The Forward Kinematics of Doubly-Planar Gough-Stewart Platforms and the Position Analysis of Strips of Tetrahedra

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Abstract. A strip of tetrahedra is a tetrahedron-tetrahedron truss where any tetrahedron has two neighbors except those in the extremes which have only one. The problem of finding all the possible lengths for an edge in the strip compatible with a given distance imposed between the strip end-points has been revealed of relevance due to the large number of possible applications. In this paper, this is applied to solve the forward kinematics of 6-6 Gough-Stewart platforms with planar base and moving platform, a problem which is known to have up to 40 solutions (20 if we do not consider mirror configurations with respect to the base as different solutions).

Keywords: Position analysis, closed-form solutions, Distance Geometry, Gough-Stewart platform.

1 Introduction

During the last years, different methods have been developed, based on the use of Distance Geometry, to solve position analysis problems which includes the inverse kinematics and the forward kinematics of serial and parallel robots, respectively. The strategy followed in all these methods is simple: the problem is reduced to find some distances so that, once known, the problem can be trivially solved by a sequence of trilaterations. Position analysis problems are thus essentially reduced to find a set of closure polynomials whose variables are the unknown distances. Since no arbitrary reference frames have to be introduced to obtain these polynomials, these methods can be classified as intrinsic.

Contrarily to what happens when using kinematic loop-closure equations to solve a position analysis problem [1], the number of generated equations using distance-based methods does not depend on the number of kinematic loops. Nice examples of this fact are the planar Watt-Baranov trusses [2] and a remarkable class of spatial variable geometry trusses [3]. Actually, the number of unknown distances that have to be introduced, which is independent of the number of kinematic loops, is the most relevant factor in these methods. Obviously, this number has to coincide with the number of generated equations in well-defined problems.

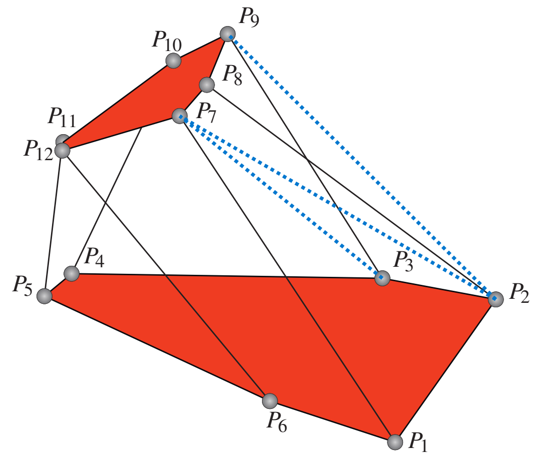

Fig. 1. A 6-6 Gough-Stewart platform. If the distances of the dotted blue segments are known, its forward kinematics can be straightforwardly solved by a sequence of trilaterations.

This paper can be seen as a continuation of the work presented in [4] where a simple procedure to obtain the distances between the end-points of a strip of tetrahedra is detailed. The resulting formula can be manipulated to obtain any edge length of the strip as the roots of a polynomial whose coefficients depend on the distance between the strip end-points and all other edge lengths. This technique was applied to solve the forward kinematics of a 3-4 and a 4-4 Gough-Stewart platform because in these two cases the problem can be solved by introducing a single distance or, in other words, the problem can be reduced to the position analysis of a single strip of tetrahedra.

The algorithm presented in [4] assumes that the tetrahedra in the strip have no orientation constraints. Although the technique can be extended to take into account such constraints, here we still adhere to this assumption as our main goal is explaining how the method extends to the case in which we have to consider several strips of tetrahedra at the same time.

To solve the forward kinematics of the general 6-6 Gough-Stewart platform, using a sequence of trilaterations, we have to introduce three unknown distances. Indeed, let us consider the parallel platform in Fig. 1. If the unknown distances $d_{2,7}$, $d_{3,7}$, $d_{2,9}$ are taken as variables, we can apply a trilateration process to locate P_{10}, P_{11}, and P_{12}. As a consequence, since the distances $d_{4,10}$, $d_{5,11}$, $d_{6,12}$ are known, three closure conditions can straightforwardly be obtained. A single univariate condition can then be derived by elimination. Now, observe that, in the general case, the orientation of the moving platform is constrained with respect to that of the base. Nevertheless, if the moving platform is planar, its mirror reflection can be described as a rotation in three dimensions. In this case, no orientation constraints have to be considered and the problem can be entirely described in terms of distances. In this paper, we study the particular case in which both base and moving platform are planar.

