastel, 2005) is denoted as $\downarrow_\ell (\cdot)$, where ℓ specifies the maximum number of column of segment matrix H after reduction. Thus, $\downarrow_\ell (H)$ is computed as follows:

- Sort the column of segment matrix H on decreasing order:
$$\downarrow (H) = [h_1, h_2, \ldots, h_m], \;\; \|h_j\|_2 \geq \|h_{j+1}\|_2,$$
where $\|\cdot\|_2$ denotes the 2-norm.
- Keep the first ℓ-column of $\downarrow (H)$ and enclose the set $H_<$ generated by remaining columns into a smallest box (interval hull) computed by using $rs(\cdot)$:
$$\begin{aligned}&\textbf{If } m \leq \ell \textbf{ then } \downarrow_\ell (H) = \downarrow (H),\\ &\textbf{Else } \downarrow_\ell (H) = [H_>, rs(H_<)] \in \mathbb{R}^{n\times \ell},\\ &\quad H_> = [h_1, \ldots, h_\ell], \; H_< = [h_{\ell+1}, \ldots, h_m],\\ &\quad rs(H_<) = \sum_{j=\ell+1}^{m} |h_j|.\end{aligned}$$

Definition 1. (Covariation). The covariation of a zonotope $\langle p, H\rangle$ is defined as $\mathrm{cov}(\langle p, H\rangle) = HH^T$, where H^T denotes the transpose matrix of H.

Definition 2. (F-radius). The F-radius of a zonotope $\langle p, H\rangle$ is defined by the Frobenius norm of H as $\|H\|_F = \mathrm{tr}\left(HH^T\right) = \mathrm{tr}\left(H^T H\right)$, where $\mathrm{tr}(\cdot)$ denotes the trace of a matrix.

2.2 Matrix Calculus

X, A, B and C denote matrices with suitable dimensions. Some necessary calculus properties regarding the matrix trace are expressed as follows:

$$\mathrm{tr}(A) = \mathrm{tr}\left(A^T\right), \tag{4}$$

$$\mathrm{tr}(AB) = \mathrm{tr}\left(B^T A^T\right), \tag{5}$$

$$\mathrm{tr}(ABC) = \mathrm{tr}(BCA) = \mathrm{tr}(CAB), \tag{6}$$

$$\frac{\partial}{\partial X}\mathrm{tr}(AX^T B) = A^T B^T, \tag{7}$$

$$\frac{\partial}{\partial X}\mathrm{tr}(AXBX^T C) = BX^T CA + B^T X^T A^T C^T. \tag{8}$$

3. PROBLEM STATEMENT

Consider the following discrete-time descriptor system with additive actuator fault

$$Ex(k+1) = Ax(k) + Bu(k) + Fn(k) + D_w\omega(k), \tag{9a}$$

$$y(k) = Cx(k) + D_v\upsilon(k), \tag{9b}$$

where $x \in \mathbb{R}^{n_x}$ and $u \in \mathbb{R}^{n_u}$ denote the system state vector and control vector, $\omega \in \mathbb{R}^{n_w}$ and $\upsilon \in \mathbb{R}^{n_v}$ denote the unknown-but-bounded system disturbance vector and measurement noise vector, $y \in \mathbb{R}^m$ denotes the measurement output vector, $n \in \mathbb{R}^q$ denote the additive actuator-fault vector. A, B, C, D_w and D_v are known system matrices of appropriate dimension. $F \in \mathbb{R}^{n_x\times q}$ is the known fault distribution matrix that represents the directions of the fault vector. In terms of the general descriptor systems, $E \in \mathbb{R}^{n_x\times n_x}$ might be a singular matrix, that is $\mathrm{rank}(E) = r \leq n_x$.

