Robust Periodic Economic Predictive Control based on Interval Arithmetic for Water Distribution Networks

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Abstract: This paper addresses a robust periodic economic model predictive control (EMPC) based on interval arithmetic with unknown-but-bounded additive disturbances for the management of water distribution networks (WDNs). The system constraints in presence of system disturbances are tightened along the prediction horizon by means of the proposed interval arithmetic and considering that system variables in the WDN model are also subject to some algebraic equations. These algebraic equations should be satisfied when the disturbances have effects on those system variables. The EMPC controller is designed with adding the terminal state constraint. The periodically optimal steady states are obtained by employing a periodic EMPC planner with the nominal model. This periodic steady states are subsequently used to be terminal states. The on-line robust constraint satisfactions are implemented into the robust periodic EMPC controller. Finally, the proposed control strategy is verified using a case study.

Keywords: Economic model predictive control, interval arithmetic, robust constraint satisfactions, algebraic equations, water distribution networks.

1. INTRODUCTION

Model predictive control (MPC) offers a flexible and effective framework for addressing engineering requirements in the controller design (Rawlings and Mayne, 2009; Mayne, 2014). MPC is designed by solving an optimization problem to minimize a multi-objective function and satisfy system constraints. Robust MPC is one of well-known research fields in the MPC framework (Pereira et al., 2016). In order to guarantee the recursive feasibility of a robust MPC controller, further system states in presence of unknown disturbances need to be bounded in a desirable set and corresponding control inputs should be adapted accordingly. Moreover, optimal control actions can be found in a compatible input set. On the other hand, economic MPC (EMPC) has attracted a lot of attention in recent years, which differs from the classical MPC strategy based on tracking a given reference by mainly reaching economic performance of the controlled system. The optimal control actions of EMPC are often found by means of economic cost function that measures the economic performance of the control systems. Hence, the stage cost of EMPC is usually not in a quadratic form involving states and controls but in a time-varying manner depending on a exogenous cost signal.

Interval analysis and set-based approaches have been widely considered for state estimation and fault diagnosis during the last decade for many applications (see, e.g. Jaulin et al. (2001); Puig et al. (2001)). Considering that a system model has finite constraints and unknown-but-bounded uncertainties, constraint satisfaction can be implemented using a proper interval arithmetic, where all the system variables are bounded in the tightened subsets. The interval-based constraint satisfaction problem can be usually used for robust fault diagnosis (Tornil-Sin et al., 2014). In this work, the tightened constraints are applied into the EMPC prediction loop for assessing the robust constraint satisfactions on both system states and control inputs.

Water distribution networks (WDNs) are one of critical infrastructures in modern cities, where a number of people are living and working. Satisfying the water demands required by all the customers is significantly important to be guaranteed. Meanwhile, the economic cost of the operations is needed to be taken into account. For that reason, EMPC is a quite suitable control strategy for WDNs (see, e.g. Ocampo-Martinez et al. (2013), Wang et al. (2016)). The linear model of a WDN is built using the mass balance at storage tanks and nodes. In general, the algebraic equations are usually used in this model relating system states, control inputs and water demand variables. On the other hand, the system disturbances are usually present in the network mainly related to the water demands. Besides, in terms of a predictive strategy, water demand forecasts are required to be computed. The unknown system disturbances should be adapted in order to guarantee the recursive feasibility and robustness of the EMPC controller.

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This paper proposes a robust periodic EMPC strategy based on interval arithmetic for operational management of WDNs. The linear control-oriented model of a WDN is utilized. The nominal steady and reachable trajectories are computed by a EMPC planner. The constraints on system states and control inputs along the prediction horizon are obtained using the interval arithmetic with a sequence of uncertainty propagations. The closed-loop convergence is guaranteed by employing the terminal equality constraints as the periodically time-varying steady states found by the EMPC planner. The tightened constraint on system states and control inputs can be obtained by using the on-line robust constraint-satisfaction method based on the interval arithmetic. Finally, the proposed strategy is tested in a case study to show its effectiveness.

Notation: M^t denotes the transpose matrix of the matrix M. $\|\cdot\|_1$ denotes the 1-norm function and $\|\cdot\|_{2,W}$ denotes the W-weighted 2-norm function. $z_{\langle k\rangle_T}$ denotes z_i at time instant i, where i is equal to k mod T.

2. PROBLEM STATEMENT

Consider the discrete-time linear WDN model as follows (Ocampo-Martinez et al., 2013):

$$\bar{\mathbf{x}}_{k+1} = \mathbf{A}\bar{\mathbf{x}}_k + \mathbf{B}\mathbf{u}_k + \mathbf{B}_d\mathbf{d}_k, \tag{1a}$$
$$0 = \mathbf{E}_x\bar{\mathbf{x}}_k + \mathbf{E}_u\mathbf{u}_k + \mathbf{E}_d\mathbf{d}_k, \tag{1b}$$

where $\bar{\mathbf{x}}_k \in \mathbb{R}^{n_x}$ denotes the vector of nominal water volumes of the storage tanks as system states at time instant k, $\mathbf{u}_k \in \mathbb{R}^{n_u}$ denotes the vector of flows of actuators (pumps and valves) as control inputs at time instant k, $\mathbf{d}_k \in \mathbb{R}^{n_d}$ denotes the vector of water demands as system disturbances at time instant k. $\mathbf{A}$, $\mathbf{B}$, $\mathbf{B}_d$, $\mathbf{E}_x$, $\mathbf{E}_u$ and $\mathbf{E}_d$ are linear system matrices of appropriate dimensions. (3a) describes the system dynamics described by the mass balance at storage tanks and (3b) presents the static relations between some system variables at non-storage nodes.