Algebraic geometric methods have shown that the forward kinematics of the general 6-6 Gough-Stewart platform has up to 40 solutions [5–9], all of which can be real [10]. Husty [7] obtained a 40th-degree univariate polynomial by finding the greatest common divisor of the intermediate polynomials of degree 320. Innocenti [8] derived it from the two 56th-degree univariate polynomials. Dhingra *et al.* [9] used Gröbner-Sylvester hybrid method to obtain a 40th-degree polynomial directly from the 64×64 Sylvester's matrix formed by the calculated Gröbner basis. Any of these methods can obviously be applied to solve the particular case in which the base and the platform are planar, but this case has interest on its own, both from practical and theoretical points of view. This is why Lee and Shim presented a method adapted to solve its forward kinematics [11], and Borras *et al.* studied its singularities [12].

The paper is organized as follows. Section 2 gives some basic facts on the distance geometry of strips of tetrahedra. These concepts are needed to understand the procedure, detailed in Section 3, to generate the three distance-based closure conditions associated with a 6-6 Gough-Stewart platform. For comparison purposes, Section 4 solves the same problem as the one reported in [11] using these three closure conditions. Finally, Section 5 offers some conclusions and prospects for further research.

2 Background

Let us consider two tetrahedra that share the face defined by points P_i, P_j, and P_k. The squared distance between their apexes, say P_l and P_m, can be expressed as follows (see [13, 14] for details):

$$s_{l,m} = \frac{2}{D(i,j,k;i,j,k)} \Big(D(i,j,k,l;i,j,k,m)\big|_{s_{l,m}=0} \pm \sqrt{D(i,j,k,l;i,j,k,l)\, D(i,j,k,m;i,j,k,m)} \Big), \quad (1)$$

where

$$D(i_1,\dots,i_n;j_1,\dots,j_n) = 2\left(-\tfrac{1}{2}\right)^n \begin{vmatrix} 0 & 1 & \dots & 1 \\ 1 & s_{i_1,j_1} & \dots & s_{i_1,j_n} \\ 1 & \vdots & \ddots & \vdots \\ 1 & s_{i_n,j_1} & \dots & s_{i_n,j_n} \end{vmatrix}, \quad (2)$$

$s_{i,j}$ being the squared distance between P_i and P_j.

By iterating this operation, it is possible to obtain the distance between the end-points of a strip of tetrahedra as a function of all edge-lengths in the strip. The result is an expression with nested square roots. As explained in [3, 4], these squared roots can be cleared to obtain a polynomial expression in terms of all involved distances.

3 Obtaining Closure Polynomials

We can derive the strip of tetrahedra that appears in Fig. 2 from the 6-6 Gough-Stewart platform in Fig. 1. The end-points of this strip can either be, on one side, P_4, P_5, or P_6,

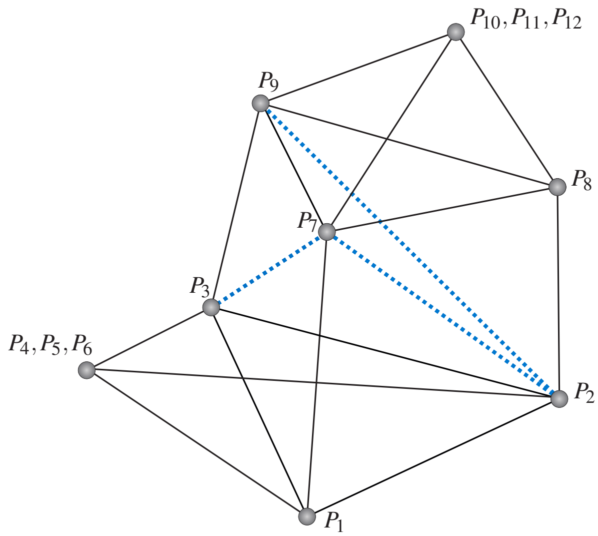

Fig. 2. This strip of five tetrahedra can be derived from the 6-6 Gough-Stewart platform in Fig. 1. The distances in dotted blue lines are unknown.

