Computing an Inner Approximation of the Viability Kernel using guaranteed integration tubes

Eduard Codres[1], Joaquim Blesa [2] , Mario Martinez Guerrero[1], Alexandru Stancu[1].

[1] The University of Manchester, Faculty of Engineering and Physical Sciences, Oxford Road, Manchester, United Kingdom
{eduard.codres, mario.martinezguerrero}@manchester.ac.uk
[2] Institut de Robòtica i Informàtica Industrial (IRI), CSIC-UPC, C/. Llorens i Artigas 4-6, 08028 Barcelona, Spain
jblesa@iri.upc.edu

Keywords: Viability kernel, Interval analysis, Guaranteed Integration.

Introduction

Viability theory [1] is a promising area of research for the design of reliable control systems in the presence of uncertainties and faults. In this work, following the ideas presented in [2], we propose a method to compute the inner and outer approximation of the viability kernel using interval analysis and guaranteed integration techniques.

Basic properties

Following the notation in [1] we will consider a dynamic system $\mathcal{S}$ defined by

$$\begin{aligned} &\dot{x}(t) = f(x(t), u(t)) \\ &u(t) \in \mathbb{U} \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $\mathbb{U}$ is a compact subset of $\mathbb{R}^m$, $u \in \mathcal{U} = u : \mathbb{R}^+ \longmapsto \mathbb{U}$, $f : \mathbb{R}^n \times \mathbb{U} \longmapsto \mathbb{R}^n$ being f a continuous and locally Lipschitzian function bounded in $\mathbb{R}^n \times \mathbb{U}$ and φ is the flow map of $\mathcal{S}$ that computes the reached state $\varphi(t, x_0, u)$ given an initial state $x_0 = x(t)$ and a control function $u(t)$.

Then, the viability kernel [1] in [2] is defined as
Definition 1 *Let $\mathcal{S}$ a system defined by Eq.* (1) *and let $\mathbb{K} \subseteq \mathbb{R}^n$ be a compact set. The viability kernel of $\mathbb{K}$ under $\mathcal{S}$, is the set $Viab\mathcal{S}(\mathbb{K})$ of initial states $x \in \mathbb{K}$ from which at least one evolution does not leave $\mathbb{K}$ for all $t \geq 0$. We have*

$$Viab\mathcal{S}(\mathbb{K}) = \{x_0 \in \mathbb{K} | \exists u \in \mathcal{U}, \forall t \geq 0, \varphi(t, x, u) \in \mathbb{K}\},$$

with the purpose of computing an inner approximation of the viability kernel using interval analysis we propose:

Proposition 1. *Given a system $\mathcal{S}$ defined by Eq.* (1)*, an unknown initial state x_0 bounded by a box $[x_0]$ (i.e. $x_0 \in [x_0]$), an interval time horizon t_H and a control vector u in the time horizon t_H; the evolution of the state x of the system $\mathcal{S}$ can be bounded by a tube $\mathbb{T}_\mathcal{S}([x_0], t, u)$ such that*

$$\varphi(t, x_0, u) \in \mathbb{T}_\mathcal{S}([x_0], t, u) \quad \forall x_0 \in [x_0] \quad \forall t \in [0, t_H].$$

This Tube can be computed by discretizing Eq. (1) and using guaranteed integration techniques.

Proposition 2. *From the Tube $\mathbb{T}_\mathcal{S}([x_0], t, u)$ a boundary set $\mathbb{B}_\mathcal{S}([x_0], t_H, u)$ can be obtained, by using the final points of the tube at t_H (i.e. $\mathbb{T}_\mathcal{S}([x_0], t_H, u)$).*

