Chapter 16
Partitioning Approaches for Large-scale Water Transport Networks

Abstract Large-scale systems (LSS), such as WTNs, present control theory with new challenges due to the large size of the plant and of its model. In order to apply decentralised or distributed control approaches to LSS, there is a prior problem to be solved: the system decomposition into subsystems. The importance of this issue has already been reported in the general literature of decentralised control of LSS. The decomposition of the system in subsystems could be carried out during the modelling of the process by identifying subsystems as parts of the system on the basis of physical insight, intuition or experience. But, when a large-scale complex system with many states, inputs and outputs is considered, it may be difficult, even impossible, to obtain partitions by physical reasoning. A more appealing alternative is to develop systematic methods, which can be used to decompose a given system by extracting information from its structure and representing it as a graph. Then, this structural information can be analysed by using methods coming from graph theory. This chapter discusses partitioning approaches towards the development of subsystem decomposition methods for LSS by reviewing automatic decomposition algorithms and techniques based on graph partitioning. The general aim of the discussed methods is to provide decompositions consisting of sets of non-overlapping subgraphs whose number of vertices is as similar as possible and the number of interconnecting edges between them is minimal. The real case study based on the Barcelona WTN in Chapter 2 is used to exemplify the discussed decomposition methodologies.

16.1 Introduction

Large-scale systems (LSS) present control theory with new challenges due to the large size of the plant and of its model [13, 21]. The goal to be achieved with control methods for this kind of systems is to obtain a reasonable solution with a reasonable effort in modelling, designing and implementing the controller.

As discussed in previous chapters, MPC has been proved to be suitably applied for the control of LSS as drinking water networks [3], sewer networks [14], open-flow channel networks [17] or electrical networks [15]. Nevertheless, the main hurdle for MPC control (as any other control technique) when applied to LSS in a centralised way, is the non-scalability. The reason is that a huge control model is needed, being difficult to maintain/update and which needs to be rebuilt on every change of the system configuration, e.g., when some part of the system should be stopped because of maintenance actions or malfunctions. Subsequently, a model change would require re-tuning the centralised controller. It is obvious that the cost of setting up and maintaining the monolithic solution of the control problem is prohibitive. A way of circumventing these issues might be by looking into *decentralised* MPC (DMPC) or *distributed* MPC techniques, where networked local MPC controllers are in charge of controlling part of the entire system. The main difference between distributed and decentralized MPC is that the former *uses* negotiations and re-computations of local control actions within the sampling period to increase the level of cooperation, whereas the latter does not (at the benefit of computation time, but at the cost of optimality).

The industrial success of the traditional centralised MPC (CMPC) drives now a new interest in this old area of distributed control, and distributed MPC has become one of the hottest topics in process control in the early 21st century, worldwide. Thus, two research projects (HDMPC [10] and WIDE [23]) are currently being carried out in Europe, both focused on the development of decentralised and distributed MPC techniques. Few works have been recently published in this area; see, e.g., [6, 11, 16, 18, 19, 22], among others.

However, in order to apply decentralised or distributed MPC approaches to LSS, there is a prior problem to be solved: the system decomposition into subsystems. The importance of this issue has already been noticed in classic control books addressing the decentralised control of LSS as [13, 21]. The decomposition of the system in subsystems could be carried out during the modelling of the process by identifying subsystems as parts of the system on the basis of physical insight, intuition or experience. But, when a large-scale complex system with many states, inputs and outputs is considered, it may be difficult, even impossible, to obtain partitions by physical reasoning. A more appealing alternative is to develop systematic methods, which can be used to decompose a given system by extracting information from its structure and representing it as a graph. Then, this structural information can be analysed by using methods coming from graph theory. Consequently, the problem of system decomposition into subsystems leads to the problem of graph partitioning, i.e., the decomposition of graph into subgraphs.

Graph partitioning is an important problem with extensive application in scientific computing [12], optimisation, very-large-scale integration (VLSI) design [8], task partitioning for parallel processing, control of cascading failures, among others. However, the development of graph partitioning algorithms that allow the decomposition of LSS into subsystems for being used in decentralised or distributed MPC is still very incipient and available methods are quite limited. In [21], a hierarchical LBT decomposition that leads to a input-reachable hierarchy for some particular

systems is presented. A more general approach is based on the ε-decomposition method, which is based on decomposing the system in weakly coupled subsystems (see also [21]). The algorithm proceeds sequentially disconnecting the edges of the system graph that are smaller than a prescribed threshold ε and identifying the disconnected subgraph of the resulting graph. The obtained subsystems correspond to the subsystems with mutual coupling smaller or equal than ε. However, the tuning of this parameter is not a trivial issue and only a trial and error approach is currently available.

16.2 Problem statement

A graph can be defined as an abstract representation of a set of objects from a certain collection, where some pairs of objects are connected by links. The interconnected elements are typically called *vertices* while the connection links are called *edges*. These latter elements may be *directed* (asymmetric) or *undirected* (symmetric) according to their connection features, what makes that the whole graph is directed or undirected as well. It is also possible to distinguish graphs whether or not their vertices and edges are weighted (weighted/unweighted graphs).