and, on the other, either P_{10}, P_{11}, or P_{12}. Then, since only $s_{2,7}$, $s_{3,7}$, and $s_{2,9}$ are unknown distances, using the procedure mentioned in the previous section and detailed in [3, 4], we have that:

$$s_{i,j} = f_{i,j}(s_{2,7}, s_{3,7}, s_{2,9}), \tag{3}$$

where $i \in \{4,5,6\}$ and $j \in \{10,11,12\}$. Thus, in particular, we have

$$s_{4,10} = f_{4,10}(s_{2,7}, s_{3,7}, s_{2,9}), \tag{4}$$

$$s_{5,11} = f_{5,11}(s_{2,7}, s_{3,7}, s_{2,9}), \tag{5}$$

$$s_{6,12} = f_{6,12}(s_{2,7}, s_{3,7}, s_{2,9}). \tag{6}$$

Since the distances on the left-hand sides of the above three equations are known, we have three equations in three unknowns. Now, since the right-hand sides contain radicals, we have to use the procedure mentioned above to clear them [3, 4].

Assuming that we are dealing with a Gough-Stewart platform with coplanar leg attachments in general position within their planes, we obtain the following system of three polynomial closure conditions, of degree 12 each, in three variables

$$\left.\begin{array}{l} p_1(s_{2,7}, s_{3,7}, s_{2,9}) = 0 \\ p_2(s_{2,7}, s_{3,7}, s_{2,9}) = 0 \\ p_3(s_{2,7}, s_{3,7}, s_{2,9}) = 0 \end{array}\right\} \tag{7}$$

Now, we can eliminate, for example, $s_{3,7}$ from $p_1(\cdot)$ and $p_2(\cdot)$, and from $p_1(\cdot)$ and $p_3(\cdot)$. Then, we obtain two polynomials in $s_{2,7}$ and $s_{2,9}$, say $p_{1,2}(s_{2,7}, s_{2,9})$ and $p_{1,3}(s_{2,7}, s_{2,9})$. These two polynomials, of degree 128 each, factor into four terms of degree 20, 28, 36 and 44.

By computing the resultant of pairs of these factors, one from each polynomial, to eliminate $s_{2,7}$, we obtain polynomials in one variable: $s_{2,9}$. These univariate polynomials also factorize into terms of different degrees as detailed in Table 1:

		$p_{1,2}(s_{3,7},s_{2,9})$			
		20	**28**	**36**	**44**
$p_{1,3}(s_{3,7},s_{2,9})$	**20**	20 380	60 500	60 660	100 780
	28	60 500	2 (32) 60 628	116 892	2 (32) 116 1020
	36	60 660	116 892	2 (16) 60 1108	2 (16) 116 1340
	44	100 780	2 (32) 116 1020	2 (16) 116 1340	2 (16) 2 (32) 2(32) 100 1548

Table 1. Degrees of all polynomial factors in $s_{2,9}$ resulting from computing the resultant of the four factors of $p_{1,2}(s_{3,7},s_{2,9})$ and $p_{1,3}(s_{3,7},s_{2,9})$. The multiplicity of each factor appears in parenthesis, if not 1.

While some quadratic factors in Table 1 are repeated (actually, there are only three different quadratic factors), all other factors of higher degree are different.

Since we know that the maximum number of assembly modes of a general Gough-Stewart platform is 40, all factors of higher degree than this bound necessarily encompass extraneous roots. Moreover, the quadratic factors with multiplicity 16 and 32 are neither part of the solution because this would mean that the analyzed platform is quadratically solvable; in other words, it would be *trilaterable* [15]. In this context, trilaterable means that it would be possible to generate a single strip of tetrahedra involving all vertices such that all edge lengths in it are known, which is impossible in this problem. As a result, only one factor survives: the one of degree 20 which appears when factorizing the resultant of the factors of degree 20 of both $p_{1,2}(\cdot)$ and $p_{1,3}(\cdot)$.