Usually, the water demand signal $\mathbf{d}_k$ is composed of two parts: deterministic part denoted by $\bar{\mathbf{d}}_k$ and stochastic part denoted by $\mathbf{w}_k$. Depending on the engineering experience, the deterministic demand $\bar{\mathbf{d}}_k$ is assumed to be known with the periodically time-varying behavior of a period T, that is $\bar{\mathbf{d}}_k = \bar{\mathbf{d}}_{k+T}$.

Assumption 1. The stochastic demand $\mathbf{w}_k$ for $\forall k \in \mathbb{N}_+$ are unknown in practice but bounded in a known set, which can be formulated as

$$\mathbf{w}_k \in \mathcal{W}. \tag{2}$$

Therefore, the WDN model (1) can be reformulated by

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{B}_d(\bar{\mathbf{d}}_k + \mathbf{w}_k), \tag{3a}$$
$$0 = \mathbf{E}_x\mathbf{x}_k + \mathbf{E}_u\mathbf{u}_k + \mathbf{E}_d(\bar{\mathbf{d}}_k + \mathbf{w}_k), \tag{3b}$$

where $\mathbf{x}_k \in \mathbb{R}^{n_x}$ denotes the vector of uncertain system states. Furthermore, according to the limits of storage tanks and capacity of actuators inside the WDN, system variables $\mathbf{x}_k$ and $\mathbf{u}_k$ are subject to their physical constraints as follows:

$$\mathbf{x}_k \in \mathcal{X}, \;\; \mathbf{u}_k \in \mathcal{U}, \;\; \forall k \in \mathbb{N}_+. \tag{4}$$

In this work, a EMPC controller is designed to operate a WDN with a periodic operation. The economic performance is measured by a parameter-varying economic cost function $\ell(k, \mathbf{p}_k, \mathbf{u}_k)$. The exogenous parameter $\mathbf{p}_k$ is usually regarded as a price with a periodically time-varying behavior.

Assumption 2. $\mathbf{p}_k$ are known with a period of T. Hence, the following condition holds:

$$\mathbf{p}_k = \mathbf{p}_{k+T}. \tag{5}$$

Specifically, the economic cost function may be defined to measure economic costs by

$$\ell_e(k, \mathbf{p}_k, \mathbf{u}_k) \triangleq \left\|\mathbf{p}_k^t\mathbf{u}_k\right\|_1. \tag{6}$$

If there are no disturbances in the system model (1), the optimal control action can be selected by solving the following optimization problem:

$$(\bar{\mathbf{x}}^s, \mathbf{u}^s) = \arg\min \; L_e(k, \mathbf{p}, \mathbf{u}) \triangleq \sum_{i=0}^{T-1} \ell_e(k+i, \mathbf{p}_k, \mathbf{u}_k), \tag{7a}$$

subject to

$$\bar{\mathbf{x}}_{k+i+1|k} = \mathbf{A}\bar{\mathbf{x}}_{k+i|k} + \mathbf{B}\mathbf{u}_{k+i|k} + \mathbf{B}_d\bar{\mathbf{d}}_{k+i}, \tag{7b}$$
$$0 = \mathbf{E}_x\bar{\mathbf{x}}_{k+i|k} + \mathbf{E}_u\mathbf{u}_{k+i|k} + \mathbf{E}_d\bar{\mathbf{d}}_{k+i}, \tag{7c}$$
$$\bar{\mathbf{x}}_{k+i+1|k} \in \mathcal{X}, \tag{7d}$$
$$\mathbf{u}_{k+i|k} \in \mathcal{U}, \tag{7e}$$
$$\bar{\mathbf{x}}_{k+T|k} = \mathbf{x}^p_{\langle k\rangle_T}, \tag{7f}$$
$$\bar{\mathbf{x}}_{k|k} = \mathbf{x}_k. \tag{7g}$$

The feasible solutions of the optimization problem (7) are denoted by $\mathbf{x}^s$ and $\mathbf{u}^s$. The terminal equality constraint (7f) is used to achieve the periodic operation, which can be reformulated as follows:

$$\bar{\mathbf{x}}^p = \left(\bar{\mathbf{x}}^p_{\langle 1\rangle_T}, \ldots, \bar{\mathbf{x}}^p_{\langle T\rangle_T}\right)^t, \tag{8a}$$
$$\mathbf{u}^p = \left(\mathbf{u}^p_{\langle 0\rangle_T}, \ldots, \mathbf{u}^p_{\langle T-1\rangle_T}\right)^t, \tag{8b}$$

where $\bar{\mathbf{x}}^p_{\langle i\rangle_T}$ and $\mathbf{u}^p_{\langle i-1\rangle_T}$ for $\forall i = 1, \ldots, T$ satisfy the system model in (1).

However, the feasible solutions of the optimization problem (7) might be violated and then be infeasible when the uncertain system (3) is included in the closed loop due to that the optimal states and inputs are selected taking into account physical constraints (7d) and (7e) with robust constraint satisfaction.

3. ROBUST CONSTRAINT SATISFACTIONS USING INTERVAL ARITHMETIC

In this section, the tightened constraints of (7d) and (7e) is computed by using the proposed interval arithmetic with a prior trajectory found by the optimization problem (7) as an on-line planner. Then, the tightened time-varying robust sets $\widetilde{\mathcal{U}}_i$ and $\widetilde{\mathcal{X}}_i$ will be implemented in the EMPC controller design.

