Zonotopic Unknown Input Observer of Discrete-time Descriptor Systems for State Estimation and Robust Fault Detection ⋆

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Abstract:
This paper studies a set-based unknown input observer based on zonotopes for discrete-time descriptor systems affected by uncertainties with application to state estimation and robust fault detection. In this paper, two types of uncertainties are considered: (i) disturbances and noise both bounded by zonotopes; (ii) unknown inputs that can be decoupled. In terms of different applications, the observer gain for state estimation is designed to minimize the effects of unknown-but-bounded disturbances and noise as well as state estimation errors. On the other hand, for robust fault detection, in addition to attenuating uncertainties, the designed observer gain is also expected to be sensitive to faults. To achieve this goal, we propose an iterative algorithm to design the fault detection gain. Finally, some illustrative results in an application example show the effectiveness of the proposed algorithms.

Keywords: unknown input observer, zonotopes, state estimation, robust fault detection, discrete-time descriptor systems.

1. INTRODUCTION

Unknown input observers (UIOs) play a significant role in control domain during past several decades with a large amount of applications, such as state estimation [Hou and Muller, 1992], robust fault diagnosis [Chen and Patton, 2012, Rotondo et al., 2016] and fault-tolerant control [Cristofaro and Johansen, 2014]. For a real system, the mathematical model includes uncertainties to describe system dynamics and output behavior. These uncertainties can be modeled as unknown inputs allowing to represent modeling errors, unknown system disturbances, measurement noise, or faults among others.

To design an UIO, the key point is to remove or reduce the effects of unknown inputs. As an extension of UIO with set theory proposed in Xu et al. [2016, 2017], a set-based UIO is designed for discrete-time dynamical systems, where the unknown inputs are divided into two categories: one that can be decoupled and another which can be bounded by a deterministic set. Zonotopes are symmetric polytopic sets so that the zonotope can be determined by a center and a generator matrix, and the computational load of the operations required for implementing a recursive algorithm is low.

Descriptor systems, also known as singular, differential-algebraic systems, have been well-known in a variety of applications, such as water distribution networks [Wang et al., 2017a] and electrical circuits [Duan, 2010]. In addition to describe system dynamics, the static relationships among system variables are formulated by means of algebraic equations. When descriptor systems are affected by uncertainties, these can be separated in those that can be bounded as disturbance and noise, and unknown inputs that can be decoupled. The subject of fault diagnosis for descriptor systems is of interest and importance [Varga, 2017], especially to guarantee the safety and reliability of critical infrastructures, such as complex water networks and power systems.

The main contribution of this paper is to design a zonotopic UIO of discrete-time descriptor systems subject to unknown-but-bounded system disturbances and measurement noise as well as unknown inputs that can be decoupled. Considering the uncertainty boundedness, we recursively construct a zonotopic UIO for state estimation and robust fault detection (FD). All system uncertainties are propagated by operating the zonotopic sets. For state estimation, the observer gain for zonotopic UIO is designed

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to minimize the effects of uncertainties, that is, to minimize the size of the state bounding zonotope. Based on the results presented in Wang et al. [2018], we can obtain the optimal Kalman observer gain by using the F-radius to measure the size of the zonotope. For robust FD, we propose an optimization-based method to maximize the fault sensitivity and minimize the effects of uncertainties. Following the proposed computation procedure, an FD observer gain can be found. Finally, the proposed algorithms are tested in a numerical example.

The paper structure begins with some preliminary results on notations, definitions and properties in Section 2. The problem formulation is expressed in Section 3. The main results including UIO structure and observer gain designs are presented in Section 4. The proposed methods are tested with an illustrative example in Section 5. The paper is concluded in Section 6.

2. PRELIMINARIES

In this section, we introduce some preliminary results including notation, some definitions and properties.

Definition 1. (Zonotope). An r-order zonotope $\mathcal{Z}$ in n-dimensional space is defined by

$$\mathcal{Z} = \langle p, H\rangle = \{p + Hz, z \in \mathbf{B}^r\},$$

where $p \in \mathbb{R}^n$ is the center and $H \in \mathbb{R}^{n\times r}$ is the generator matrix, $\mathbf{B}^r = [-1, +1]^r$ is an r-order hypercube.