The following assumptions are considered along the paper:

Assumption 1. The unknown system disturbance vector, measurement noise vector and initial system state vector in (9) are assumed to be bounded in known zonotopic sets as follows

$$\omega_k \in \langle 0, I_{n_w}\rangle, \upsilon_k \in \langle 0, I_{n_v}\rangle, x_0 \in \langle p_0, H_0\rangle.$$

Assumption 2. For the descriptor system in (9), it is assumed that $\text{rank}(C) = m$ and $\text{rank}(F) = q$ with $q < m$. Moreover, matrices E and C satisfy the following condition (Dai, 1989)

$$\text{rank} \begin{bmatrix} E \\ C \end{bmatrix} = n_x. \tag{10}$$

The aim of this paper is to design a set-based fault estimation filter to estimate the fault magnitudes. To this end, we propose the following filter

$$\begin{cases} z(k+1) = TA\hat{x}(k) + TBu(k) + G_k\left(y(k) - C\hat{x}(k)\right) \\ \hat{x}(k) = z(k) + Ny(k) \\ r(k) = M\left(y(k) - C\hat{x}(k)\right) \end{cases} \tag{11}$$

where $\hat{x} \in \mathbb{R}^{n_x}$ and $z \in \mathbb{R}^{n_x}$ respectively denote the state observation vector and intermediate state vector, $r \in \mathbb{R}^q$ is an enhanced residual vector used for fault estimation, and $T \in \mathbb{R}^{n_x \times n_x}$, $N \in \mathbb{R}^{n_x \times m}$, $G_k \in \mathbb{R}^{n_x \times m}$, and $M \in \mathbb{R}^{q \times m}$ are matrices to be designed. Firstly, matrices T and N should be chosen such that (Wang et al., 2015a)

$$TE + NC = I_{n_x}. \tag{12}$$

Based on the equation constraint in (12), T and N can be determined by

$$[T \;\; N] = \begin{bmatrix} E \\ C \end{bmatrix}^{\dagger} + S\left(I_{n_x+m} - \begin{bmatrix} E \\ C \end{bmatrix}\begin{bmatrix} E \\ C \end{bmatrix}^{\dagger}\right) \tag{13}$$

where $\begin{bmatrix} E \\ C \end{bmatrix}^{\dagger}$ denotes the Moore-Penrose pseudoinverse matrix of $\begin{bmatrix} E \\ C \end{bmatrix}$. $S \in \mathbb{R}^{n_x \times (n_x+m)}$ is a freely chosen matrix that provides the design degrees of freedom.

After T and N are determined, the optimal parameterized filter gain G_k and the weighting matrix M will be designed such that the proposed filter (11) provides fault estimation results in a zonotopic framework. The minimization criterion of the corresponding zonotope is also provided.

4. ZONOTOPIC FAULT ESTIMATION FILTER DESIGN

4.1 Structural Residual

Before the filter design, it is necessary to introduce the concept of fault detectability indexes and matrix (Liu and Si, 1997; Keller, 1999).

For better illustration, we first write F and $n(k)$ as follows

$$F = [f_1, f_2, \cdots, f_q] \tag{14}$$

$$n(k) = [n_1(k), n_2(k), \cdots, n_q(k)]^T \tag{15}$$

Definition 3. (Fault detectability indexes). The discrete-time descriptor system (9) is said to have fault detectability indexes $\rho = \{\rho_1, \rho_2, \ldots, \rho_q\}$ if

$$\rho_i = \min\left\{\sigma \mid C(TA)^{\sigma-1}Tf_i \neq 0, i = 1, 2, \ldots\right\}. \tag{16}$$

and $s = \max\{\rho_1, \rho_2, \ldots, \rho_q\}$ denotes the maximum of fault detectability indexes.

Definition 4. (Fault detectability matrix). With the fault detectability indexes of the descriptor system (9) defined as $\rho = \{\rho_1, \rho_2, \ldots, \rho_q\}$, the fault detectability matrix is given by

$$\Upsilon = C\Psi, \tag{17}$$

with

$$\Psi = \left[(TA)^{\rho_1-1}Tf_1 \;\; (TA)^{\rho_2-1}Tf_2 \;\; \cdots (TA)^{\rho_q-1}Tf_q\right]. \tag{18}$$