Given the initial state $\widetilde{\mathcal{X}}_0 = \mathbf{x}_k$, the optimal nominal states $\bar{\mathbf{x}}^s$ and inputs $\mathbf{u}^s$ from (8) as the on-line planner, the goal is to compute the sets $\widetilde{\mathcal{U}}_i$ and $\widetilde{\mathcal{X}}_{i+1}$ for $i = 0, \ldots H_p - 1$ with $H_p \leq T$ using interval arithmetic in such a way these sets are as close as possible to feasible solutions $\mathbf{u}^s$ and $\bar{\mathbf{x}}^s$ and guarantee the feasibility of (3) considering demand uncertainties in (2) and its propagation along the prediction horizon by (3a).

Assumption 3. With the purpose of tackling the problem of considering the uncertainty in the constrained system described by (3), it will be considered that not all the inputs are involved in the algebraic equation (3b), i.e:

$$\mathbf{E}_u = \left(\mathbf{0}, \mathbf{E}_u^H\right), \tag{9}$$

where $\mathbf{0} \in \mathbb{R}^{n_v \times n_L}$ is a zero matrix and $\mathbf{E}_u^H \in \mathbb{R}^{n_v \times n_H}$ is a partitioned matrix from $\mathbf{E}_u$ with $n_L + n_H = n_u$.

Therefore, matrix $\mathbf{B}^L$ in (3a) will be decomposed in two parts:

$$\mathbf{B} = \left(\mathbf{B}^L, \mathbf{B}^H\right), \tag{10}$$

with $\mathbf{B}^L \in \mathbb{R}^{n_x \times n_L}$ and $\mathbf{B}^H \in \mathbb{R}^{n_x \times n_H}$. In the same way, we have the partitioned $\mathbf{u}_k$ as

$$\mathbf{u}_k = \left((\mathbf{u}_k^L)^t, (\mathbf{u}_k^H)^t\right)^t. \tag{11}$$

Therefore, the T vectors of the feasible solutions $\mathbf{u}^s$ can be also split as

$$\mathbf{u}_i^s = \left((\mathbf{u}_i^{s,L})^t, (\mathbf{u}_i^{s,H})^t\right)^t \quad \forall i = 0, ..., T-1. \tag{12}$$

Assumption 4. In order to have the degree of freedom for dealing with uncertainties, the rank of matrix $\mathbf{B}^L$ is assumed to be different from zero, that is

$$rank(\mathbf{B}^L) > 0. \tag{13}$$

Then, with (9) and (10), the discrete-time constrained system (3) can be rewritten as follows:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}^L\mathbf{u}_k^L + \mathbf{B}^H\mathbf{u}_k^H + \mathbf{B}_d\bar{\mathbf{d}}_k + \mathbf{B}_d\mathbf{w}_k, \tag{14a}$$
$$0 = \mathbf{E}_x\mathbf{x}_k + \mathbf{E}_u^H\mathbf{u}_k^H + \mathbf{E}_d\bar{\mathbf{d}}_k + \mathbf{E}_d\mathbf{w}_k. \tag{14b}$$

Assumption 5. The feasible state set $\mathcal{X}$ is characterized in a box/hypercube as

$$\mathcal{X} = [\underline{\mathbf{x}}^1, \overline{\mathbf{x}}^1] \times \cdots \times [\underline{\mathbf{x}}^{n_x}, \overline{\mathbf{x}}^{n_x}]. \tag{15}$$

Assumption 6. In the similar way, the robust state set at each time instant $\widetilde{\mathcal{X}}_i$ will be computed as the following box:

$$\widetilde{\mathcal{X}}_i = [\bar{\mathbf{x}}_i^{s,1} + \underline{\delta}\mathbf{x}_i^1, \bar{\mathbf{x}}_i^{s,1} + \overline{\delta}\mathbf{x}_i^1] \times \cdots$$
$$\cdots \times [\bar{\mathbf{x}}_i^{s,n_x} + \underline{\delta}\mathbf{x}_i^{n_x}, \bar{\mathbf{x}}_i^{s,n_x} + \overline{\delta}\mathbf{x}_i^{n_x}]. \tag{16}$$

where $\bar{\mathbf{x}}_i^{s,j}$ is the j^{th} component of vector i ($\bar{\mathbf{x}}_i^s$) of feasible solutions $\bar{\mathbf{x}}^s$ and $\underline{\delta}\mathbf{x}_i^j$ and $\overline{\delta}\mathbf{x}_i^j$ define the bounds of the state uncertainty.

Assumption 7. The feasible input set $\widetilde{\mathcal{U}}$ can be obtained by using the Cartesian product of two boxes

$$\widetilde{\mathcal{U}} = \widetilde{\mathcal{U}}^L \times \widetilde{\mathcal{U}}^H, \tag{17}$$

where

$$\widetilde{\mathcal{U}}^L = [\underline{\mathbf{u}}^{L_1}, \overline{\mathbf{u}}^{L_1}] \times \cdots \times [\underline{\mathbf{u}}^{L_{n_L}}, \overline{\mathbf{u}}^{L_{n_L}}], \tag{18a}$$
$$\widetilde{\mathcal{U}}^H = [\underline{\mathbf{u}}^{H_1}, \overline{\mathbf{u}}^{H_1}] \times \cdots \times [\underline{\mathbf{u}}^{H_{n_H}}, \overline{\mathbf{u}}^{H_{n_H}}]. \tag{18b}$$

Finally, the robust input set $\widetilde{\mathcal{U}}_i$ and the robust state set $\widetilde{\mathcal{X}}_{i+1}$ will be computed iteratively for $i = 0, ...H_p - 1$ by the following procedure:

- **First step**: Given $\widetilde{\mathcal{X}}_i$ and considering (14b) compute $\widetilde{\mathcal{U}}_i^H$
- **Second step**: Given $\widetilde{\mathcal{X}}_i$ and $\widetilde{\mathcal{U}}_i^H$ and considering (14a) compute $\widetilde{\mathcal{U}}_i^L$ and $\widetilde{\mathcal{X}}_{i+1}$

The details of the computations of the sets involved in previous steps will be described in detail in the following.