Consider a dynamical system represented in general form by the state-space equations

$$\mathbf{x}(k+1) = \mathbf{g}(\mathbf{x}, \mathbf{u}, \mathbf{d}), \tag{16.1a}$$

$$\mathbf{y} = \mathbf{h}(\mathbf{x}, \mathbf{u}, \mathbf{d}), \tag{16.1b}$$

where $\mathbf{x} \in \mathbb{R}^{n_x}$ and $\mathbf{x}(k+1) \in \mathbb{R}^{n_x}$ are, respectively, the current and successor system states in discrete time, $\mathbf{u} \in \mathbb{R}^{n_u}$ is the system input and $\mathbf{d} \in \mathbb{R}^{n_d}$ is a bounded process disturbance. Moreover, $\mathbf{g} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_d} \to \mathbb{R}^{n_x}$ is the states mapping function and $\mathbf{h} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_d} \to \mathbb{R}^{n_y}$ corresponds with the output mapping function. Suppose now that it is desired to decompose (16.1) into subsystems. With this aim, the graph representation of the system model (16.1) is determined (by using the system topology) and incidence matrix $\mathbf{B}_{ij}$ is then stated, which describes the connections (edges) between the graph vertices (system inputs, outputs and states). Without loss of generality, $\mathbf{B}_{ij}$ and the directonality of the edges are derived from the relation between system equations (rows of $\mathbf{B}_{ij}$) and system variables (columns of I_M), as proposed by [21, 24, 25]. There are alternative matrix representations for a (directed) graph such as the *adjacency matrix* and the *Laplacian matrix* (see [2]), which are related to the matrix representation used in this paper. Once $\mathbf{B}_{ij}$ has been obtained from the system directed graph (digraph), the problem of the decomposition into subsystems can be formulated in terms of partitioning the corresponding graph into subgraphs. Since such partitioning is oriented to the application of a decentralised control strategy (in particular, DMPC), the resultant subgraphs should have the following features (see [13, 21]):

- nearly the same number of vertices;

- few connections between the subgraphs.

These features guarantee that the obtained subgraphs have a similar size, fact that balances computations between subsystem controllers and allows minimising communications between them. Hence, the problem of graph partitioning can be more formally established as follows:

Problem 16.1 (Standard Graph Partitioning). Given a graph $G(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ denotes the set of vertices, $\mathcal{E}$ is the set of edges, and $M \in \mathbb{Z}_{\geq 1}$, find M subsets $\mathcal{V}_1$, $\mathcal{V}_2, \ldots, \mathcal{V}_M$ of $\mathcal{V}$ such that

1. $\bigcup_{i=1}^{M} \mathcal{V}_i = \mathcal{V}$,
2. $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$, for $i \in \{1, 2, \ldots, M\}$, $j \in \{1, 2, \ldots, M\}$, $i \neq j$,
3. $|\mathcal{V}_1| \approx |\mathcal{V}_2| \approx \cdots \approx |\mathcal{V}_M|$,
4. the *cut size*, i.e., the number of edges with endpoints in different subsets $\mathcal{V}_i$, is minimised.

Remark 16.1. Defining the *vertex-based weight* of a subset $\mathcal{V}_i$ as

$$\Omega_i \triangleq \sum_{j=1}^{|\mathcal{V}_i|} \omega_i^j, \tag{16.2}$$

where ω_i^j corresponds to the weight of the j-th vertex of the subset $\mathcal{V}_i$, the following condition should be added to Problem 16.1 in the case of *weighted graph partitioning*:

- $\Omega_i \approx \Omega/M$, with $i \in \{1, 2, \ldots, M\}$, where

$$\Omega \triangleq \sum_{i=1}^{M} \Omega_i. \tag{16.3}$$

Remark 16.2. Conditions 3 and 4 of Problem 16.1 are of high interest from the decentralised control point of view since they are related to the degree of interconnection between resultant subsystems and their size balance, respectively.

Graph partitioning is considered as a $\mathcal{NP}$-complete problem [21]. However, it can be solved in polynomial time for $|\mathcal{V}_i| = 2$ (Kernighan-Lin algorithm) [4, 7]. Since this condition is quite restrictive for large-scale graphs, alternatives for graph partitioning based on fundamental heuristics are properly accepted. Two main classes of successful heuristics have evolved over the years, trying to achieve the proper trade off between partitioning speed and quality. They are the *minimum-degree-based* ordering algorithms (MDB), and the *graph-partitioning-based* ordering algorithms (GPB) [9].

16.3 Proposed approaches

16.3.1 Using graph theory

This approach consists in proposing a partitioning algorithm, as much automatised as possible, through which a partition of a dynamical system can be found, which allows its decomposition in subsystems. This algorithm requires to represent the dynamical system as a graph, which can obtained from the system structure [21].

Main algorithm

The partitioning algorithm proposed in this chapter follows some ideas developed in [9] for graph partitioning purposes. However, some refining steps have been added as well as some of the original procedures have been drastically changed in order to find partitions oriented to split dynamical networked systems. Hence, the different parts/routines of the main proposed algorithm are presented and explained in sections below. The current version of the algorithm is though to be used off-line, i.e., the partitioning of the system is not carried out on-line. A further improvement could be to adapt the proposed algorithm such that the partitioning could be done on-line when some structural change of the network occurs. In this way, the potential benefit of using a DMPC approach described in the Introduction could be fully exploited.

Start up: This procedure requires the definition of the graph, i.e., *the incidence matrix*[13] $\mathbf{B}_{ij}$, which describes the connections between the graph vertices, their directionality and, in some cases, the weight of each edge.

Preliminary partitioning: This procedure performs a preliminary automatic partitioning of the graph as follows. The vertex $v_j \in \mathcal{V}$, for $j \in \{1, 2, \ldots, |\mathcal{V}|\}$, with maximum weight ω is found and defined as the centre of the first subgraph G_1. Then, all vertices connected to this vertex of maximum weight are assigned to G_1. At this point, the set of non-selected vertices is defined as

$$\mathcal{V}_r \triangleq \{v_j \in \mathcal{V} : v_j \notin \mathcal{V}_1\}.$$

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[13] The *incidence matrix* of a directed graph $G(\mathcal{V}, \mathcal{E})$, denoted as $\mathbf{B}_{ij}$, is defined such that

$$\mathbf{B}_{ij} = \begin{cases} -1 & \text{if the edge } z_j \text{ leaves vertex } v_i, \\ 1 & \text{if the edge } z_j \text{ enters vertex } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

This matrix has dimensions $\varphi \times \eta_e$, where φ corresponds with the total number of vertices and η_e denotes de total number of edges [2]. Additionally, the weight of the j-th vertex, denoted as ω^j, for $j = 1, 2, \ldots, \varphi$, where $\varphi \triangleq |\mathcal{V}|$, is computed. The weight ω^j represents the number of edges connected to this vertex. Moreover, ω^j is also known as the *vertex degree* [5].