Definition 2. (Interval Hull). Given a zonotope $\mathcal{Z} = \langle p, H\rangle$, the interval hull $rs(H) \in \mathbb{R}^{n\times n}$ is defined as an aligned minimum box, where $rs(H)$ is a diagonal matrix with diagonal elements of $rs(H)_{i,i} = \sum_{j=1}^{r} |H_{i,j}|$ for $i = 1, 2, \ldots, n$.

Definition 3. (F-radius). Given a zonotope $\mathcal{Z} = \langle p, H\rangle$, the F-radius is defined by the Frobenius norm of H, and $\|H\|_F^2 = \mathrm{tr}\left(HH^\top\right) = \mathrm{tr}\left(H^\top H\right)$, where $\mathrm{tr}(\cdot)$ denotes the trace of a matrix and $H^\top$ denotes the transpose matrix of H.

Definition 4. (Covariation). Given a zonotope $\mathcal{Z} = \langle p, H\rangle$, the covariation is defined by $P = HH^\top$.

Denote the Minkowski sum and the linear image as $\oplus$ and $\odot$. Therefore, the following properties hold:

$$\langle p_1, H_1\rangle \oplus \langle p_2, H_2\rangle = \langle p_1 + p_2, [H_1 \ \ H_2]\rangle,$$
$$L \odot \langle p, H\rangle = \langle Lp, LH\rangle,$$
$$\langle p, H\rangle \subset \langle p, rs(H)\rangle,$$

where L is a matrix with appropriate dimension.

For a zonotope $\mathcal{Z} = \langle p, H\rangle$, the weighted reduction operator proposed in Combastel [2015] is denoted as $\downarrow_{q,W}(H)$, where q specifies the maximum number of column of H and W is a weighting matrix of appropriate dimension. $\downarrow_{q,W}(H)$ can be obtained by the following procedure:

- Sort the column of segment matrix H on decreasing order: $\downarrow_W(H) = [h_1, h_2, \ldots, h_r]$, $\|h_j\|_W^2 \geq \|h_{j+1}\|_W^2$, where $\|h_j\|_W$ is the weighted 2-norm of h_j.
- Take the first q-column of $\downarrow_W(H)$ and enclose a set $H_<$ generated by remaining columns into a smallest box (interval hull) computed by using $rs(\cdot)$:

  If $r \leq q$, **then** $\downarrow_{q,W}(H) = \downarrow_W(H)$,

  Else $\downarrow_{q,W}(H) = [H_>, rs(H_<)] \in \mathbb{R}^{n\times q}$,

  $H_> = [h_1, \ldots, h_q]\,,\ H_< = [h_{q+1}, \ldots, h_r]\,.$

Besides, I_r denotes an identity matrix with the dimension r. $\mathrm{rank}(X)$ denotes the rank of a matrix X.

3. PROBLEM FORMULATION

Consider a discrete-time descriptor linear time-invariant system with additive faults as

$$Ex^+ = Ax + Bu + D_w w + Dd + F_a f, \quad (1a)$$
$$y = Cx + D_v v + F_s f, \quad (1b)$$

where $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^{n_y}$ denote the state, known input and output vectors. $w \in \mathbb{R}^{m_w}$, $v \in \mathbb{R}^{m_v}$ are vectors of unknown system disturbances and measurement noise that are assumed to be bounded by a centered zonotope $\mathcal{W} = \langle 0, I_{m_w}\rangle$ and $\mathcal{V} = \langle 0, I_{m_v}\rangle$. $d \in \mathbb{R}^{m_d}$ denotes the unknown input vector that includes system disturbances that cannot be bounded in $\mathcal{W}$. $f \in \mathbb{R}^{m_f}$ denotes the normalized additive fault vector with $f \in \mathcal{F} = \langle 0, I_{m_f}\rangle$ and the magnitudes of actuator and sensor faults are defined by $F_a \in \mathbb{R}^{n_x\times m_f}$ and $F_s \in \mathbb{R}^{n_y\times m_f}$. $A \in \mathbb{R}^{n_x\times n_x}$, $B \in \mathbb{R}^{n_x\times m}$, $C \in \mathbb{R}^{n_y\times n_x}$, $D \in \mathbb{R}^{n_x\times m_d}$, $D_w \in \mathbb{R}^{n_x\times m_w}$ and $D_v \in \mathbb{R}^{n_y\times m_v}$. Besides, by definition of descriptor systems, $E \in \mathbb{R}^{n_x\times n_x}$ satisfies $\mathrm{rank}(E) \leq n_x$. For notation simplicity, the time instant k on all vectors in (1) is omitted and we replace the time instant $k+1$ by the superscript $+$.