3.1 Computation of $\widetilde{\mathcal{U}}_i^H$

$\widetilde{\mathcal{U}}_i^H$ can be computed as the minimum box

$$\widetilde{\mathcal{U}}_i^H = [\mathbf{u}_i^{s,H_1} + \underline{\delta}\mathbf{u}_i^{H_1}, \mathbf{u}_i^{s,H_1} + \overline{\delta}\mathbf{u}_i^{H_1}] \times$$
$$\cdots \times [\mathbf{u}_i^{s,H_{n_H}} + \underline{\delta}\mathbf{u}_i^{H_{n_H}}, \mathbf{u}_i^{s,H_{n_H}} + \overline{\delta}\mathbf{u}_i^{H_{n_H}}], \tag{19}$$

where $\mathbf{u}_i^{s,H_j}$ is the j^{th} component of vector i ($\mathbf{u}_i^{s,H}$) of feasible solutions $\mathbf{u}^s$ and $\underline{\delta}\mathbf{u}_i^{H_j}$ and $\overline{\delta}\mathbf{u}_i^{H_j}$ define the bounds of the input feasibility set such that

$$\forall \mathbf{x}_i \in \widetilde{\mathcal{X}}_i, \quad \forall \mathbf{w}_i \in \mathcal{W}, \quad \exists \mathbf{u}_i^H \in \widetilde{\mathcal{U}}_i^H, \tag{20}$$

it holds

$$0 = \mathbf{E}_x\mathbf{x}_i + \mathbf{E}_u^H\mathbf{u}_i^H + \mathbf{E}_d\bar{\mathbf{d}}_i + \mathbf{E}_d\mathbf{w}_i. \tag{21}$$

Considering (16) and (19), (21) can be rewritten as

$$0 = \mathbf{E}_x(\mathbf{x}_i^s + \delta\mathbf{x}_i) + \mathbf{E}_u^H(\mathbf{u}_i^{s,H} + \delta\mathbf{u}_i^H) + \mathbf{E}_d\bar{\mathbf{d}}_i + \mathbf{E}_d\mathbf{w}_i. \tag{22}$$

where

$$\delta\mathbf{x}_i \in [\underline{\delta}\mathbf{x}_i^1, \overline{\delta}\mathbf{x}_i^1] \times \cdots \times [\underline{\delta}\mathbf{x}_i^{n_x}, \overline{\delta}\mathbf{x}_i^{n_x}], \tag{23}$$
$$\delta\mathbf{u}_i^H \in [\underline{\delta}\mathbf{u}_i^{H_1}, \overline{\delta}\mathbf{u}_i^{H_1}] \times \cdots \times [\underline{\delta}\mathbf{u}_i^{H_{n_H}}, \overline{\delta}\mathbf{u}_i^{H_{n_H}}], \tag{24}$$

where the box defined by (23) is given by $\widetilde{\mathcal{X}}_i$ and the box defined by (24) has to be computed to obtain $\widetilde{\mathcal{U}}_i^H$ using (19).

Then, the problem defined by (19)-(21) can be reformulated as a set-membership parameter identification problem (Blesa et al. (2012b)). Firstly, (22) can be rewritten as

$$\mathbf{y}_i = \mathbf{\Phi}_i\delta\mathbf{u}_i^H + \mathbf{e}_i, \tag{25}$$

where

- $\mathbf{y}_i = -\left(\mathbf{E}_x\mathbf{x}_i^s + \mathbf{E}_u^H\mathbf{u}_i^{s,H} + \mathbf{E}_d\bar{\mathbf{d}}_i\right)$,
- $\mathbf{\Phi}_i = \mathbf{E}_u^H$,
- $\mathbf{e}_i = \mathbf{E}_x\delta\mathbf{x}_i + \mathbf{E}_d\mathbf{w}_i$, which is the unknown-but-bounded error whose components fulfil $\mathbf{e}_i^j \in [\underline{\sigma}_i^j, \bar{\sigma}_i^j]$ $\forall j = 1, ..., n_v$, where the bounds can be found considering (23) and $\mathbf{w}_i \in \mathcal{W}$.

Then, the bounded error set-membership approach (Blesa et al., 2011) can be used to find the *Feasible Parameter Set* (FPS) that contains of the vector parameters that fulfills

$$\delta\mathbf{u}_i^H \in \mathbb{R}^{n_H} \quad | \quad \underline{\sigma}_i^j \leq y_i^j - \boldsymbol{\varphi}_i^j\delta\mathbf{u}_i^H \leq \bar{\sigma}_i^j, \quad \forall j = 1, ..., n_v, \tag{26}$$

where y_i^j denotes the j^{th} component of vector $\mathbf{y}_i$ and $\boldsymbol{\varphi}_i^j$ denotes the j^{th} row of matrix $\mathbf{\Phi}_i$. The feasible parameter set is a polytope in the space of parameter vector $\delta\mathbf{u}_i^H$. According to Casini et al. (2016), the minimum box (24) containing this polytope can be obtained by means of solving $2n_H$ linear programming optimization problems:

$$\underline{\delta}\mathbf{u}_i^{H_j} = \min \ \mathbf{c}_j^t\delta\mathbf{u}_i^H, \tag{27a}$$

subject to

$$\begin{pmatrix} \mathbf{\Gamma}_i^{H0} \\ \mathbf{\Gamma}_i^H \end{pmatrix} \delta\mathbf{u}_i^H \leq \begin{pmatrix} \mathbf{b}_i^{H0} \\ \mathbf{b}_i^H \end{pmatrix}, \ \forall j = 1, ..., n_H. \tag{27b}$$