Alternatively, to conclude that the 20th-degree factor is the only factor containing the desired solutions, we have also generated a different set of strips of tetrahedra containing three different variable edges, except P_2,P_9. Then, we have verified that the sought factor is the common factor resulting from using both sets of strips.

Once the roots of the 20th-degree factor in $s_{2,9}$ are computed, each of these roots can be substituted back in any of the two factors of degree 20 of $p_{1,2}(\cdot)$ or $p_{1,3}(\cdot)$ to obtain the corresponding solutions for $s_{3,7}$. Then, the results can be substituted in either $p_1(\cdot)=0$, $p_2(\cdot)=0$, or $p_3(\cdot)=0$, to finally obtain the solutions for $s_{2,7}$.

4 Example

To properly compare our results with those reported in [11], we use here the same example. Nevertheless, it is important to notice that the length of the sixth leg in this latter reference is incorrect. Using the same notation as in Fig. 1, instead of $s_{6,12} = 117.805089939638$, it should be $s_{6,12} = 119.2390086191201$. Then, the considered Gough-Stewart platform defines the following distance matrix:

$$\mathbf{S} = \begin{bmatrix}
0.0 & 3844.0 & 3965.0 & 3208.0 & 2545.0 & 218.0 & 9889.0 & s_{1,8} & s_{1,9} & s_{1,10} & s_{1,11} & s_{1,12} \\
3844.0 & 0.0 & 121.0 & 1844.0 & 2421.0 & 3194.0 & s_{2,7} & 14977.47 & s_{2,9} & s_{2,10} & s_{2,11} & s_{2,12} \\
3965.0 & 121.0 & 0.0 & 1129.0 & 1684.0 & 3029.0 & s_{3,7} & s_{3,8} & 24340.67 & s_{3,10} & s_{3,11} & s_{3,12} \\
3208.0 & 1844.0 & 1129.0 & 0.0 & 101.0 & 1850.0 & s_{4,7} & s_{4,8} & s_{4,9} & 23700.59 & s_{4,11} & s_{4,12} \\
2545.0 & 2421.0 & 1684.0 & 101.0 & 0.0 & 1301.0 & s_{5,7} & s_{5,8} & s_{5,9} & s_{5,10} & 18569.53 & s_{5,12} \\
218.0 & 3194.0 & 3029.0 & 1850.0 & 1301.0 & 0.0 & s_{6,7} & s_{6,8} & s_{6,9} & s_{6,10} & s_{6,11} & 14217.94 \\
9889.0 & s_{2,7} & s_{3,7} & s_{4,7} & s_{5,7} & s_{6,7} & 0.0 & 196.0 & 2378.0 & 2845.0 & 2554.0 & 2020.0 \\
s_{1,8} & 14977.47 & s_{3,8} & s_{4,8} & s_{5,8} & s_{6,8} & 196.0 & 0.0 & 1258.0 & 1753.0 & 2106.0 & 1768.0 \\
s_{1,9} & s_{2,9} & 24340.67 & s_{4,9} & s_{5,9} & s_{6,9} & 2378.0 & 1258.0 & 0.0 & 197.0 & 1600.0 & 1802.0 \\
s_{1,10} & s_{2,10} & s_{3,10} & 23700.59 & s_{5,10} & s_{6,10} & 2845.0 & 1753.0 & 197.0 & 0.0 & 853.0 & 1125.0 \\
s_{1,11} & s_{2,11} & s_{3,11} & s_{4,11} & 18569.53 & s_{6,11} & 2554.0 & 2106.0 & 1600.0 & 853.0 & 0.0 & 58.0 \\
s_{1,12} & s_{2,12} & s_{3,12} & s_{4,12} & s_{5,12} & 14217.94 & 2020.0 & 1768.0 & 1802.0 & 1125.0 & 58.0 & 0.0
\end{bmatrix},$$

where $\mathbf{S}(i,j) = s_{i,j} = d_{i,j}^2$.