For the descriptor system (1), the following assumption are made.

Assumption 1. The descriptor system (1) is assumed to be admissible, that is the matrix pair (E, A) is regular, causal and stable [Chadli and Darouach, 2012].

Assumption 2. The descriptor system (1) is assumed to be observable and matrices E and C satisfy the rank condition in [Wang et al., 2018, Eq. (5)].

Thus, we can find matrices $T \in \mathbb{R}^{n_x\times n_x}$ and $N \in \mathbb{R}^{n_x\times n_y}$ satisfying

$$TE + NC = I_{n_x}, \quad (2a)$$
$$TD = 0. \quad (2b)$$

Consider the initial state vector $x_0 \in \mathcal{X}_0 = \langle p_0, H_0\rangle$. In this work, we are interested in designing a zonotopic UIO of the descriptor system (1) with two applications:

(i) **State estimation** (with $f = 0$): design a zonotopic UIO and find an observer gain to minimize the effects of system uncertainties.

(ii) **Robust FD** (with $f \neq 0$): design a zonotopic UIO and find an observer gain to minimize the effects of system uncertainties and meanwhile maximize the sensitivity to the faults.

4. MAIN RESULTS

In this section, we first introduce the zonotopic UIO structure for the descriptor system (1). Then, the observer gain is designed with two different criteria for state estimation and robust FD. Finally, the implementations of these two applications are summarized by two algorithms.

4.1 Zonotopic UIO Structure for Descriptor Systems

According to Xu et al. [2016], we consider a basic UIO structure as

$$z^{+} = Mz + Ku + Gy, \tag{3a}$$
$$\hat{x} = z + Ny, \tag{3b}$$
$$\hat{y} = C\hat{x}, \tag{3c}$$
where $z \in \mathbb{R}^{n_x}$, $\hat{x} \in \mathbb{R}^{n_x}$ and $\hat{y} \in \mathbb{R}^{n_y}$ denote vectors of the observer state, the estimated state and output. $M \in \mathbb{R}^{n_x \times n_x}$, $K \in \mathbb{R}^{n_x \times n_u}$, $N \in \mathbb{R}^{n_x \times n_y}$ and $G \in \mathbb{R}^{n_x \times n_y}$.

Define the state estimation error $e = x - \hat{x}$. For the descriptor system (1), with a pair of matrices T and N satisfying (2a) and (3), the state estimation error dynamics for the descriptor system (1) can be expressed as
$$\begin{aligned} e^{+} = {} & Me + (TA - GC - M)\,x + MNy + (TB - K)\,u \\ & + TD_w w + TDd - GD_v v - ND_v v^{+} \\ & + (TF_a - GF_s)\,f - NF_s f^{+}. \end{aligned} \tag{4}$$
Since matrices T and N satisfy (2a), the effects of unknown inputs in (4) can be removed.

We now define the zonotopic UIO structure of descriptor systems assuming that the state vector x of the descriptor system (1) satisfies the inclusion $x \in \langle p, H\rangle$ at a given time instant $k \in \mathbb{N}$. When $k = 0$, the initial state vector $x_0 \in \langle p_0, H_0\rangle$ also holds.

Theorem 1. (Zonotopic UIO Structure). Consider the descriptor system (1) and $x \in \langle p, H\rangle$. The zonotopic UIO of the descriptor system (1) is recursively defined by $x^{+} \in \langle p^{+}, H^{+}\rangle$, where
$$\begin{cases} p^{+} &= (TA - GC)\,p + TBu + Gy + Ny^{+}, \\ H^{+} &= \left[(TA - GC)\,\bar{H},\ TD_w,\ -GD_v,\ -ND_v\right], \end{cases} \tag{5}$$
with $\bar{H} = \downarrow_{q,W}(H)$.

Proof. For $x \in \langle p, H\rangle$ at time instant k, according to the inclusion property in Section 2, $x \in \langle p, H\rangle \subset \langle p, \bar{H}\rangle$ holds. With $\hat{x} = p$, we also have $e = x - \hat{x} \in \langle 0, \bar{H}\rangle$.