Similarly, $\overline{\delta}\mathbf{u}_i^{H_j}$ can be obtained replacing "min" by "max" in (27a). The vector $\mathbf{c}_j$ is the j^{th} column of the identity matrix, $\mathbf{\Gamma}_i^{H0}$ and $\mathbf{b}_i^{H0}$ define the constraints of the box $\widetilde{\mathcal{U}}^H$. The matrix $\mathbf{\Gamma}_i^H$ and vector $\mathbf{b}_i^H$ define constrains of (26). More details can be found in Blesa et al. (2012a).

Remark 1. Note that it is possible that not all the $2n_H$ linear programming optimization problems (27a) are feasible. In this case, $\widetilde{\mathcal{U}}_i^H = \varnothing$ and the iterative computation is stopped.

3.2 Computation of $\widetilde{\mathcal{U}}_i^L$ *and* $\widetilde{\mathcal{X}}_{i+1}$

$\widetilde{\mathcal{U}}_i^L$ can be computed as the vector

$$\widetilde{\mathcal{U}}_i^L = \{\hat{\mathbf{u}}_i^{s,L_1}\} \times \cdots \times \{\hat{\mathbf{u}}_i^{s,L_{n_L}}\} = \hat{\mathbf{u}}_i^L, \tag{28}$$

where

$$\hat{\mathbf{u}}_i^L = \mathbf{u}_i^{s,L} + \Delta\mathbf{u}_i. \tag{29}$$

From (29), we have $\Delta\mathbf{u}_i = \hat{\mathbf{u}}_i^L - \mathbf{u}_i^{s,L}$. The objective is to provide an output vector as close as possible to the one provided by the planner that guarantees feasibility. So, the problem of obtaining vector $\hat{\mathbf{u}}_i^L$ can be defined as

$$\hat{\mathbf{u}}_i^L = \arg\min \ \|\Delta\mathbf{u}_i\|_2, \tag{30a}$$

subject to

$$\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}^L\hat{\mathbf{u}}_i^L + \mathbf{B}^H\hat{\mathbf{u}}_i^H + \mathbf{B}_d\bar{\mathbf{d}}_i + \mathbf{B}_d\mathbf{w}_k, \tag{30b}$$

$$\mathbf{x}_{i+1} \in \mathcal{X}, \quad \forall\mathbf{x}_i \in \widetilde{\mathcal{X}}_i, \quad \forall\hat{\mathbf{u}}_i^H \in \widetilde{\mathcal{U}}_i^H, \quad \forall\mathbf{w}_k \in \mathcal{W}. \tag{30c}$$

Considering (29) in (30b)

$$\begin{aligned}\mathbf{x}_{i+1} &= \mathbf{A}\mathbf{x}_i + \mathbf{B}^L(\mathbf{u}_i^{s,L} + \Delta\mathbf{u}_i) + \mathbf{B}^H\hat{\mathbf{u}}_i^H + \mathbf{B}_d\bar{\mathbf{d}}_i + \mathbf{B}_d\mathbf{w}_k \\ &= \hat{\mathbf{x}}_{i+1}^s + \mathbf{B}^L\Delta\mathbf{u}_i,\end{aligned} \tag{31}$$

where the term

$$\hat{\mathbf{x}}_{i+1}^s = \mathbf{A}\mathbf{x}_i + \mathbf{B}^L\mathbf{u}_i^{s,L} + \mathbf{B}^H\hat{\mathbf{u}}_i^H + \mathbf{B}_d\bar{\mathbf{d}}_i + \mathbf{B}_d\mathbf{w}_k \tag{32}$$

can be exactly bounded by a zonotope obtained by the sum of the three zonotopes obtained from the linear transformation of boxes $\widetilde{\mathcal{X}}_i$, $\widetilde{\mathcal{U}}_i^H$ and $\mathcal{W}$ with matrices $\mathbf{A}$, $\mathbf{B}^H$, $\mathbf{B}_d$, and considering an extra translation provided by the term $\mathbf{B}^L\mathbf{u}_i^{s,L} + \mathbf{B}_d\bar{\mathbf{d}}_i$. This zonotope can be described by

$$\mathcal{X}_{i+1}^0 = \mathbf{x}_{i+1}^0 \oplus \mathbf{\Omega}\mathbb{I}^{n\Omega}, \tag{33}$$

where $\mathbf{x}_{i+1}^0$ is the center of the zonotope, $\oplus$ denotes the Minkowski sum, $\mathbf{\Omega} \in \mathbb{R}^{n_x} \times \mathbb{R}^{n\Omega}$ and $\mathbb{I}$ is the unitary interval $[-1, 1]$. Then, the interval hull $\square\mathcal{X}_{i+1}^0$ defined as the smallest interval box that contains $\mathcal{X}_{i+1}^0$ (Montes De Oca et al. (2012)) can be computed as

$$\begin{aligned}\square\mathcal{X}_{i+1}^0 = {} & [\mathbf{x}_{i+1}^{0,1} - \|\Omega^1\|_1, \mathbf{x}_{i+1}^{0,1} + \|\Omega^1\|_1] \times \cdots \\ & \cdots \times [\mathbf{x}_{i+1}^{0,n_x} - \|\Omega^{n_x}\|_1, \mathbf{x}_{i+1}^{0,n_x} + \|\Omega^{n_x}\|_1],\end{aligned} \tag{34}$$

where Ω^i is the i^{th} row of matrix $\mathbf{\Omega}$.