To design observer (11), we define the state estimation error $e(k)$ and output estimation error $\varepsilon(k)$ as

$$e(k) = x(k) - \hat{x}(k), \tag{19}$$

$$\varepsilon(k) = y(k) - C\hat{x}(k). \tag{20}$$

By using (9), (11), and (12), we obtain the following error dynamic system

$$\begin{cases} e(k+1) = (TA - G_kC)e(k) + TFn(k) \\ \qquad\qquad +TD_w\omega(k) - G_kD_v\upsilon(k) \\ \qquad\qquad -ND_v\upsilon(k+1) \\ \varepsilon(k) = Ce(k) + D_v\upsilon(k) \end{cases} \tag{21}$$

To facilitate the design of fault estimation filter (11), the state estimation error $e(k)$ and output estimation error $\varepsilon(k)$ are split as follows

$$e(k) = e_f(k) + e_w(k) \tag{22}$$

$$\varepsilon(k) = \varepsilon_f(k) + \varepsilon_w(k) \tag{23}$$

where $e_f(k)$ and $e_w(k)$ are respectively obtained by

$$\begin{cases} e_f(k+1) = (TA - G_kC)e_f(k) + TFn(k) \\ \varepsilon_f(k) = Ce_f(k) \end{cases} \tag{24}$$

and

$$\begin{cases} e_w(k+1) = (TA - G_kC)e_w(k) + TD_w\omega(k) \\ \qquad\qquad -G_kD_v\upsilon(k) - ND_v\upsilon(k+1) \\ \varepsilon_w(k) = Ce_w(k) + D_v\upsilon(k) \end{cases} \tag{25}$$

with the following initial conditions

$$e_f(0) = 0, \; e_w(0) = e(0). \tag{26}$$

Theorem 1. The effect of the faults on the output estimation error $\varepsilon(k)$ can be expressed as

$$\varepsilon(k) = \varepsilon_w(k) + C\Psi\left[n_1^T(k-\rho_1) \;\; \cdots \;\; n_q^T(k-\rho_q)\right]^T \tag{27}$$

if there exists a matrix G_k satisfying the following condition

$$(TA - G_kC)\Psi = 0. \tag{28}$$

Proof. By using (24), we can derive that

$$\begin{aligned} \varepsilon_f(k) = &\; C\Phi_ke_f(0) + C\Phi_{k-1}TFn(0) + \cdots \\ &+ C\Phi_1TFn(k-1) \end{aligned} \tag{29}$$

where

$$\Phi_k = \prod_{j=1}^{k}(TA - G_jC) \tag{30}$$

Along the same lines of the proof of Theorem 1 in (Wang et al. 2015), we can obtain that

$$C\Phi_jTf_i = \begin{cases} C(TA)^{\rho_i-1}Tf_i & j = \rho_i \\ 0 & j \neq \rho_i \end{cases} \tag{31}$$

Substituting (31) into (29) yields

$$\begin{aligned} \varepsilon_f(k) &= C\Phi_ke_f(0) + C(TA)^{\rho_1-1}Tf_1n_1(k-\rho_1) \\ &\quad + \cdots + C(TA)^{\rho_q-1}Tf_qn_q(k-\rho_q) \\ &= C\Phi_ke_f(0) + C\Psi\begin{bmatrix} n_1(k-\rho_1) \\ \vdots \\ n_q(k-\rho_q) \end{bmatrix} \end{aligned} \tag{32}$$

Since $e_f(0) = 0$, equation (32) follows that

$$\varepsilon_f(k) = C\Psi\begin{bmatrix} n_1(k-\rho_1) \\ \vdots \\ n_q(k-\rho_q) \end{bmatrix} \tag{33}$$

By substituting (33) into (23), we obtain (27). □

Based on *Theorem 1*, we have the following corollary.

