Computing an Inner Approximation of the Viability Kernel using guaranteed integration tubes

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Introduction

Viability theory \cite{1} is a promising area of research for the design of reliable control systems in the presence of uncertainties and faults. In this work, following the ideas presented in \cite{2}, we propose a method to compute the inner and outer approximation of the viability kernel using interval analysis and guaranteed integration techniques.

Basic properties

Following the notation in \cite{1} we will consider a dynamic system $S$ defined by

$$\dot{x}(t) = f(x(t), u(t))$$

$$u(t) \in \mathbb{U}$$

(1)
where \( x(t) \in \mathbb{R}^n \), \( U \) is a compact subset of \( \mathbb{R}^m \), \( u \in \mathcal{U} = u : \mathbb{R}^+ \to U \), \( f : \mathbb{R}^n \times U \to \mathbb{R}^n \) being \( f \) a continuous and locally Lipschitzian function bounded in \( \mathbb{R}^n \times U \) and \( \varphi \) is the flow map of \( \mathcal{S} \) that computes the reached state \( \varphi(t, x_0, u) \) given an initial state \( x_0 = x(t) \) and a control function \( u(t) \).

Then, the viability kernel [1] in [2] is defined as

**Definition 1** Let \( \mathcal{S} \) a system defined by Eq. (1) and let \( K \subseteq \mathbb{R}^n \) be a compact set. The viability kernel of \( K \) under \( \mathcal{S} \), is the set \( \text{Viab}_\mathcal{S}(K) \) of initial states \( x \in K \) from which at least one evolution does not leave \( K \) for all \( t \geq 0 \). We have

\[
\text{Viab}_\mathcal{S}(K) = \{ x_0 \in K | \exists u \in \mathcal{U}, \forall t \geq 0, \varphi(t, x, u) \in K \}
\]

with the purpose of computing an inner approximation of the viability kernel using interval analysis we propose:

**Proposition 1.** Given a system \( \mathcal{S} \) defined by Eq. (1), an unknown initial state \( x_0 \) bounded by a box \([x_0]\) (i.e. \( x_0 \in [x_0] \)), an interval time horizon \( t_H \) and a control vector \( u \) in the time horizon \( t_H \); the evolution of the state \( x \) of the system \( \mathcal{S} \) can be bounded by a tube \( T_\mathcal{S}([x_0], t, u) \)

such that

\[
\varphi(t, x_0, u) \in T_\mathcal{S}([x_0], t, u) \quad \forall x_0 \in [x_0] \quad \forall t \in [0, t_H].
\]

This Tube can be computed by discretizing Eq. (1) and using guaranteed integration techniques.

**Proposition 2.** From the Tube \( T_\mathcal{S}([x_0], t, u) \) a boundary set \( B_\mathcal{S}([x_0], t_H, u) \) can be obtained, by using the final points of the tube at \( t_H \) (i.e. \( T_\mathcal{S}([x_0], t_H, u) \)).

The Tube \( T_\mathcal{S}([x_0], t, u) \) and the Boundary \( B_\mathcal{S}([x_0], t_H, u) \) satisfy the following conditions:

\[
\varphi(t, x_0, u) \cap T_\mathcal{S}([x_0], t, u) = \emptyset, \quad (2) \\
\varphi(t, x_0, u) \cap B_\mathcal{S}([x_0], t_H, u) \neq \emptyset. \quad (3)
\]
For a $\mathbb{R}^2$ system this can be depicted as the line segment between the two final points of the tube in Figure 1.

![Integration Tube](image)

**Figure 1: Integration Tube.**

**Proposition 3.** Given that the set $\mathcal{V}_{inner}$ is non-convex, if there exists a set $\mathcal{V}' \subset \mathbb{K}$ generated by the tube $T_S([x], t_H, u)$ and the set $\mathcal{V}_{inner}$, then the set $\mathcal{V}' \subset \mathcal{V}_{inner}$.

**Proof.** Let’s suppose:

$$\varphi(t, x, u) \cap (\mathcal{V}_{inner} \cup T_S([x], t, u)) = \emptyset, \quad \forall x \in \mathcal{V}'$$

(4)

Taking into account the properties of the system and proposition 2, $\forall x \in \mathcal{V}'$.
\[ \varphi(t, x, u) \cap V_{\text{inner}} \neq \emptyset, \quad (5) \]
\[ \text{or } \varphi(t, x, u) \cap \mathbb{T}_S([x], t, u) \neq \emptyset, \quad (6) \]
\[ \text{or } \varphi(t, x, u) \cap V' \neq \emptyset. \quad (7) \]

Where Eq. (5) and Eq. (6) contradicts Eq. (4), Eq. (7) states the posibility of an equilibrium point inside the set \( V' \). Therefore, \( V' \subset V_{\text{inner}} \). Figure 2 depicts a graphical illustration for proposition 3 in a \( \mathbb{R}^2 \) system.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Case defined in Proposition 3.}
\end{figure}

**Main results**

Given an initial inner approximation of the viability kernel \( V_{\text{inner}} \), the tube \( \mathbb{T}_S([x_0], t, u) \) and its boundary set \( \mathbb{B}_S([x_0], t_H, u) \) defined in previous Section can be used to compute an inner approximation of the viability kernel \( \text{ViabS}(K) \) by means of algorithm 1 that follows the ideas proposed in [2]. Figures 4 and 3 depict the two different cases presented in algorithm 1.
Algorithm 1 Computation of an inner approximation of $\text{ViabS}(K)$

Require: $S, U, K, t_H$ and initial set $V_{\text{inner}}$

1: $H = \emptyset$ and $S = K \setminus V_{\text{inner}}$

2: while $S \neq \emptyset$ do

3:   for $[x_i] \in S$ do

4:       Choose $u \in U$

5:       if $T_S([x_i], t_H, u) \subseteq K$ and $B_S([x_i], t_H, u) \subseteq V_{\text{inner}}$ then

6:           $V_{\text{inner}} := V_{\text{inner}} \cup [x_i]$, $S := S \setminus [x_i]$

7:           Compute $V_T = (T_S([x_i], t_H, u) \cap S)_{\text{inner}}$

8:           $V_{\text{inner}} := V_{\text{inner}} \cup V_T$, $S := S \setminus V_T$

9:       end if

10:  end for

11:  Bisect boxes of $S$

12: end while

13: return $V_{\text{inner}}$ and $H$
Figure 3: Box evolving towards the Viable Set
Figure 4: Box evolving towards the Non Viable Set

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