Set $M = TA - GC$ and $K = TB$. With $TD = 0$ and $f = 0$, (4) becomes
$$\begin{aligned} e^{+} = {} & (TA - GC)\,e + (TA - GC)\,Ny \\ & + TD_w w - GD_v v - ND_v v^{+}. \end{aligned} \tag{6}$$

And from (3), we have
$$\hat{x}^{+} = (TA - GC)\,p + TBu + (G - (TA - GC)\,N)y + Ny^{+}.$$

Since $w \in \mathcal{W} = \langle 0, I_{m_w}\rangle$ and $v, v^{+} \in \mathcal{V} = \langle 0, I_{m_v}\rangle$, we derive $x^{+} = e^{+} + \hat{x}^{+}$ with
$$\begin{aligned} x^{+} & \in \langle p^{+}, H^{+}\rangle \\ & = \left((TA - GC) \odot \langle 0, \bar{H}\rangle\right) \oplus \langle (TA - GC)\,Ny, 0\rangle \\ & \quad \oplus (TD_w \odot \langle 0, I_{m_w}\rangle) \oplus (-GD_v \odot \langle 0, I_{m_v}\rangle) \\ & \quad \oplus (-ND_v \odot \langle 0, I_{m_v}\rangle) \oplus \langle \hat{x}^{+}, 0\rangle. \end{aligned}$$

Thus, by using the zonotope properties, we obtain p^{+} and H^{+} as in (5). □

Corollary 1. Consider the descriptor system (1) with $x \in \langle p, H\rangle$. The residual vector $r = y - Cx - D_v v$ can be enclosed by the residual zonotope $r \in \langle p_r, H_r\rangle$, where
$$\begin{cases} p_r &= y - Cp, \\ H_r &= \left[-CH,\ -D_v\right]. \end{cases} \tag{7}$$

Proof. From Theorem 1, $x \in \langle p, H\rangle$ can be computed recursively at time instant $k \in \mathbb{N}$. Let us define $r = y - Cx - D_v v$. With $v \in \mathcal{V} = \langle 0, I_{m_v}\rangle$, we derive
$$r \in \langle y, 0\rangle \oplus (-C \odot \langle p, H\rangle) \oplus (-D_v \odot \langle 0, I_{m_v}\rangle).$$
Thus, we obtain the residual zonotope in (7). □

Remark 1. In (5), G is a time-varying observer gain. Since the residual zonotope is a linear projection of the zonotopic UIO, the observer gain also has an effect on the residual zonotope.

Remark 2. For the descriptor system (1), if $f = 0$, then the output equation gives $0 = y - Cx - D_v v$ that implies $0 \in \langle p_r, H_r\rangle$ when no fault has occurred.

4.2 Observer Gain Design for State Estimation

For state estimation, the observer gain G for the zonotopic UIO in (5) is designed to minimize the effects of unknown-but-bounded system uncertainties. Inspired by the zonotopic Kalman filter in Combastel [2015], the size of a zonotope can be measured by the F-radius (see Definition 3) and to minimize the F-radius of a zonotope $\langle p, H\rangle$ involves minimizing the trace of its covariation $P = HH^T$ (see Definition 4). According to [Combastel, 2015, Section 4.3], an optimal Kalman gain is independent to any weighting matrix for the F-radius. Hence, for state estimation, we choose $J_s = \|H\|_F^2 = \mathrm{tr}(P)$ as a convex function with respect to G. For the zonotopic UIO in (5), we give the explicit result to compute the optimal Kalman gain for descriptor systems in the following theorem.

Theorem 2. (Optimal Kalman Gain). Given the zonotopic UIO of descriptor systems in (5), the optimal Kalman gain $G^{*} = \arg\min_G J_s$ with $J_s = \mathrm{tr}(P^{+})$ and $P^{+} = H^{+}(H^{+})^{\top}$ is computed by the following procedure:
$$G^{*} = TA\bar{K}, \tag{8a}$$
$$\bar{K} = LS^{-1}, \tag{8b}$$
$$L = \bar{P}C^{\top}, \tag{8c}$$
$$S = C\bar{P}C^{\top} + D_v D_v^{\top}, \tag{8d}$$
with $\bar{P} = \bar{H}\bar{H}^{\top}$.