Corollary 1. Under $\mathrm{rank}(C\Psi) = q$, the enhanced residual $r(k)$ can be expressed as

$$r(k) = M\varepsilon_w(k) + \left[n_1^T(k-\rho_1) \ \cdots \ n_q^T(k-\rho_q)\right]^T \quad (34)$$

if there exists a matrix G_k satisfying (28) and M is chosen as

$$M = (C\Psi)^\dagger \quad (35)$$

Proof. It is shown in Theorem 1 that $\varepsilon(k)$ can be expressed as (27) if G_k satisfies (28). Moreover, if $\mathrm{rank}(C\Psi) = q$, we have

$$MC\Psi = I_q \quad (36)$$

By using the relation $r(k) = M\varepsilon(k)$ and (27), we obtain (34). □

4.2 Zonotopic Error Dynamics

From (22)-(26), it is known that $\varepsilon_f(k)$ is only driven by the faults while $\varepsilon_w(k)$ characterizes the effect of uncertainties on the residual. In order to guarantee the estimation accuracy, it is necessary to attenuate the effect of uncertainties in the filter. To this end, the system errors are analyzed in a zonotopic framework, which will serve as the basis of the filter gain design in the sequel.

Theorem 2. (Zonotopic Error Dynamics). Given the descriptor system with additive actuator faults (9), considering that *Assumption 1 and 2* hold, the partial state estimation error $e_w(k)$ is bounded into the zonotope $\langle 0, R_k\rangle$ as the prior information, then there exists the suitable time-varying filter gain G_k such that $e_w(k) \in \langle 0, R_{k+1}\rangle$ can be propagated by

$$R_{k+1} = \left[(TA - G_kC)\bar{R}_k \ TD_w \ -G_kD_v \ -ND_v\right], \quad (37)$$

with

$$\bar{R}_k = \downarrow_\ell (R_k). \quad (38)$$

Then, the output estimation error on system disturbances is bounded in the zonotope $\varepsilon_w(k+1) \in \langle 0, R_{k+1}^{\varepsilon_w}\rangle$ with

$$R_{k+1}^{\varepsilon_w} = [CR_{k+1} \ \ D_v]. \quad (39)$$

Proof. According to the error dynamics of $e_w(k+1)$ in (25) with $\omega_k \in \langle 0, I_{n_w}\rangle$ and $\upsilon_k \in \langle 0, I_{n_v}\rangle$, it implies

$$\begin{aligned} e_w(k+1) &\in \langle 0, R_{k+1}\rangle \\ &= \Big((TA - G_kC) \odot \langle 0, \bar{R}_k\rangle\Big) \\ &\quad \oplus \Big(TD_w \odot \langle 0, I_{n_w}\rangle\Big) \\ &\quad \oplus \Big(-G_kD_v \odot \langle 0, I_{n_v}\rangle\Big) \\ &\quad \oplus \Big(-ND_v \odot \langle 0, I_{n_v}\rangle\Big). \end{aligned} \quad (40)$$

By using the zonotope properties in (2), (37) can be obtained.

Similarly, from (25), it can also be derived

$$\begin{aligned} \varepsilon_w(k+1) &\in \langle 0, R_{k+1}^{\varepsilon_w}\rangle \\ &= \Big(C \odot \langle 0, R_{k+1}\rangle\Big) \oplus \Big(D_v \odot \langle 0, I_{n_v}\rangle\Big). \end{aligned} \quad (41)$$

and then $R_{k+1}^{\varepsilon_w} = [CR_{k+1} \ \ D_v]$ is obtained. □

Corollary 2. With the suitable time-varying filter gain G_k and the state error zonotope found in *Theorem 2*, the residual error $e_r(k+1) = M\varepsilon_w(k+1)$ for estimating the fault magnitude can be also bounded in a zonotope as $e_r(k+1) \in \langle 0, R_{k+1}^r\rangle$ with

$$R_{k+1}^r = \left[MC\bar{R}_{k+1} \ MD_v\right]. \quad (42)$$

Proof. Considering the fault detectability indexes and G_k satisfying the algebraic condition (28), the effect of the faults has been divided into two parts. According to (34), the residual error $e_r(k+1)$ can be written as

$$\begin{aligned} e_r(k+1) &= r(k+1) \\ &\quad - \left[n_1^T(k+1-\rho_1) \ \cdots \ n_q^T(k+1-\rho_q)\right]^T \\ &= M\varepsilon_w(k+1) \end{aligned} \quad (43)$$