The Tube $\mathbb{T}_\mathcal{S}([x_0], t, u)$ and the Boundary $\mathbb{B}_\mathcal{S}([x_0], t_H, u)$ satisfy the following conditions:

$$\varphi(t, x_0, u) \cap \mathbb{T}_\mathcal{S}([x_0], t, u) = \emptyset, \tag{2}$$

$$\varphi(t, x_0, u) \cap \mathbb{B}_\mathcal{S}([x_0], t_H, u) \neq \emptyset. \tag{3}$$

For a $\mathbb{R}^2$ system this can be depicted as the line segment between the two final points of the tube in Figure 1

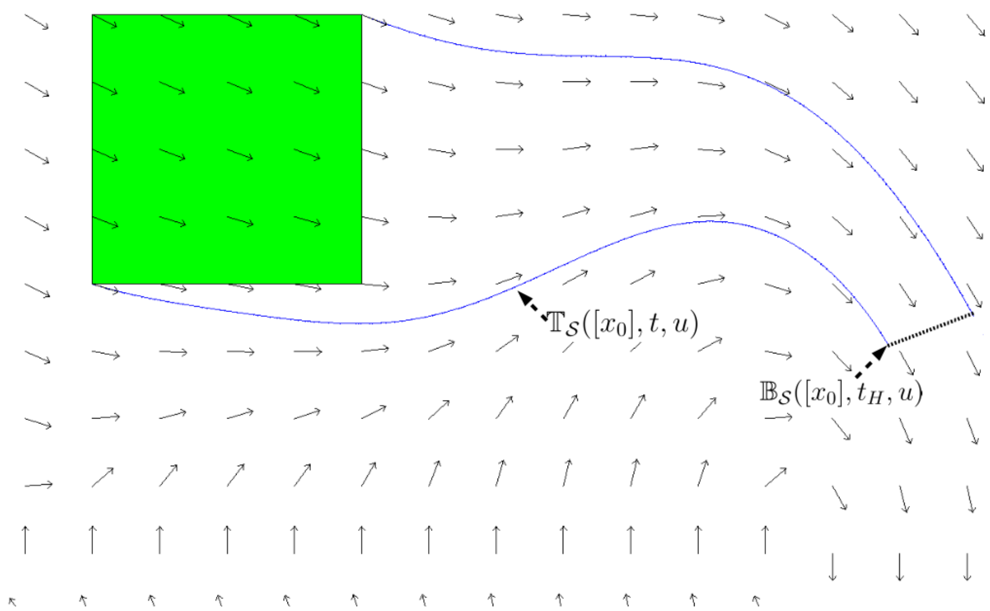

Figure 1: Integration Tube.

Proposition 3. *Given that the set $\mathbb{V}_{inner}$ is non-convex, if there exists a set $\mathbb{V}' \subset \mathbb{K}$ generated by the tube $\mathbb{T}_{\mathcal{S}}([x], t_H, u)$ and the set $\mathbb{V}_{inner}$, then the set $\mathbb{V}' \subset \mathbb{V}_{inner}$.*

Proof. Lets suppose:

$$\varphi(t, x, u) \cap (\mathbb{V}_{inner} \cup \mathbb{T}_{\mathcal{S}}([x], t, u)) = \emptyset, \quad \forall x \in \mathbb{V}' \tag{4}$$

Taking into account the properties of the system and proposition 2, $\forall x \in \mathbb{V}'$

$$\varphi(t,x,u) \cap \mathbb{V}_{inner} \neq \emptyset,, \quad (5)$$

$$or \quad \varphi(t,x,u) \cap \mathbb{T}_{\mathcal{S}}([x],t,u)) \neq \emptyset,, \quad (6)$$

$$or \quad \varphi(t,x,u) \cap \mathbb{V}' \neq \emptyset. \quad (7)$$

Where Eq. (5) and Eq. (6) contradicts Eq. (4), Eq. (7) states the posibility of an equilibrium point inside the set $\mathbb{V}'$. Therefore, $\mathbb{V}' \subset \mathbb{V}_{inner}$. Figure 2 depicts a graphical illustration for proposition 3 in a $\mathbb{R}^2$ system. ■