Proof. For the zonotope $\langle p^{+}, H^{+}\rangle$ defined in (5), $J_s = \|H^{+}\|_F^2 = \mathrm{tr}(P^{+})$ is convex with respect to G. The optimal observer gain G^{*} satisfies $\frac{\partial}{\partial G}\mathrm{tr}\,(P^{+}) = 0$. Hence, following the proof in [Combastel, 2015, Theorem 5], we evaluate the derivative of J_s with respect to G. Set L and S as in (8). We have
$$\frac{\partial}{\partial G}\mathrm{tr}\left(GSG^{\top}\right) - 2\frac{\partial}{\partial G}\mathrm{tr}\left(TALG^{\top}\right) = 0,$$
And by simplifying the above equation, we obtain G^{*} in (8). □

4.3 Observer Gain Design for Robust Fault Detection

For robust FD, we consider the fault sensitivity under the assumption of $f \in \mathcal{F}$, $\forall k \in \mathbb{N}$. From (3), the state estimation error e^{+} is also affected by the faults f and f^{+}. Hence, we decompose the zonotope (5) with two sets: one is only affected by system uncertainties and the other by faults.

Theorem 3. (Zonotopic UIO Decomposition). Consider the descriptor system (1), $f \in \mathcal{F}$ and $x \in \{\langle p_e, H_e\rangle \oplus \langle p_f, H_f\rangle\}$. The zonotopic UIO can be recursively defined in the decomposition form as $x^{+} \in \{\langle p_e^{+}, H_e^{+}\rangle \oplus \langle p_f^{+}, H_f^{+}\rangle\}$, where

$$\begin{cases} p_e^+ &= (TA - GC)\, p_e + TBu + Gy + Ny^+, \\ H_e^+ &= \left[(TA - GC)\, \bar{H}_e,\ TD_w,\ -GD_v,\ -ND_v\right], \end{cases} \tag{9}$$

and

$$\begin{cases} p_f^+ &= (TA - GC)\, p_f, \\ H_f^+ &= \left[(TA - GC)\, \bar{H}_f,\ TF_a - GF_s,\ -NF_s\right], \end{cases} \tag{10}$$

with $\bar{H}_e = \downarrow_{q,W} (H_e)$, $\bar{H}_f = \downarrow_{q,W} (H_f)$, $H_e^+ \in \mathbb{R}^{n_x \times n_e}$, and $H_f^+ \in \mathbb{R}^{n_x \times n_f}$.

Proof. Given $x \in \{\langle p_e, H_e\rangle \oplus \langle p_f, H_f\rangle\}$ we have

$$\begin{aligned} x^+ &\in \{\langle p_e^+, H_e^+\rangle \oplus \langle p_f^+, H_f^+\rangle\} \\ &= \left((TA - GC) \odot \langle p_e + p_f, [\bar{H}_e,\ \bar{H}_f]\rangle\right) \oplus \langle TBu, 0\rangle \\ &\quad \oplus \langle Gy, 0\rangle \oplus \langle Ny^+, 0\rangle \oplus (TD_w \odot \langle 0, I_{m_w}\rangle) \\ &\quad \oplus (-GD_v \odot \langle 0, I_{m_v}\rangle) \oplus (-ND_v \odot \langle 0, I_{m_v}\rangle) \\ &\quad \oplus \left((TF_a - GF_s) \odot \langle 0, I_{m_f}\rangle\right) \oplus \left(-NF_s \odot \langle 0, I_{m_f}\rangle\right). \end{aligned}$$

For $f \in \mathcal{F}$, the above equation satisfies $x^+ = e^+ + \hat{x}^+$ as defined in (4). □

According to [Ding, 2013, Chapter 7], for designing an FD observer, the residual should be sensitive to faults and at the same time robust against uncertainties. For the zonotopic UIO structure in (5), we assume that we can always find T and G such that the observer (5) is stable. The observer gain is designed to maximize the fault sensitivity and minimize the effects of uncertainties. From Corollary 1, the residual zonotope is a projection of the zonotopic UIO in (5). As the zonotopic UIO is decomposed in the presence of faults as in Theorem 3, we define the corresponding criteria as $J_e = \mathrm{tr}(P_e^+)$ and $J_f = \mathrm{tr}(P_f^+)$ with $P_e^+ = H_e^+(H_e^+)^\top$ and $P_f^+ = H_f^+(H_f^+)^\top$. To maximize J_f and minimize J_e at time instant k, we can use the following optimization problem:

$$\tilde{G} = \arg\max_{G} \frac{\beta}{\gamma}, \tag{11a}$$

subject to

$$J_e = \mathrm{tr}(P_e^+) < \gamma, \tag{11b}$$

$$J_f = \mathrm{tr}(P_f^+) > \beta, \tag{11c}$$

where γ and β are positive scalars. Therefore, the performance index for the FD observer is given by $J_d = \frac{J_f}{J_e}$.