From the output error zonotope $\varepsilon_w(k+1) \in \langle 0, R_{k+1}^{\varepsilon_w}\rangle$, the residual error zonotope can be found by applying the zonotope operation properties in (2) as follows:

$$\begin{aligned} e_r(k+1) &\in \langle 0, R_{k+1}^r\rangle \\ &= M\langle 0, R_{k+1}^{\varepsilon_w}\rangle. \end{aligned} \quad (44)$$

Then, the generator matrix R_{k+1}^r can be formulated as

$$R_{k+1}^r = MR_{k+1}^{\varepsilon_w} = \left[MC\bar{R}_{k+1} \ MD_v\right]. \quad (45)$$

Hence, the proof is completed. □

As a result, the fault magnitudes can be estimated in a zonotopic set defined by $\langle c_{k+1}^r, R_{k+1}^r\rangle = r(k+1) \oplus \langle 0, R_{k+1}^r\rangle$ and meanwhile the individual bound for each fault can be obtained by using the interval hull $\langle c_{k+1}^r, R_{k+1}^r\rangle \subset \langle c_{k+1}^r, rs(R_{k+1}^r)\rangle$.

4.3 Optimal Filter Gain

The criteria for finding the optimal fault estimation filter gain G_k are

- G_k should satisfy the algebraic condition (28).
- G_k should minimize the state estimation error on system disturbances $e_w(k+1)$ or the output estimation error $\varepsilon_w(k+1)$.

Because the state and output estimation errors are bounded using the zonotopes, the F-radius of the zonotope is considered to be the minimization criterion. The optimal filter gain can be obtained by applying the following theorem.

Theorem 3. (Optimal Filter Gain). Consider the fault detectability matrix Υ with $rank(\Upsilon) = q$, the time-varying fault estimation filter gain G_k can be computed in a parameterized form:

$$G_k = \Phi M + \bar{G}_k\Omega, \quad (46)$$

with

$$\Phi = TA\Psi, M = \Upsilon^\dagger, \Omega = \alpha\left(I_m - \Upsilon M\right), \quad (47)$$

where $\alpha \in \mathbb{R}^{(m-q)\times m}$ is an arbitrary matrix such that Ω has full rank. Moreover, $\bar{G}_k \in \mathbb{R}^{n_x\times(m-q)}$ is a reduced-order filter gain that can be computed iteratively as follows:

$$\bar{G}_k = \tilde{L}_k\tilde{S}_k^{-1}, \quad (48)$$

$$\tilde{L}_k = (TA - \Phi MC)\bar{P}_kC^T\Omega^T - \Phi MV\Omega^T, \quad (49)$$

$$\tilde{S}_k = \Omega\left(C\bar{P}_kC^T + V\right)\Omega^T, \quad (50)$$

with

$$\bar{P}_k = \bar{R}_k \bar{R}_k^T, V = D_v D_v^T. \tag{51}$$

Proof. From the definition of the fault detectability matrix, $rank(\Upsilon) = q$ implies that two matrices M and Ω are chosen such that

$$M\Upsilon = I_q, \Omega\Upsilon = 0. \tag{52}$$

and then from the algebraic condition (28), the following equation can be satisfied:

$$\begin{aligned}(TA - G_k C)\Psi &= \Big(TA - \left(\Phi M + \bar{G}_k \Omega\right) C\Big)\Psi \\ &= TA\Psi - TA\Psi MC\Psi - \bar{G}_k \Omega C\Psi \\ &= TA\Psi - TA\Psi M\Upsilon - \bar{G}_k \Omega\Upsilon = 0. \end{aligned} \tag{53}$$