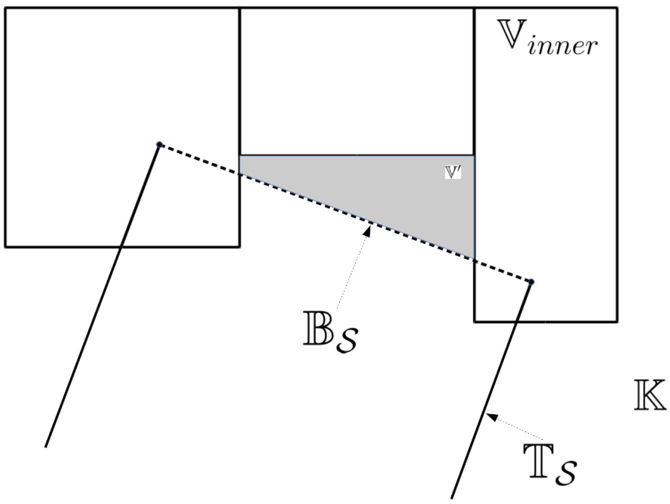

Figure 2: Case defined in Proposition 3.

Main results

Given an initial inner approximation of the viability kernel $\mathbb{V}_{inner}$, the tube $\mathbb{T}_{\mathcal{S}}([x_0],t,u)$ and its boundary set $\mathbb{B}_{\mathcal{S}}([x_0],t_H,u)$ defined in previous Section can be used to compute an inner approximation of the viability kernel $Viab\mathcal{S}(\mathbb{K})$ by means of algorithm 1 that follows the ideas proposed in [2]. Figures 4 and 3 depict the two different cases presented in algorithm 1.

Algorithm 1 Computation of an inner approximation of $Viab\mathcal{S}(\mathbb{K})$

Require: $\mathcal{S}, \mathcal{U}, \mathbb{K}, t_H$ and initial set $\mathbb{V}_{inner}$

$\mathbb{H} = \emptyset$ and $\mathbb{S} = \mathbb{K} \setminus \mathbb{V}_{inner}$
while $\mathbb{S} \neq \emptyset$ **do**
  for $[x_i] \in \mathbb{S}$ **do**
    Choose $u \in \mathcal{U}$
    if $\mathbb{T}_{\mathcal{S}}([x_i], t_H, u) \subseteq \mathbb{K}$ and $\mathbb{B}_{\mathcal{S}}([x_i], t_H, u) \subseteq \mathbb{V}_{inner}$ **then**
      $\mathbb{V}_{inner} := \mathbb{V}_{inner} \cup [x_i], \quad \mathbb{S} := \mathbb{S} \setminus [x_i]$
      Compute $\mathbb{V}_T = (\mathbb{T}_{\mathcal{S}}([x_i], t_H, u) \cap \mathbb{S})_{inner}$
      $\mathbb{V}_{inner} := \mathbb{V}_{inner} \cup \mathbb{V}_T, \quad \mathbb{S} := \mathbb{S} \setminus \mathbb{V}_T$
    end if
  end for
  Bisect boxes of $\mathbb{S}$
end while
return $\mathbb{V}_{inner}$ and $\mathbb{H}$