This procedure is now repeated for all vertices $v_j \in \mathcal{V}_r$ (now for $j = \{1, 2, \ldots, |\mathcal{V}_r|\}$) until $\mathcal{V}_r$ is empty, after the corresponding updating. This routine highlights the subgraphs of higher connectivity. The resultant subgraphs with just one vertex are merged to the closest subgraph. Once a set of subgraphs $G_i(\mathcal{V}_i, \mathcal{V}_i)$, for $i = 1, 2, \ldots, M$, is obtained, it is possible to determine some useful indexes for the entire graph and each one of the resu subgraphs. These indexes are:

- $\varphi_i \triangleq |\mathcal{V}_i|$ (from now on called *subgraph internal weight* of G_i);
- ε_i, denoted as the *cut size*[14] of the subgraph G_i (from now on called *subgraph external weight* of G_i);
- $\varphi^{\max} \triangleq \max_i \varphi_i$, for $i = 1, 2, \ldots, M$;
- $\bar{\varphi} \triangleq \frac{1}{M} \sum_{i=1}^{M} \varphi_i$ (arithmetic mean).

Notice that at this stage, the number M of subgraphs is obtained in an automatic way so it is not imposed.

Remark 16.3. Notice that introducing the set $\tilde{\mathcal{E}}_a \subset \mathcal{E}$, defined as the set of edges with endpoints in other subgraphs different to G_a, the representation of subgraphs G_i such that

$$\bigcup_{i=1}^{M} G_i = G,$$

can be slightly modified to $G_i(\mathcal{V}_i, \mathcal{E}_i, \tilde{\mathcal{E}}_i)$ for completeness purposes. Also notice that $\varepsilon_i \triangleq |\tilde{\mathcal{E}}_i|$.

Uncoarsening - Internal balance: This procedure aims at the reduction of the number of subgraphs, trying to achieve similar internal weights for all of them. This process starts determining the set

$$\mathcal{L} = \{G_i, i = 1, 2, \ldots, m : \varphi_i \leq \bar{\varphi}\}, \tag{16.4}$$

with $m \in$
$mathbbZ_+$ and $m < M$. For each $G_i \in \mathcal{L}$, the set of neighbour[15] subgraphs, denoted as $\mathcal{N}_i$, is determined and expressed as

$$\mathcal{N}_i = \{G_j, j = 1, 2, \ldots, h_i : G_j \text{ is neighbour of } G_i\}, \tag{16.5}$$

with $h_i = |\mathcal{L}_i|$. If the condition

$$\varphi_i + \varphi_j \leq \bar{\varphi}, \quad i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, h_i\} \tag{16.6}$$

holds for $G_i \in \mathcal{L}$ and $G_j \in \mathcal{N}_i$, then these two subgraphs are merged. If there are two or more subgraphs $G_j \in \mathcal{N}_i$ such that (16.6) holds, the subgraph $G_j \in \mathcal{N}_i$

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[14] See Problem 16.1.

[15] Two subgraphs are called *neighbours* if they are contiguous and share edges (see, e.g., [1] among many others).

with minimum internal weight is selected. Once two subgraphs are merged, $\bar{\varphi}$ is updated.

This procedure is iterated until no additional merging was possible. It is considered that the internal balance has been achieved when either

- $\bar{\varphi} \leq \varphi_i \leq \varphi^{\max}$, for $i = 1, 2, \ldots, M$, or
- G_i with $\varphi_i \leq \bar{\varphi}$ cannot be merged with any of its neighbours since the φ associated to the resultant subgraph might be greater than $\varphi^{\max}$.

Refining - External balance: This procedure aims at the reduction of the cut size of the resultant subgraphs. To achieve this goal, define ω_i^j as the degree of the j-th vertex of the i-th subgraph, with $j \in \{1, 2, \ldots, \varphi_i\}$ and $i \in \{1, 2, \ldots, M\}$. From this definition, two indexes can be stated:

- the *vertex internal degree*, denoted as $\hat{\omega}_i^j$, which represents the number of connections of the vertex $v_j \in \mathcal{V}_i$, for $j \in \{1, 2, \ldots, \varphi_i\}$, $i \in \{1, 2, \ldots, M\}$, with other vertices $v_p \in \mathcal{V}_i$, $p \in \{1, 2, \ldots, \varphi_i\}$, $p \neq j$;
- the *vertex external degree*, denoted as $\breve{\omega}_i^j$, which represents the number of connections of the vertex $v_j \in \mathcal{V}_i$, for $j \in \{1, 2, \ldots, \varphi_i\}$, $i \in \{1, 2, \ldots, M\}$, with other vertices $v_p \in \mathcal{V}_q$, $p \in \{1, 2, \ldots, \varphi_q\}$, $q \in \{1, 2, \ldots, M\}$, $q \neq i$.

Hence, for a given vertex $v_j \in \mathcal{V}_i$, if $\hat{\omega}_i^j < \breve{\omega}_i^j$, then vertex v_j is moved from subgraph $G_i(\mathcal{V}_i, \mathcal{E}_i, \tilde{\mathcal{E}}_i)$ to the subgraph in which most of its edges have their endpoint (like in the AVL tree algorithm [5]). All indexes should be updated for the M subgraphs and the next vertex is analysed. This procedure will last until each subgraph vertex fulfils $\hat{\omega}_i^j \geq \breve{\omega}_i^j$.