Hence, the parameterized filter gain G_k satisfies the first criterion. On the other hand, the reduced-order filter gain can be computed by minimizing the size of the zonotopic set $e_w(k+1) \in \langle 0, R_{k+1}\rangle$ of the output estimation error considering system disturbances. The covariance matrix of $\langle 0, R_{k+1}\rangle$ is defined as

$$P_{k+1} = R_{k+1} R_{k+1}^T. \tag{54}$$

Note that P_{k+1} is a symmetric matrix. Therefore, the F-radius of $\langle 0, R_{k+1}\rangle$ can be formulated as

$$J = \mathrm{tr}\left(P_{k+1}\right). \tag{55}$$

Then, the optimal reduced-order filter gain $\bar{G}_k$ can be found by $\frac{\partial \mathrm{tr}(P_{k+1})}{\partial \bar{G}_k} = 0$. Set $\tilde{L}_k$, $\tilde{S}_k$, $\bar{P}_k$ and V as in (49), (50) and (51) we have

$$\frac{\partial \mathrm{tr}}{\partial \bar{G}_k}\left(\bar{G}_k \tilde{S}_k \bar{G}_k^T\right) - 2\frac{\partial \mathrm{tr}}{\partial \bar{G}_k}\left(\tilde{L}_k \bar{G}_k^T\right) = 0. \tag{56}$$

By means of the matrix calculus in (4), (56) can be simplified as

$$\tilde{S}_k \bar{G}_k^T + \tilde{S}_k^T \bar{G}_k^T - 2\tilde{L}_k^T = 0. \tag{57}$$

Considering that $\tilde{S}_k$ is also a symmetric matrix. Hence, (48) is obtained. Thus, the proof is completed. □

By applying *Theorem 3*, the optimal parameterized filter gain G_k can be obtained with a specific iterative approach. The reduced-order filter gain $\bar{G}_k$ is computed by minimizing the zonotope $\langle 0, R_{k+1}\rangle$. Thus, the zonotope of the residual vector r for estimation fault vector n can be obtained.

4.4 Zonotopic Fault Estimation Algorithm

As a result, the overall procedure of the zonotopic fault estimation can be summarized in **Algorithm 1**.

5. ILLUSTRATIVE EXAMPLE

Motivated by the example in Wang et al. (2015b), the system matrices of the discrete-time descriptor system in (9) are given as follows:

$$E = \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0 \end{bmatrix}, A = \begin{bmatrix} 0.9 & 0.005 & -0.095 & 0\\ 0.005 & 0.995 & 0.0997 & 0\\ 0.095 & -0.0997 & 0.99 & 0\\ 1 & 0 & 1 & 1 \end{bmatrix},$$

$$B = F = [f_1\ \ f_2] = \begin{bmatrix} 0.1 & 0\\ 1 & 1\\ -0.1 & 1\\ -1 & 0 \end{bmatrix}, D_w = \begin{bmatrix} 0.01\\0.01\\0.01\\0 \end{bmatrix},$$

Algorithm 1 Zonotopic fault estimation algorithm

Data: Given the system matrices E, A, B, C, D_w, D_v and F and the initial state bounded in $x_0 \in \langle p_0, H_0\rangle$;
Solve the equation (12) obtaining T and N;
Compute the fault detectability indexes ρ_i;
Compute the fault detectability matrix $\Upsilon = C\Psi$;
$\Phi \longleftarrow TA\Psi$;
$M \longleftarrow \Upsilon^{\dagger}$;
$\Omega \longleftarrow \alpha\left(I_m - \Upsilon M\right)$ with the selection of α by guaranteeing the full rank Ω;
for *k=1:t_{end}* **do**
  Compute the optimal reduced-order gain $\bar{G}_k$ by using (48)-(50);
  Obtain the parameterized filter gain G_k;
  Compute the residual zonotope $\langle c_{k+1}^r, R_{k+1}^r\rangle = r(k+1) \oplus \langle 0, R_{k+1}^r\rangle$;
  Obtain the fault estimation and its bounds from $\langle c_{k+1}^r, rs(R_{k+1}^r)\rangle$;
end

$$D_v = \begin{bmatrix} 0.01 & 0 & 0\\ 0 & 0.01 & 0\\ 0 & 0 & 0.01 \end{bmatrix}, C = \begin{bmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix}.$$