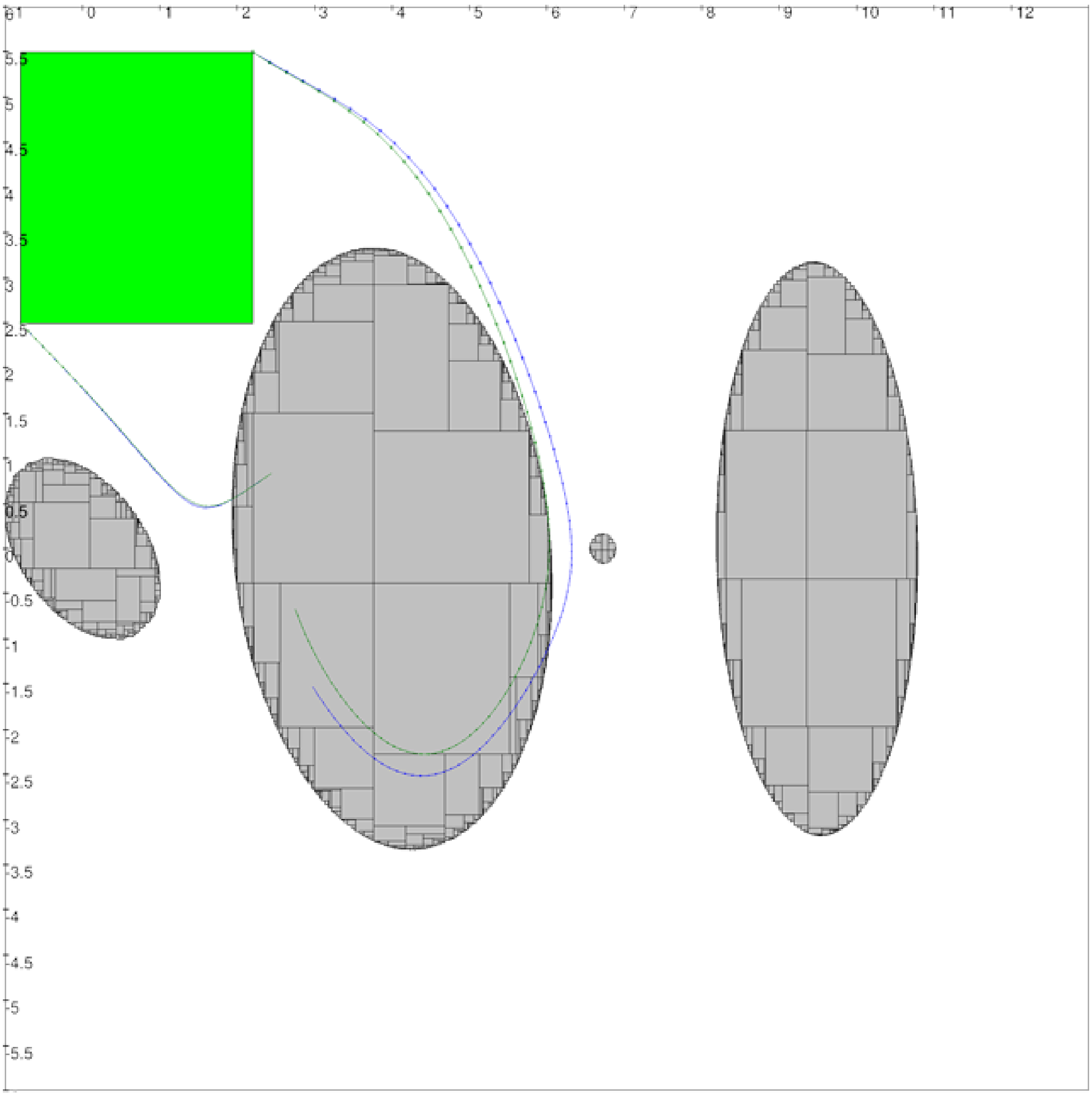

Figure 3: Box evolving towards the Viable Set

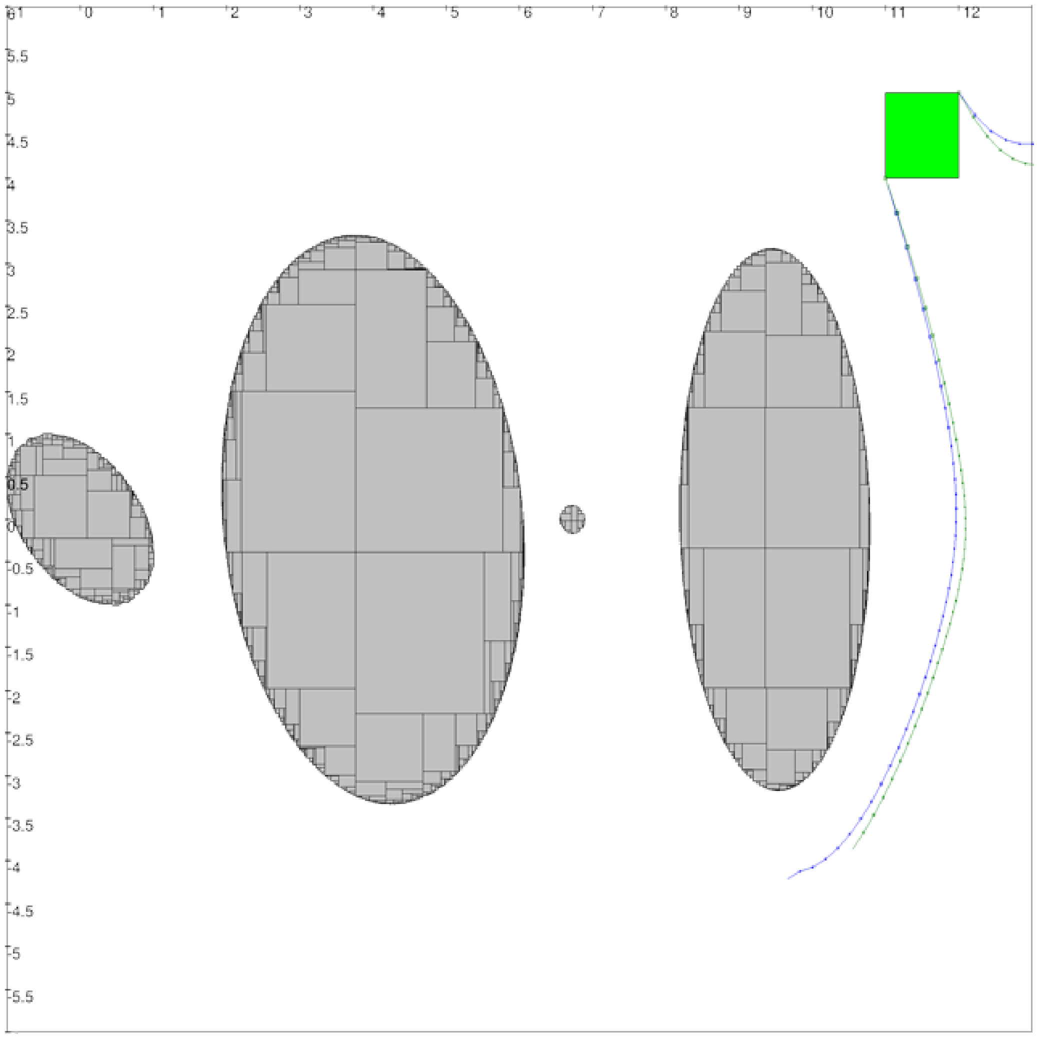

Figure 4: Box evolving towards the Non Viable Set

Acknowledgement

Thanks to the University of Manchester, Autonomous Research Theme for the support during the realization of the project.

This work has been partially funded by the Ministerio de Economía, Industria y Competitividad (MEICOMP) of the Spanish Government and FEDER through the project DEOCS (ref. DPI2016-76493-C3-3-R) and through the grant IJCI-2014-20801.

Thanks to the National Council of Science and Technology (CONACyT) for their support in the realization of this project.

References

[**1**] JP. Aubin, *Viability Theory*, Birkhauser, Boston, 1991.

[**2**] D. Monnet, J. Ninin and L. Jaulin, Computing an inner and an outer approximation of viability kernels, *Reliable Computing* 22, 138-148, 2016.