Following the steps detailed in the previous section, the 20th-degree polynomial in $s_{2,9}$ is:

$$s_{2,9}^{20} - 3.8124\times10^{5}\, {s_{2,9}}^{19} + 1.0636\times10^{10}\, {s_{2,9}}^{18} + 1.2896\times10^{16}\, {s_{2,9}}^{17} - 2.7847\times10^{21}\, {s_{2,9}}^{16} +$$
$$3.0704\times10^{26}\, {s_{2,9}}^{15} - 2.2140\times10^{31}\, {s_{2,9}}^{14} + 1.1390\times10^{36}\, {s_{2,9}}^{13} - 4.3622\times10^{40}\, {s_{2,9}}^{12} +$$
$$1.2726\times10^{45}\, {s_{2,9}}^{11} - 2.8611\times10^{49}\, {s_{2,9}}^{10} + 4.9798\times10^{53}\, {s_{2,9}}^{9} - 6.7075\times10^{57}\, {s_{2,9}}^{8} +$$
$$6.9694\times10^{61}\, {s_{2,9}}^{7} - 5.5643\times10^{65}\, {s_{2,9}}^{6} + 3.3970\times10^{69}\, {s_{2,9}}^{5} - 1.5529\times10^{73}\, {s_{2,9}}^{4} +$$
$$4.7760\times10^{76}\, {s_{2,9}}^{3} - 5.1150\times10^{79}\, {s_{2,9}}^{2} - 2.6566\times10^{83}\, s_{2,9} + 8.9169\times10^{86}.$$

The positive reals roots of this polynomial are: 7451.80, 9587.28, 17271.4, 24044.1, 24511.3, 24579.0, 26132.2, 27680.5, 28332.5, 28809.9, 44848.6, and 251456. Obviously, complex or negative real squared distances have no physical meaning in our problem and they can be discarded. Moreover, these ten solutions do not necessarily lead to real solutions for $s_{2,7}$ and $s_{3,7}$. In this case, only the roots 24511.3 and 24579.0 lead to real configurations of the strip of tetrahedra. The corresponding parallel platform configurations appear in Fig.3. They coincide with the two real solutions reported in [11].

In this example, the quadratic factors appearing in Table 1 are:

$$s_{2,9}^2 - 49578.5\, s_{2,9} + 598299000.0 = 0, \tag{8}$$

$$s_{2,9}^2 - 32415.2\, s_{2,9} + 188224000.0 = 0, \tag{9}$$

$$s_{2,9}^2 - 48923.3\, s_{2,9} + 586592000.0 = 0. \tag{10}$$

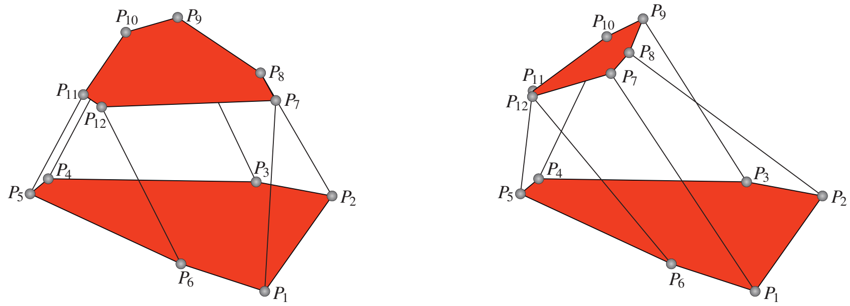

Fig. 3. Forward kinematics solutions of the analyzed 6-6 Gough-Stewart platform. The two mirror configurations with respect to the fixed base are also solutions but they are not represented.

The last factor is identically equal to $D(2,3,9;2,3,9)$ which vanishes when P_2, P_3, and P_9 are aligned [16]. For symmetry reasons, some of the other terms of higher degree should correspond to other alignments. Nevertheless, since they are expressed in terms of $s_{2,9}$, instead of a variable edge length connecting two of the three points involved in an alignment, their expressions are very complicated.

5 Conclusions

It has been shown how to derive three closure polynomials for 6-6 Gough-Stewart platforms with planar base and moving platform in terms of three unknown distances, and how the solution to this system of equations permits solving the forward kinematics of the analyzed Gough-Stewart platform by a trivial sequence of trilaterations.