The matrices T and N satisfying the condition (12) are chosen as

$$T = \begin{bmatrix} 1&0&0&0\\0&0.5&0&0\\0&0&0&1 \end{bmatrix}, N = \begin{bmatrix} 0&0&0\\0.5&0&0\\0&0.5&0\\0&0&1 \end{bmatrix},$$

and the matrix S is selected in the general solution (13) by

$$S = \begin{bmatrix} 1&0&0&0&0&0&0\\0&1&0&0&1&0&0\\0&0&1&0&0&1&0\\0&0&0&1&0&0&1 \end{bmatrix}.$$

Since $rank(F) = rank(CF) = 2$, then $CTf_1 \neq 0$ and $CTf_2 \neq 0$. So the fault detectability indexes are $\rho_1 = 1$ and $\rho_2 = 1$ and the fault detectability matrix is given as

$$\Upsilon = C\Psi = \begin{bmatrix} 0.5 & 0.5\\ -0.05 & 0.5\\ -1 & 0 \end{bmatrix},$$

with

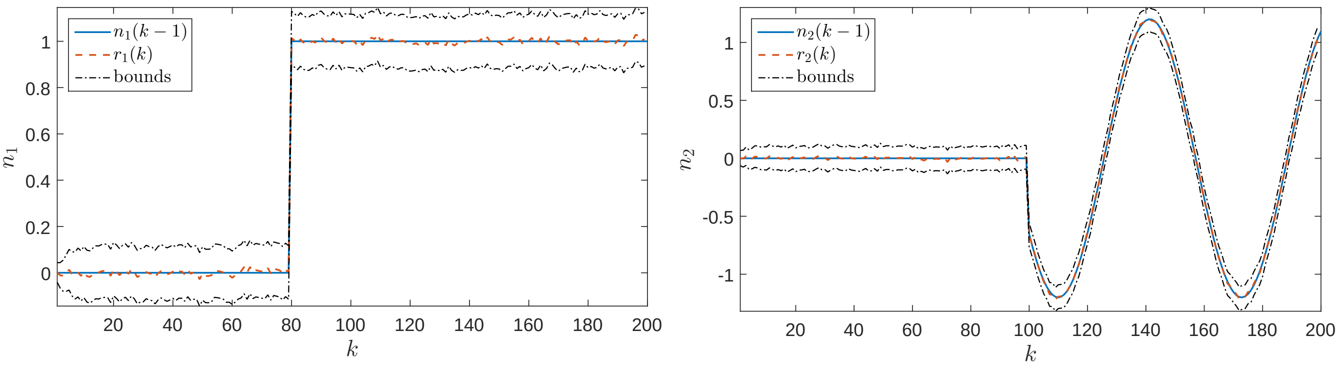
$$\Psi = [Tf_1\ \ Tf_2] = \begin{bmatrix} 0.1 & 0\\ 0.5 & 0.5\\ -0.05 & 0.5\\ -1 & 0 \end{bmatrix}.$$

Assume that there are two possible actuator faults:

$$n_1(k) = \begin{cases} 0 & k < 80\\ 1 & k \geq 80 \end{cases},$$

$$n_2(k) = \begin{cases} 0 & k < 100\\ 1.2\sin(0.1k) & k \geq 100 \end{cases}.$$

The simulation results by applying the zonotopic fault estimation filter for two faults are plotted in Fig. 1. Because of $\rho_1 = 1$ and $\rho_2 = 1$, there is one-step delay for estimating the faults n_1 and n_2, respectively. Under the assumption of unknown-but-bounded system disturbances and measurement noise, the center of each fault estimation has a small difference than the actual value and the worst-case bounds of each fault is given in the estimation interval (black dashed lines).



(a) n_1 (b) n_2

Fig. 1. Actuator-fault estimation results

6. CONCLUSION

In this paper, a zonotopic fault estimation filter for uncertain discrete-time descriptor systems is proposed. Based on the fault detectability indexes and matrix, the actuator faults can be estimated and the worst-case fault estimation can be also obtained in a zonotopic set. The optimal filter gain is computed based on the F-radius minimization criterion. Finally, the proposed method is demonstrated by numerical simulations.

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