The Complete Algorithm: Algorithm 13 collects all the procedures/routines mentioned and explained before. Hence, applying this algorithm to the graph associated to a given dynamical system, the expected result consists of a set of subgraphs which determines a particular system decomposition. This set $\mathcal{P}$ is then defined as

$$\mathcal{P} = \left\{ G_i, i = 1, 2, \ldots, M : \bigcup_{i=1}^{M} G_i = G \right\}. \tag{16.7}$$

Algorithm 13 Graph partitioning algorithm

1: $\mathbf{B}_{ij} \leftarrow$ System topology
2: $G(\mathcal{V}, \mathcal{E}) \leftarrow \mathbf{B}_{ij}$
3: **for** $j = 1$ to φ **do**
4:     Compute ω^j
5: **end for**
6: $\mathcal{V}_r \leftarrow \mathcal{V}$, $i = 1$
7: **repeat**
8:     Find $v \in \mathcal{V}_r$ with maximum ω
9:     $\mathcal{V}_i \leftarrow v$ and all its neighbour vertices
10:     $\mathcal{V}_r \triangleq \mathcal{V} - \left\{ \bigcup_{h=1}^{i} \mathcal{V}_h \right\}$
11:     $i = i + 1$
12: **until** $\mathcal{V}_r = \emptyset$
13: **for** $i = 1$ to M **do** {Compute some indexes}
14:     $\varphi_i \triangleq |\mathcal{V}_i|$ {internal weight}
15:     $\varepsilon_i \triangleq |\tilde{\mathcal{E}}_i|$ {external weight}
16: **end for**
17: $\varphi^{\max} \triangleq \max\limits_i \varphi_i$
18: $\bar{\varphi} \triangleq \frac{1}{M} \sum\limits_{i=1}^{M} \varphi_i$ {arithmetic mean}
19: Compute $\mathcal{L}$ {see (16.4)}
20: $b_{\text{int}} = $ false   {Internal balance}
21: **while** $b_{\text{int}} = $ false **do**
22:     **for** $i = 1$ to m **do**
23:         Compute $\mathcal{N}_i$ {see (16.5)}
24:         **for** $j = 1$ to h **do**
25:             **if** $\varphi_i + \varphi_j \leq \bar{\varphi}$ **then** {see (16.6)}
26:                 $G_* = G_i \cup G_j$
27:                 $G_{\text{new}} \leftarrow G_*$ with minimum φ_*
28:                 Update $\bar{\varphi}$
29:             **end if**
30:         **end for**
31:     **end for**
32:     Update φ_i
33:     $b_{\text{ext}} = $ false   {External balance}
34:     **while** $b_{\text{ext}} = $ false **do**
35:         **for** $i = 1$ to M **do**
36:             **for** $j = 1$ to φ_i **do**
37:                 Compute $\hat{\omega}_i^j$ and $\breve{\omega}_i^j$
38:                 **if** $\hat{\omega}_i^j < \breve{\omega}_i^j$ **then**
39:                     Move v^j from G_i to its neighbour
40:                 **end if**
41:                 Update φ_i, $\bar{\varphi}$, $\varphi^{\max}$
42:             **end for**
43:         **end for**
44:         Update all indexes
45:         Check external balance (nodes)
46:     **end while**
47:     Check internal balance (subgraphs)
48: **end while**
49: **return** $\mathcal{P}$ {see (16.7)}

Auxiliary routines

Despite Algorithm 13 yields an automatic partitioning of a given graph, it does not imply that the resultant set P follows the pre-established requirements stated in Problem 16.1. In this sense, complementary routines can be useful for improving the partitioning process according to the considered application. Additional auxiliary routines could be added such that the generated partitioning takes into account the control performance that would be achieved when used in decentralised or distributed MPC control.

Pre-filtering: In general, the resultant solution given by the Algorithm 14 is nearly appropriate in terms of $\hat{\omega}$ and $\breve{\omega}$, but it highly depends on the topology and complexity of the graph. For this reason, in order to obtain a better graph partitioning, sometimes it can be useful to make a *Pre-filtering* routine, where all the vertexes with $\omega = 1$ are virtually merged to this vertex that shares its unique edge. This procedure creates *supranodes*, which should be properly recognised at the moment of determining the partitioning of the dynamical system from the decomposition of its associated graph. Moreover, doing the manual merging of those vertices reduces the work done by subsequent routines.

Post-filtering: On the other hand, suppose that after partitioning a given graph $G(\mathcal{V}, \mathcal{E})$ by using Algorithm 13, all the M resultant subgraphs fulfil

$$\bar{\varphi} \leq \varphi_i \leq \varphi^{\text{max}}, \quad \text{for} \quad i \in \{1, 2, \ldots, M\}. \tag{16.8}$$

However, the following situation could occur. Suppose a subgraph G_a with $\varphi_a \ll \bar{\varphi}$, which is placed next to a subgraph G_b and fulfils (16.8). The merging of subgraphs G_a and G_b, expressed as $G_c \triangleq G_a \cup G_b$, is not allowed since $\varphi_c \geq \varphi_{\text{max}}$. The *Post-filtering* routine implements an approximation and a parametrisation, i.e., by adding a small tolerance δ, the existence of the resultant subgraph G_c is now allowed since $\varphi_c \leq \varphi^{\text{max}} + \delta$. This relaxation allows to have less subgraphs but with higher complexity and internal weight.

Anti-oscillation: This procedure leads to solve a possible issue when the *refining (external balance)* routine is run. When a vertex is moved from one subgraph to another according to its internal and external degrees, there exists the possibility of doing this movement during an infinite time if there is no specification of routine ending. Therefore, the refining routine is then run within a for loop and the parameter ρ is set as the maximum number of iterations that this procedure is executed. Afterwards, since the resulting set of subgraphs is stored at each iteration $t' \in \mathbb{Z}_+$, $t' = \{1, 2, \ldots, \rho\}$, the configuration of M subgraphs with minor ε_i, for $i = 1, 2, \ldots, M$, can be chosen.

Some practical issues

Given that the partitioning algorithm proposed in this chapter is mainly thought for performing decentralised control of LSS, several features could be taken into

account to achieve a convenient system partitioning and less complex controller designs. For instance, an additional routine that would restrict the connection of subgraphs with unidirectional edges would be very useful since a pure hierarchical control scheme can be straightforwardly implemented, decreasing the inherent loss of performance of a decentralised control scheme.