The elimination process to generate the 20th-degree final polynomial in a single distance from the three closure polynomials leads to extraneous factors. The geometric meaning of some of them have been identified and that of some others, conjectured. The analysis of the groups of substitution of finite order associated with the symmetries of the analyzed strips of tetrahedra could play a fundamental role in the identification of the geometric meaning of all these factors. They actually seem to be correlated with the Galois groups of the obtained factors. In the same way that a knot polynomial is a knot invariant in the form of a polynomial whose coefficients encode some of the properties of a given knot, the study of all polynomial factors derived from a set of strips of tetrahedra in terms of invariants opens a fascinating new field in Kinematics.

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References

1. G. Sandor, "On the loop equation in Kinematics," *Trans. of the 7th Conference on Mechanisms*, Purdue University, West Lafayette, Ind., pp. 49-56, 1962.
2. N. Rojas and F. Thomas, "Closed-form solution to the position analysis of Watt-Baranov trusses using the bilateration method," *ASME Journal of Mechanisms and Robotics*, Vol. 3, No. 3, 031001 (10 pages), 2011.
3. J.M. Porta and F. Thomas, "Closed-form position analysis of variable geometry trusses," *Mechanism and Machine Theory*, Vol. 109, pp. 14-21, 2017.
4. F. Thomas and J.M. Porta, "Closure polynomials for strips of tetrahedra," Advances in Robot Kinematics (ARK2016), Grasse, France, 2016.
5. F. Ronga and T. Vust, "Gough-Stewart platforms without computer?" in *Proc. of the 1992 Int. Conf. in Real Analytic and Algebraic Geometry*, Walter de Gruyter, pp. 196-212, 1995.
6. C. Wampler, "Forward displacement analysis of general six-in-parallel SPS (Stewart) platform manipulators using soma coordinates," *Mechanism and Machine Theory*, Vol. 31, No. 3, pp. 331-337, 1996.
7. M.L. Husty, "An algorithm for solving the direct kinematics of general Stewart-Gough platforms," *Mechanism and Machine Theory*, Vol. 31, No. 4, pp. 365-380, 1996.
8. C. Innocenti, "Forward kinematics in polynomial form of the general Stewart platform," *25th Biennal Mechanisms Conference*, Atlanta, DETC98/MECH-5894, 1998.
9. A. Dhingra, A. Almadi, D. Kohli, "A Gröbner-Sylvester hybrid method for closed-form displacement analysis of mechanisms," *25th Biennial Mechanisms Conference*, Atlanta, DECTC98/MECH-5969, 1998.
10. P. Dietmaier, "The Stewart-Gough platform of general geometry can have 40 real postures," In: Lenarcic J., Husty M.L. (eds) *Advances in Robot Kinematics: Analysis and Control*, Springer, Dordrecht, 1998.
11. T.-Y. Lee and J.-K. Shim, "Algebraic elimination-based real-time forward kinematics of the 6-6 Gough-Stewart platform with planar base and platform," *Proc. of the IEEE International Conference on Robotics and Automation*, pp. 1301-1306,2001.
12. J. Borras, F. Thomas, and C. Torras, "Singularity-invariant leg rearrangements in doubly-planar Gough-Stewart-Gough platforms," in Robotics: Science and Systems 2010, Zaragoza, Spain, June 27-30, 2010.
13. J.M. Porta, F. Thomas, L. Ros, C. Torras, "A branch-and-prune algorithm for solving systems of distance constraints" *Proc. of the 2003 IEEE International Conference on Robotics and Automation*, Taipei, Taiwan, 2003.
14. N. Rojas and F. Thomas, "The closure condition of the double banana and its application to robot position analysis," *Proc. International Conference on Robotics and Automation*, pp. 4641-4646, 2013.
15. J.M. Porta, L. Ros, and F. Thomas, "On the trilaterable six-degree-of-freedom parallel and serial manipulators," *Proc. of the 2005 IEEE International Conference on Robotics and Automation*, Barcelona, Spain, 2005.
16. F. Thomas and L. Ros, "Revisiting trilateration for robot localization," *IEEE Transactions on Robotics*, Vol. 21, No. 1, pp. 93-101, 2005.