Assuming the matrix $(TA - GC)$ is Schur stable, we rewrite constraints (11b) and (11c) as matrix inequalities.

Theorem 4. If there exists a diagonal matrix $\Upsilon \in \mathbb{R}^{n_x \times n_x}$ with $\Upsilon \succ 0$ such that the following inequality hold:

$$\begin{bmatrix} \Upsilon & H_e^+ \\ (H_e^+)^\top & I_{n_e} \end{bmatrix} \succ 0, \tag{12}$$

then the zonotopic UIO (5) guarantees the uncertainty attenuation performance $J_e < \gamma$ with $\gamma = \mathrm{tr}(\Upsilon)$.

Proof. From (11b), there exists a diagonal matrix $\Upsilon \in \mathbb{R}^{n_x \times n_x}$, $\Upsilon \succ 0$ such that

$$H_e^+(H_e^+)^\top \prec \Upsilon.$$

By using the Schur complement to the inequality above, we obtain (12). □

Theorem 5. If there exist matrices $Q \in \mathbb{R}^{n_f \times n_f}$, $R \in \mathbb{R} \in \mathbb{R}^{n_x \times n_f}$ and a diagonal matrix $\Omega \in \mathbb{R}^{n_x \times n_x}$ with $\Omega \succ 0$ such that the following inequality holds:

$$\begin{bmatrix} -\Omega + R(H_f^+)^\top + H_f^+ R^\top & -R - H_f^+ Q^\top \\ -R^\top - Q(H_f^+)^\top & Q + Q^\top + I_{n_f} \end{bmatrix} \succ 0, \tag{13}$$

then the zonotopic UIO (5) guarantees the fault sensitivity performance $J_f > \beta$ with $\beta = \mathrm{tr}(\Omega)$.

Proof. From (11c), there exists a diagonal matrix $\Omega \in \mathbb{R}^{n_x \times n_x}$ with $\Omega \succ 0$ such that

$$H_f^+(H_f^+)^\top \succ \Omega.$$

If there exists a matrix $Q \in \mathbb{R}^{n_f \times n_f}$ such that $Q + Q^\top + I_{n_f} \succ 0$, then we have

$$\begin{bmatrix} -\Omega + H_f^+(H_f^+)^\top & 0 \\ 0 & Q + Q^\top + I_{n_f} \end{bmatrix} \succ 0.$$

By pre-multiplying $\begin{bmatrix} I_{n_x} & -H_f^+ \\ 0 & I_{n_f} \end{bmatrix}$ and post-multiplying its transpose to the inequality above, we obtain (13) by setting $R = H_f^+ + H_f^+ Q$. □

Due to the terms $R(H_f^+)^\top$ and $Q(H_f^+)^\top$ coupled in (13), we thus propose an iterative procedure to find a solution of the observer gain $\tilde{G}$. We use an optimal Kalman observer gain G^* to find the initialization of the multipliers, namely matrices R and Q.

Computation of the zonotopic FD observer gain:

- *Step 1*: Obtain the optimal Kalman observer gain G^* following the procedure in (8).
- *Step 2*: For the fixed G, solve the following optimization problem to obtain the optimal solutions of the multipliers Q and R:

  $$J_{d,1} = \max_{Q,R} \mathrm{tr}(\Omega) - \mathrm{tr}(\Upsilon), \tag{14}$$

  subject to (12) and (13).
- *Step 3*: For the fixed Q and R, solve the following optimization problem to obtain the optimal solution of the observer gain G

  $$J_{d,2} = \max_{G} \mathrm{tr}(\Omega) - \mathrm{tr}(\Upsilon), \tag{15}$$

  subject to (12) and (13).
- *Step 4*: If $J_{d,1} - J_{d,2} < \varepsilon$ with a sufficient small scalar ε, then stop and obtain the observer gain $\tilde{G} = G$. Otherwise, go to *Step 2*.

4.4 Implementation Algorithms

We now summarize the implementation procedures for state estimation and robust FD. Considering a simulation time horizon, these procedures are presented in Algorithm 1 and 2.