16.3.2 Using masks

The application of DMPC to WTN depends crucially on how the network is decomposed into subsystems. Identifying subsystems is not an easy task in a large-scale network as it involves to find automatically *sufficiently small* sections of the networked plant that are not *too coupled* among them. The partitioning algorithm, proposed in this chapter, aims to obtain this decomposition automatically by identifying clusters of elements that are strongly connected with each other but weakly interconnected with the other clusters, in order to represent the whole network as a set of loosely coupled subsystems [20]. The current version of the algorithm is though to be used off-line, that is, the partitioning of the system is static and is not carried out on-line. A further improvement could be to adapt the proposed algorithm such that the partitioning could be done on-line when, for instance, some structural change of the network appears.

As a starting point, the partitioning algorithm requires the following information of the WTN:

1. The interconnection structure characterised by the matrix

$$\mathbf{I_c} = \begin{bmatrix} \mathbf{A}_{sp} & \mathbf{B}_{sp} \end{bmatrix}, \tag{16.9a}$$

   where

$$\mathbf{A}_{sp} = \begin{bmatrix} \mathbf{A} & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B}_{sp} = \begin{bmatrix} \mathbf{B} \\ \mathbf{E} \end{bmatrix}, \tag{16.9b}$$

   where $\mathbf{A}$ and $\mathbf{B}$ are the system matrices in (12.8), the subscript sp identifies the matrices used for system decomposition, and $\mathbf{E} \triangleq [\mathbf{E}_u \quad \mathbf{E}_d]$ is the matrix related to the equality constraints (16.10b). In order to take into account input bounds, new normalised inputs are introduced $\bar{\mathbf{u}} \triangleq \mathbf{u}/\mathbf{u}^{max}$ so that $\bar{\mathbf{u}} \in [0, 1]$. Thus, new matrices $\bar{\mathbf{B}}$ and $\bar{\mathbf{E}}$ are introduced in (16.9b) to take into account the rescaling. From matrix $\mathbf{I_c}$, the adjacency matrix Ψ of the network graph can be obtained by replacing the non-zero elements by ones, leaving the null elements unchanged.
2. A threshold value ε is used for determining whether a term, which takes into account the actuator capacity (maximum allowable flow) and its usage frequency, has a negligible *effect* on the entire plant. In this way the less important actuators are filtered out, in order to reduce the coupling degree of the system and identify independent subnetworks.

The partitioning algorithm proceeds by decomposing the matrix $\mathbf{I_c}$ into a set of submatrices, named as *partitions* and denoted by $\mathcal{P}_\varepsilon = \{\mathbf{I_{c1}}, \cdots, \mathbf{I_{cM}}\}$. Then, $\mathcal{P}_\varepsilon$ correspond to a set of subgraphs (subsystems) obtaining by deleting the edges corresponding to elements of $\mathbf{I_c}$ with magnitude no larger than ε. That is, the idea behind the partitioning approach is to neglect less important elements (i.e., links) in matrix $\mathbf{I_c}$ such that the resulting $\tilde{\mathbf{I}}_c$ is less coupled. Ideally, $\tilde{\mathbf{I}}_c$ should lead to a permutation matrix $\mathbf{P}$ such that $\mathbf{P'\tilde{M}P}$ is block-diagonal. This procedure is repeated iteratively by reducing ε until an enough number of partitions is obtained. Algorithm 14 summarises the steps of the proposed partitioning algorithm.

Partitions can be tuned by means of parameter ε of the proposed approach, which makes the user able to attempt matching the desired number and size of subsystems.

Typically, in the first iteration, Algorithm 14 neglects a high number of elements of $\mathbf{I_c}$, highly reducing the matrix connectivity degree and obtaining a subsystem decomposition. Then, once the sets of states/inputs relative to each partition are computed, the task of finding a suitable $\mathbf{P}$ that block-diagonalises the matrix $\mathbf{P'\tilde{M}P}$ is a matter of linear algebra implementation. Every subsystem is composed by sets of state and input variables that are linked, meaning that are in the same block in the $\mathbf{P'\tilde{M}P}$ diagonal. Let $\mathcal{X}^i$ and $\mathcal{U}^i$ be respectively the sets of state and input variables assigned to subsystem i, while $|\mathcal{X}^i|$ and $|\mathcal{U}^i|$ determine the number of variables for each set. A subsystem is created if both numbers are different than zero. All state and input variables that are not assigned to any of the currently created subsystems, i.e., that does not belong to $\mathcal{X}^i$ or $\mathcal{U}^i$, respectively, are available for the next iteration. Otherwise, variables already assigned to a subsystem in the current or in a previous iteration, are *masked*[16] to prevent their reassignment to other subsystem.

Then, a new iteration of the algorithm starts by decreasing ε (e.g., halving ε). Algorithm 14 iterates until all state variables are assigned to a subsystem. Note that the algorithm may terminate even if some inputs are not be assigned to any subsystem, which is due to automatic threshold based neglecting process. Such issue can be managed by manually include unassigned inputs to proper subsystem following engineering insight.

The importance of the mask arises from the structure of the algorithm. In fact, if not excluded, all previously assigned states and inputs would be part of the next iteration partition, introducing couplings and hence increasing the size of the resulting submodels. The aforementioned inclusion easily follows from the decreasing of ε among sequential iterations.

Few remarks on the above algorithm:

1. At any iteration of Algorithm 14, the numerical value of ε is a crucial tuning knob of the approach. A guideline is that the larger is the decreasing step, the larger is the size of the obtained subsystems. Ways for automatically determining the step size are a subject of current research.
2. Matrix $\mathbf{E}$ in (16.9b) defines a constraint among actuators that can be easily taken into account if all the actuators belong to the same subsystem. Otherwise,

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[16] Consider a variable to be masked when it does not belong to any set since it has already been classified in a previous iteration.