Then, the optimization problem defined by (30) can be rewritten as

$$\Delta\mathbf{u}_i = \arg\min \ \|\Delta\mathbf{u}_i\|_2, \tag{35a}$$

subject to

$$\mathbf{x}_{i+1} = \hat{\mathbf{x}}_{i+1}^s + \mathbf{B}^L\Delta\mathbf{u}_i, \tag{35b}$$

$$\mathbf{x}_{i+1} \in \mathcal{X}, \quad \hat{\mathbf{x}}_{i+1}^s \in \square\mathcal{X}_{i+1}^0. \tag{35c}$$

The optimization problem (35) can be solved by the following quadratic problem:

$$\hat{\mathbf{u}}_i^L = \mathbf{u}_i^{s,L} + \text{argmin} \ \|\Delta\mathbf{u}_i\|_2, \tag{36a}$$

subject to

$$\begin{pmatrix}\mathbf{\Gamma}_i^{L0} \\ \mathbf{\Gamma}_i^L\end{pmatrix}\Delta\mathbf{u}_i \le \begin{pmatrix}\mathbf{b}_i^{L0} \\ \mathbf{b}_i^L\end{pmatrix}, \tag{36b}$$

where matrix $\mathbf{\Gamma}_i^{L0}$ and vector $\mathbf{b}_i^{L0}$ define the constraints of the box $\widetilde{\mathcal{U}}^L$ and matrix $\mathbf{\Gamma}_i^L$ and vector $\mathbf{b}_i^L$ define the constrains in (35).

Finally, $\widetilde{\mathcal{X}}_{i+1}$ can be computed by

$$\begin{aligned}\widetilde{\mathcal{X}}_{i+1} = {} & [\bar{\mathbf{x}}_{i+1}^{s,1} + \underline{\delta}\mathbf{x}_{i+1}^1, \bar{\mathbf{x}}_{i+1}^{s,1} + \overline{\delta}\mathbf{x}_{i+1}^1] \times \cdots \\ & \cdots \times [\bar{\mathbf{x}}_{i+1}^{s,n_x} + \underline{\delta}\mathbf{x}_{i+1}^{n_x}, \bar{\mathbf{x}}_{i+1}^{s,n_x} + \overline{\delta}\mathbf{x}_{i+1}^{n_x}],\end{aligned} \tag{37}$$

where

$$\underline{\delta}\mathbf{x}_{i+1}^j = \mathbf{x}_{i+1}^{0,1} - \|\Omega^j\|_1 + \mathbf{B}^{L,j}\Delta\mathbf{u}_i - \bar{\mathbf{x}}_{i+1}^{s,j}, \tag{38}$$

$$\overline{\delta}\mathbf{x}_{i+1}^j = \mathbf{x}_{i+1}^{0,1} + \|\Omega^j\|_1 + \mathbf{B}^{L,j}\Delta\mathbf{u}_i - \bar{\mathbf{x}}_{i+1}^{s,j}, \tag{39}$$

for $\forall j = 1, ..., n_x$ with $\mathbf{B}^{L,j}$ is the $^{j\text{th}}$ row of $\mathbf{B}^L$.

Remark 2. It can happen that quadratic optimization problem (36a) is not feasible. In this case, $\widetilde{\mathcal{U}}_i^L = \varnothing$ and the iterative computation is stopped.

3.3 General Algorithm

The iterative computation of the sets involved in Sections 3.1 and 3.2 is described in **Algorithm 1**.

Algorithm 1 Robust Constraint Satisfaction

Algorithm $(\mathbf{x}_k, \mathbf{u}^s, \bar{\mathbf{x}}^s, \mathcal{U}, \mathcal{X}, \mathcal{W})$
  $\widetilde{\mathcal{X}}_0 \leftarrow \mathbf{x}_k$
  $\check{H}_p \leftarrow 0$
  error $\leftarrow 0$
  while $\big(\check{H}_p \le T - 1$ and *error* $= 0\big)$ **do**
    Compute $\widetilde{\mathcal{U}}_i^H$ as described in Section 3.1
    if $\widetilde{\mathcal{U}}_i^H = \varnothing$
      error $\leftarrow 1$
    else
      Compute $\widetilde{\mathcal{U}}_i^L$ and $\widetilde{\mathcal{X}}_{i+1}$ as in Section 3.2
      if $\widetilde{\mathcal{U}}_i^H = \varnothing$
        error $\leftarrow 1$
      else
        $\widetilde{\mathcal{U}}_i \leftarrow \widetilde{\mathcal{U}}_i^L \times \widetilde{\mathcal{U}}_i^H$
        $\check{H}_p \leftarrow \check{H}_p + 1$
      endif
    endif
  endwhile
    $\check{H}_p \leftarrow \check{H}_p - 1$
  Return $(\widetilde{\mathcal{U}}_0, \ldots, \widetilde{\mathcal{U}}_{H_p-1}, \widetilde{\mathcal{X}}_1, \ldots, \widetilde{\mathcal{X}}_{\check{H}_p})$
end Algorithm

4. ROBUST PERIODIC EMPC CONTROLLER DESIGN

In order to obtain the periodically time-varying steady states and inputs (8), an off-line EMPC planner is designed firstly. In Angeli et al. (2016), the two following conditions for the economic cost function $\ell_e(k, \mathbf{p}_k, \mathbf{u}_k)$ have been used:

- Periodic economic cost: $\exists T > 0$, $\ell_e(i, \mathbf{p}_i, \mathbf{u}_i) = \ell_e(i + T, \mathbf{p}_i, \mathbf{u}_i)$ holds for $i \in \mathbb{N}_+$.
- Average economic cost:

$$\bar{L}_e(i, \mathbf{p}, \mathbf{u}) \triangleq \frac{1}{T}\sum_{i=0}^{T-1}\ell_e(i, \mathbf{p}_i, \mathbf{u}_i).$$

The periodic economic cost is able to obtain under **Assumption 2** and suitable control actions. The optimal control actions that satisfy the periodic economic costs and minimum average economic cost can be found by solving an open-loop optimization problem:

$$(\bar{\mathbf{x}}^p, \mathbf{u}^p) = \arg\min\ \bar{L}_e\left(i, \mathbf{p}, \mathbf{u}\right), \tag{40a}$$

subject to

$$\bar{\mathbf{x}}_{i+1} = \mathbf{A}\bar{\mathbf{x}}_i + \mathbf{B}\mathbf{u}_i + \mathbf{B}_d\bar{\mathbf{d}}_i, \tag{40b}$$
$$0 = \mathbf{E}_x\bar{\mathbf{x}}_i + \mathbf{E}_u\mathbf{u}_i + \mathbf{E}_d\bar{\mathbf{d}}_i, \tag{40c}$$
$$\bar{\mathbf{x}}_{i+1} \in \mathcal{X}, \tag{40d}$$
$$\mathbf{u}_i \in \mathcal{U}, \tag{40e}$$
$$\bar{\mathbf{x}}_T = \bar{\mathbf{x}}_0. \tag{40f}$$

The feasible solutions of the optimization problem (40) are regarded as the optimal steady states and inputs in (8).

In order to reach these periodically optimal steady states in the closed-loop, a trade-off cost function can be defined as follows:

$$\ell_t\left(k, \mathbf{p}_k, \mathbf{x}_k, \mathbf{u}_k\right) \triangleq \theta\ell_e\left(k, \mathbf{p}_k, \mathbf{u}_k\right) + (1-\theta)\left(\left\|\mathbf{x}_k - \bar{\mathbf{x}}^p_{\langle k\rangle_T}\right\|^2_{2,P} + \left\|\mathbf{u}_k - \mathbf{u}^p_{\langle k\rangle_T}\right\|^2_{2,Q}\right), \tag{41}$$

where $\theta \in [0,1]$ is the trade-off parameter. P and Q are the weighting matrices of appropriate dimensions.

Therefore, the on-line planner can be implemented with the last measurement as initial state by solving the optimization problem (7) with a prediction horizon of H_p and under mild modification including cost function as (41) and terminal constraint as $\bar{\mathbf{x}}_{k+H_p|k} = \bar{\mathbf{x}}^p_{\langle k+H_p\rangle_T}$. After solving this, the feasible solutions of the on-line planner at time instant k can be represented by

$$\bar{\mathbf{x}}^s_k = \left(\mathbf{x}^s_{k+1|k}, \ldots, \mathbf{x}^s_{k+H_p|k}\right)^t, \tag{42a}$$
$$\mathbf{u}^s_k = \left(\mathbf{u}^s_{k|k}, \ldots, \mathbf{u}^s_{k+H_p-1|k}\right)^t. \tag{42b}$$

As presented in Section 3, the interval arithmetic with the feasible solution (42) as $\mathbf{x}^s_k$ and $\mathbf{u}^s_k$ can be implemented and the maximum prediction horizon $\check{H}_p$ determined by the interval arithmetic can be also obtained. The tightened constraints on states and inputs along the prediction horizon $\check{H}_p$ can be written as

$$\mathbf{x}_{k+j+1|k} \in \widetilde{\mathcal{X}}_j, \mathbf{u}_{k+j|k} \in \widetilde{\mathcal{U}}_j, \quad j = 1, \ldots, \check{H}_p. \tag{43}$$

Remark 3. Note that the tightened bounds on states and inputs are computed on-line after obtaining the optimal open-loop system evolutions.

Finally, the robust periodic EMPC controller can be implemented by solving the following optimization problem:

$$\left(\mathbf{x}^*_k, \mathbf{u}^*_k\right) = \arg\min\ L_c\left(k, \mathbf{p}, \mathbf{u}\right) \triangleq \sum_{j=0}^{\check{H}_p-1} \ell_e\left(k+j, \mathbf{p}_k, \mathbf{u}_k\right), \tag{44a}$$

subject to

$$\mathbf{x}_{k+j+1|k} = \mathbf{A}\mathbf{x}_{k+j|k} + \mathbf{B}\mathbf{u}_{k+i|k} + \mathbf{B}_d\bar{\mathbf{d}}_{k+i}, \tag{44b}$$
$$0 = \mathbf{E}_x\mathbf{x}_{k+j|k} + \mathbf{E}_u\mathbf{u}_{k+j|k} + \mathbf{E}_d\bar{\mathbf{d}}_{k+j}, \tag{44c}$$
$$\mathbf{x}_{k+j+1|k} \in \widetilde{\mathcal{X}}_j, \tag{44d}$$
$$\mathbf{u}_{k+j|k} \in \widetilde{\mathcal{U}}_j, \tag{44e}$$
$$\left(\mathbf{x}_{k|k}, \bar{\mathbf{d}}_k\right) = \left(\mathbf{x}_k, \mathbf{d}_k\right). \tag{44f}$$

Therefore, by using the receding horizon strategy, the optimal robust control action at time instant k can be found by solving (44) with the selection of the first element $\mathbf{u}^*_{k|k}$.