Algorithm 14 Automatic partitioning algorithm

1: Initialise masks to a neutral value
2: Initialise the sets of unassigned variables $\mathcal{X}$ and $\mathcal{U}$ with all state and input variables, respectively
3: Determine the number of unassigned states: $N_x = |\mathcal{X}|$;
4: Init ε
5: **while** $N_x > 1$ **do**
6:     Apply masks to $\mathbf{A}_{sp}$ and $\mathbf{B}_{sp}$
7:     $\mathbf{I_c} = [\mathbf{A}_{sp}\ \mathbf{B}_{sp}\bar{\mathbf{u}}]$
8:     For all elements of $\mathbf{I_c}$
9:     **if** $\mathbf{I}_{\mathbf{c}i,j} < \varepsilon$ **then**
10:         $\tilde{\mathbf{I}}_{\mathbf{c}i,j} = 0$;
11:     **else**
12:         $\tilde{\mathbf{I}}_{\mathbf{c}i,j} = 1$;
13:     **end if**
14:     Find $\mathbf{P}$ such that $\mathbf{P}'\tilde{\mathbf{M}}\mathbf{P}$ is block diagonal
15:     Identify parts satisfying $N_{x^i} = |\mathcal{X}^i| > 0$ and $N_{u^i} = L(\mathcal{U}^i) > 0$ and add to previous ones
16:     Update N_x
17:     Update masks with updated states and inputs
18:     Update ε
19: **end while**

---

    since each controller manipulates every partition independently from the others, negotiations between controllers would be required to guarantee the fulfilment of node constraints.

3. The use of masks to prevent state reassignment avoids that submodels have overlapping states and inputs: if a state variable is used in a model by a controller, no other controller can use it. The main benefit of this choice is the very low level of coupling between partitions, but the price to pay is a potential decrease of closed-loop performance.

4. The current structure of the algorithm is unsuitable to handle state overlaps because it relies on links between elements that present different degree of coupling. Hence, once the stronger couplings are eliminated (using masking), the weaker ones gain relative importance. State overlaps may be introduced a posteriori based on engineering insight, in order to increase the adherence with respect the original centralised model. Handling overlapping in an automatic way is also a current research topic.

5. In some cases even relatively small connections, i.e., capable of carrying a minor amount of water, are very important for demand satisfaction. A way of accounting for such an issue is to perform a simulation using, for instance, a CMPC controller, and compute the average percentage of use for each actuator. Thus, this information could be used to weight $\bar{\mathbf{u}}$ component-wise. The main drawback of this approach is the need of (and dependence on) simulation.

6. Note that the proposed algorithm can be customised by setting different importance levels of states vs. inputs, by weighting the related components in $\mathbf{I_c}$ from its statement at (16.9a).

7. The structure of the proposed algorithm suggests that termination is achieved if the ε value is decreased at each iteration. However, at the current status of development, the algorithm cannot guarantee any property for the resulting partitioning but the assignment of all system-state variables to a subsystem.

The decomposition process of matrix $\mathbf{I_c}$ reported here is similar to the one proposed by the ε-decomposition method in [20]. The underlying idea in both cases is to disconnect those actuators corresponding to interconnections with strength smaller than the prescribed ε, identifying the disconnected subsystems. According to [20], there are s different ε-decompositions $\mathcal{P}_\varepsilon$ that can be obtained for different values of ε satisfying

$$\max_{i\neq j}\left|m_{ij}\right| = \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_K = 0,$$

with $K \leq s$, where $s = dim(\mathbf{I_c})$. Moreover, such decompositions are nested, that is, the partitions obtained satisfy: $\mathcal{P}_{\varepsilon_1} \subset \mathcal{P}_{\varepsilon_2} \cdots \mathcal{P}_{\varepsilon_K}$ with $\mathcal{P}_{\varepsilon_1}$ being the finest and $\mathcal{P}_{\varepsilon_k}$ the coarsest. The main novelty of the algorithm presented in this chapter is the matrix normalisation taking into account actuator physical/operative limits, and the iterative threshold updating that allows one to take into account weaker coupling without being influenced by the stronger ones.

16.4 Simulations and results

16.4.1 Results using masks-based approach

Using the partitioning algorithm presented in this section, the aggregate model of the Barcelona WTN is decomposed in three subsystems, as depicted in Figure 16.1 in different colours. The resultant decomposition follows the scheme shown in Figure 16.2, where μ_i denotes the i-th vector of shared variables among the subsystems S_j, for $j = 1, \ldots, M$. The subsystems are defined by the following elements:

- *Subsystem 1:* Composed by tanks x_i, $i \in \{1,2\}$, inputs u_j, $j \in \{1:5\}$, demands d_l, $l \in \{1,2,3\}$, and nodes n_q, $q \in \{1,2\}$. It is represented in Figure 16.1 with red colour and corresponds to Subsystem S_1 in Figure 16.2.
- *Subsystem 2:* Composed by tanks x_i, $i \in \{3,4,5,12,17\}$, inputs u_j, $j \in \{7:16,18,19,25,26,32,34,40,41,47,48,56,60\}$, demands d_l, $l \in \{4:7,15,18,22\}$, and nodes n_q, $q \in \{3,4,7\}$. It is represented in Figure 16.1 with green colour and corresponds to Subsystem S_2 in Figure 16.2.
- *Subsystem 3:* Composed by tanks x_i, $i \in \{6:11,13:16\}$, the inputs u_j, $j \in \{6,17,20:24,27:31,33,35:39,42:46,49:55,57,58,59,61\}$, demands d_l, $l \in \{8:14,16,17,19,20,21,23,24,25\}$, and nodes n_q, $q \in \{5,6,8:11\}$. It is represented in Figure 16.1 with blue colour and corresponds to Subsystem S_3 in Figure 16.2.
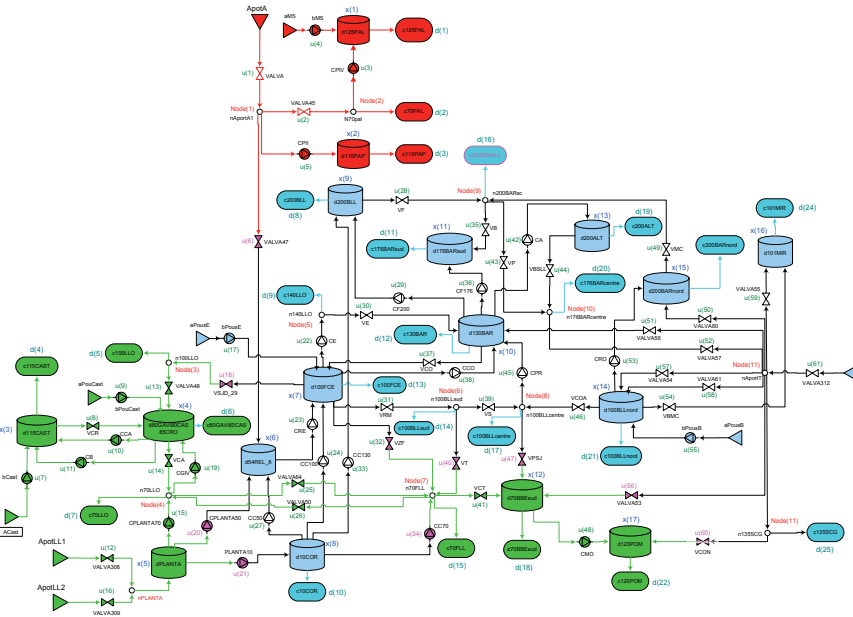
Fig. 16.1 Partition of the Barcelona WTN, aggregate model

Table 16.1 Dimension comparison between the subsystems and the whole network.

Elements	Subsystem 1	Subsystem 2	Subsystem 3	Whole Model
Tanks	2	5	10	17
Actuators	5	22	34	61
Demands	3	7	15	25
Nodes	2	3	6	11

Table 16.1 collects the resultant dimensions for each subsystem and the corresponding comparison with the dimensions of the vectors of variables for the entire aggregate network.

16.4.2 Results using graph-theory-based approach

This section presents the results of the application of Algorithm 14 for the partitioning of the Barcelona WTN into compositional subsystems. Algorithm 13 and auxiliary routines presented in Section 16.3.1 have been designed for any system. However, some particular features should be introduced depending on the considered case study and control law in order to obtain an suitable decomposition. More
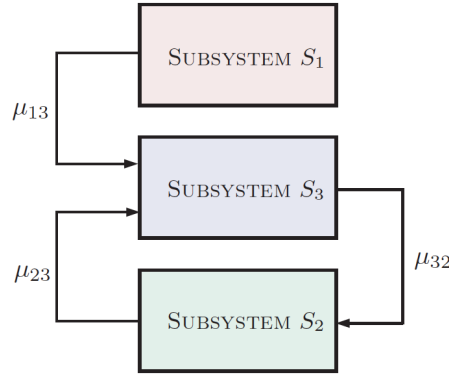
Fig. 16.2 Conceptual scheme of the partitioned Barcelona WTN

precisely, the graph of the Barcelona WTN has been derived from its mathematical model expressed in the way introduced in Chapter 12, i.e.,

$$\mathbf{x}(k+1) = \mathbf{A}\,\mathbf{x}(k) + \mathbf{B_u}\,\mathbf{u}(k) + \mathbf{B_d}\,\mathbf{d}(k), \tag{16.10a}$$
$$0 = \mathbf{E_u}\,\mathbf{u}(k) + \mathbf{E_d}\,\mathbf{d}(k), \tag{16.10b}$$

under the following considerations:

- every tank, sector of consume, water source and node is considered as a vertex of the graph;
- every pump, valve and link with a sector of consume is considered as a graph edge.

In order to evaluate the partitioning results obtained from the application of Algorithm 13 and auxiliary routines to the Barcelona WTN, the following indexes are taken into account additionally to those previously introduced:

- $\varepsilon \triangleq \sum_{i=1}^{M} \varepsilon_i,$
- $\bar{\varepsilon} \triangleq \frac{\varepsilon}{M}$ (arithmetic mean),
- $\sigma_\varphi^2 \triangleq \frac{1}{M} \sum_{i=1}^{M} (\varphi_i - \bar{\varphi})^2,$
- $\sigma_\varepsilon^2 \triangleq \frac{1}{M} \sum_{i=1}^{M} (\varepsilon_i - \bar{\varepsilon})^2.$

Remark 16.4. Notice that although ε is not directly related with the number of shared edges between subgraphs obtained by using Algorithm 14, this index gives an indirect idea about their level of interconnection. Recall that the objective of the partitioning algorithm is the minimisation of indexes σ_φ^2, ε, and ε_i (for $i = 1, 2, \ldots, M$) to obtain a graph decomposition as less interconnected as possible and with similar number of vertices for each subgraph (internal weight).

Table 16.2 Results for different partitioning approaches

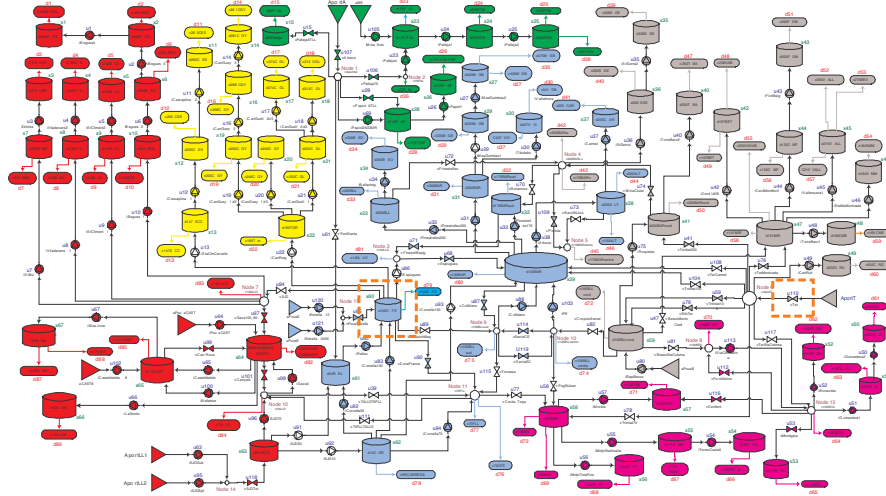| ROUTINE COMBINATION	M	$\bar{\varphi}$	$\bar{\varepsilon}$	σ_φ^2	σ_ε^2	ε
1	17	10.59	3.76	53.88	25.32	64
2	13	6.30	4.15	21.39	27.80	54
3	10	8.20	5.10	31.73	32.76	52
4	6	13.67	6.33	**14.88**	25.22	**38**

Fig. 16.3 Definitive partition of the Barcelona WTN. The *key* elements are properly featured

Table 16.2 summarises the partitioning results obtained applying Algorithm 13 (A1) combined with the auxiliary routine/filters presented in Section 16.3.1 performing the following combinations:

1. No auxiliary routines are considered.
2. A1 and Pre-filtering (Pre-F) routine only.
3. A1 in addition to Pre-F and Post-filtering (Post-F) routines.
4. A1 in addition to Pre-F, Post-F and Anti-Oscillation (AO) routines.

This distinction has been done in order to understand how the proposed routines affect the partitioning results.

Using only the Algorithm 13, the resultant partitioning $\mathcal{P}$ is comprised by 17 subgraphs. Many of them are small and cannot be merged since their neighbour subgraphs have internal weights with values quite close to $\bar{\varphi}$ (see Section 16.3.1). Moreover, there are several vertices with $\omega = 1$, which correspond to network water sources and demands, leading to unnecessarily difficult algorithm computations due to sizes of the resultant subgraphs (in terms of internal weight). By employing the Pre-F routine, the previous problems are fixed and Algorithm 13 produces 13

Table 16.3 Dimension comparison of the WTN subsystems

SUBSYSTEM	Tanks	Actuators	Demands	Nodes
1	13	36	20	5
2	11	11	11	0
3	13	22	20	3
4	9	16	12	2
5	6	10	8	2
6	15	26	17	3
Total	67	121	88	15

Table 16.4 Dimensions of shared links μ_{ij}

SET	μ_{12}	μ_{13}	μ_{14}	μ_{16}	μ_{31}	μ_{34}	μ_{51}	μ_{61}
NUMBER OF u'S	2	2	2	2	4	3	1	3

subgraphs (see Table 16.2). Additionally, if the refining routine embedded within Algorithm 13 is complemented with the Post-F routine, setting $\delta = 2$, a partitioning with ten subgraphs is reached[17]. Finally, if the AO routine is also considered, setting the refining limit to $\rho = 250$, a partitioning with six subgraphs is now reached. According to Table 16.2, this last partitioning (Combination 4) satisfies the minimisation of the average of the internal weights for all resultant subgraphs as well as the interconnection degree between subgraph measured through ε. It is important to highlight that the proposed partitioning approach automatically determines the final number of partitions M (six for this case) when the conditions 3 and 4 of Problem 16.1 are fulfilled (see Remark 16.2). The tuning parameters δ and ρ also influence in the obtained value of M.

Notice that each subgraph of the final decomposition corresponds to a subsystem of the Barcelona WTN with the number of elements presented in Table 16.3. Figure 16.3 shows, in different colours, the obtained subsystems of Barcelona WTN.

Moreover, Figure 16.4 schematically shows the disposition of the resultant subsystems S_i, for $i \in \{1, \ldots, 6\}$, and the sets μ_{ij} of shared links between the network subsystems corresponding to the control inputs u (manipulated flows), whose directionality is defined from S_i to S_j for $j \in \{1, \ldots, 6\}$, $i \neq j$. Table 16.4 collects the number of control inputs of each set μ_{ij}.

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[17] Notice that increasing the parameter δ implies that $\sigma_{\varepsilon_i}^2$ becomes bigger.
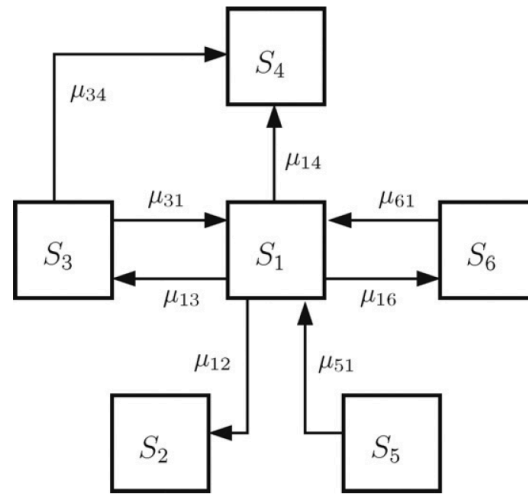
Fig. 16.4 Network subsystems S_i and their sets of shared connections μ_{ij}

16.5 Conclusions

This chapter has proposed two approaches for the automatic partitioning of a WTN into subsystems intended to be applied along with a non centralized model predictive control strategy. The algorithm transforms the dynamical model of the given system into a graph representation. Once the equivalent graph has been obtained, the problem of graph partitioning is then solved. The resultant partitions are composed of a set of non-overlapping subgraphs such that their sizes, in terms of number of vertices, are similar and the number of edges connecting them is minimal. To achieve this goal the algorithm applied a set of procedures based on identifying the highly connected subgraphs with balanced number of internal and external connections. Some additional pre-filtering and post-filtering routines are also needed to be included to reduce the number of obtained subsystems. The performance of the proposed decomposition approach has been assessed in a real case study based on the Barcelona WTN. An study of the effect of auxiliary routines on the basic partitioning algorithm has also been included showing the benefits of their use.

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