5. CASE STUDY: A TWO-TANK WATER NETWORK

5.1 System Description

The two-tank water network is shown in Fig. 1. In this network, there are 2 tanks, 3 actuators (two pumps and a valve) and 4 water demands. The water is distributed from two water sources by two pumps. The discrete-time control-oriented model of this two-tank water network can be formulated as follows:

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.99 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & \Delta t & 0 \\ \Delta t & 0 & -\Delta t \end{bmatrix},$$
$$\mathbf{B}_d = \begin{bmatrix} 0 & -\Delta t & 0 & 0 \\ -\Delta t & 0 & 0 & -\Delta t \end{bmatrix},$$

$$\mathbf{E}_x = \left[0.2/\Delta t\ \ 0.01/\Delta t\right], \mathbf{E}_u = \left[0\ \ 0\ \ 1\right], \mathbf{E}_d = \left[0\ \ 0\ \ -1\ \ 0\right],$$

where $\mathbf{x}_k \in \mathbb{R}^{n_x}$ denotes the vector of water volumes in the storage tanks, $\mathbf{u}_k \in \mathbb{R}^{n_u}$ denotes the vector of flows through actuators (two pumps and a valve), $\mathbf{d}_k \in \mathbb{R}^{n_d}$ denotes the vector of water demands at demand sectors, $\mathbf{w}_k \in \mathbb{R}^{n_d}$ denotes the vector of underlying disturbances from the water demands at time instant k. Δt is the sampling time of 24 hours (3600 seconds). The system constraints are given in the boxes as follows:

$$\mathcal{X} = [0, 170] \times [0, 560],$$
$$\mathcal{U} = [0, 0.028] \times [0, 0.02] \times [-0.05, 0.05].$$

The underlying disturbances are unknown but bounded in the known sets denoted by $\mathcal{W}$. Each disturbance is independent since water demands are measured from each demand sector. In this simulation, the bound of each disturbance is assumed to be 10% mean demands as follows:

$$\mathcal{W} = [-0.0006, 0.0006] \times [-0.0014, 0.0014] \times \cdots \cdots \times [-0.0004, 0.0004] \times [-0.0014, 0.0014].$$

Beside, in order to achieve more economic performance, the trade-off parameter θ is set as 0.8.

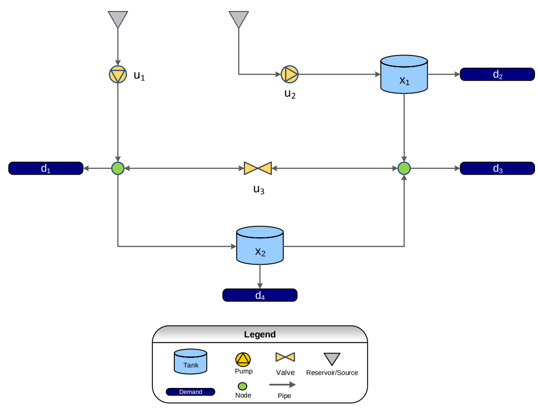
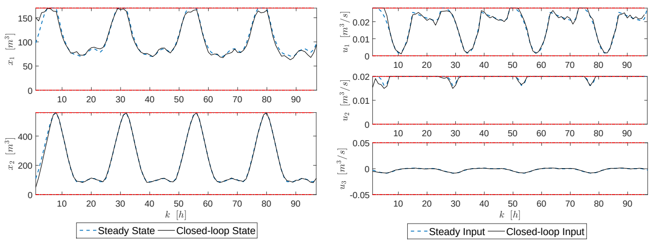


Fig. 1. A two-tank water network

5.2 Simulation results

The closed-loop simulation results are shown in Fig. 2(a) and 2(b). By using the proposed interval arithmetic, the optimal control inputs at each time step can be obtained from the robustly satisfied constraints. As shown in Fig. 2(b), the control inputs are close to the steady ones. Meanwhile, the system states are able to reach and stay around the steady states in Fig. 2(a).



(a) $\mathbf{x}$ (b) $\mathbf{u}$

Fig. 2. Simulation results of states and inputs

Fig. 3 shows the state space result. The big blue box is the physical limitation on the system state x. The optimal steady states computed by the off-line planner is characterized in red dashed line. It is clear that this trajectory is close to the physical limitation. Hence, it is necessary to do the robust constraint satisfactions in order to guarantee the recursive feasibility. By using the proposed interval arithmetic, the closed-loop trajectory is always staying inside the physical limitations. Besides, the convergence of the system states can be also guaranteed.

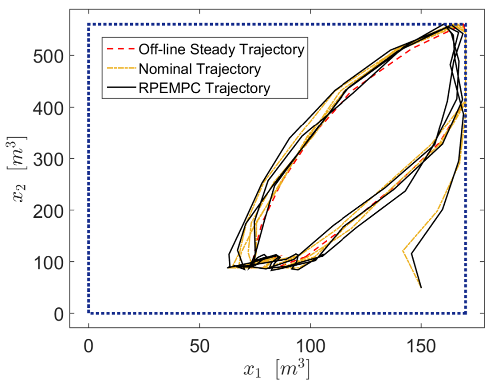


Fig. 3. Simulation results of states in $\mathbf{x}_1 - \mathbf{x}_2$ space

6. CONCLUSION

In this paper, a robust periodic EMPC controller for dynamic systems subject to algebraic constraints is designed. The robustness of the EMPC controller is guaranteed by checking whether the constraints on states and inputs are consistent in both differential and algebraic equations using an interval arithmetic. The robust constraint satisfactions are considered in the prediction horizon. Finally, the effectiveness of the proposed strategy is shown through a case study.

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