5. ILLUSTRATIVE EXAMPLE

In order to illustrate the state estimation and robust FD algorithms, we use a numerical example from Wang et al. [2017b] in the descriptor form (1) with

Algorithm 1 State Estimation based on Zonotopic UIO

Data: Given the descriptor system (1) with $f = 0$, system matrices E, A, B, C, D_w, D_v, and $x_0 \in \langle p_0, H_0 \rangle$, $w \in \langle 0, I_{m_w} \rangle$, $v \in \langle 0, I_{m_v} \rangle$, $\forall k \in \mathbb{N}_+$;

Obtain a pair of matrices T and N satisfying $TE + NC = I_{n_x}$ and $TD = 0$;

$p \longleftarrow p_0, H \longleftarrow H_0$;

while $k \geq 0$ **do**

Compute the optimal Kalman gain G^* following (8);

Measure the system outputs y and y^+;

Determine the state zonotope $x^+ \in \langle p^+, H^+ \rangle$ in (5);

Obtain the state estimation results $x_i^+ \in [\underline{x}_i^+, \overline{x}_i^+]$, $i = 1, \ldots, n_x$ by

$$\begin{cases} \underline{x}_i^+ = p_i^+ - rs(H^+)_{i,i} \\ \overline{x}_i^+ = p_i^+ + rs(H^+)_{i,i} \end{cases}$$

end

Algorithm 2 Robust FD based on Zonotopic UIO

Data: Given the descriptor system (1) with system matrices E, A, B, C, D_w, D_v, F_a, F_s, and $x_0 \in \langle p_0, H_0 \rangle$, $w \in \langle 0, I_{m_w} \rangle$, $v \in \langle 0, I_{m_v} \rangle$, $\forall k \in \mathbb{N}_+$;

Obtain a pair of matrices T and N satisfying $TE + NC = I_{n_x}$ and $TD = 0$;

$p \longleftarrow p_0, H \longleftarrow H_0$;

$p_e \longleftarrow p_0, H_e \longleftarrow H_0$;

$p_f \longleftarrow 0, H_f \longleftarrow 0$;

while $k \geq 0$ **do**

Compute the zonotopes $\langle p_e^+, H_e^+ \rangle$ by (9) and $\langle p_f^+, H_f^+ \rangle$ by (10);

Compute the FD observer gain $\tilde{G}$ for (5) following the proposed computation steps presented above;

Measure the system outputs y and y^+;

Compute the state zonotope $x^+ \in \langle p^+, H^+ \rangle$ in (5);

Compute the residual zonotope $r^+ \in \mathcal{R}^+ = \langle p_r^+, H_r^+ \rangle$ in (7);

Determine the FD alarm ($\chi^+ = 0$: no fault detected; $\chi^+ = 1$: fault detected) by checking

$$\chi^+ = \begin{cases} 0 & \text{if } 0 \in \mathcal{R}^+ \\ 1 & \text{if } 0 \notin \mathcal{R}^+ \end{cases}$$

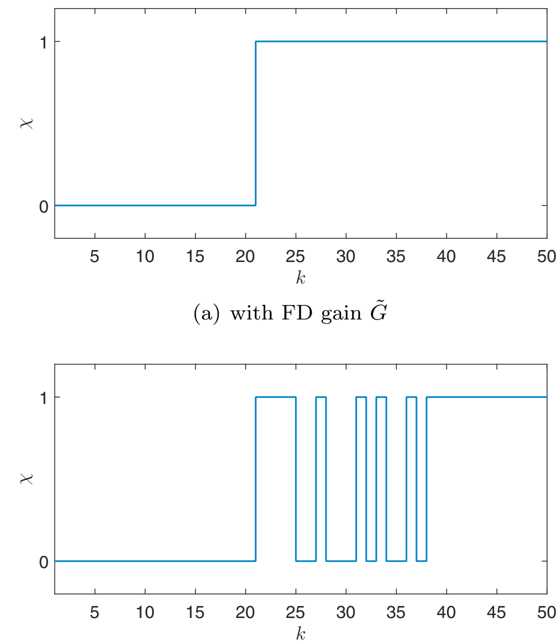
end

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0.9 & 0.005 & -0.095 & 0 \\ 0.005 & 0.995 & 0.0997 & 0 \\ 0.095 & -0.0997 & 0.99 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.1 & 0 \\ 1 & 1 \\ -0.1 & 1 \\ -1 & 0 \end{bmatrix}, D_w = \begin{bmatrix} 0.05 \\ 0.1 \\ 0.1 \\ 0 \end{bmatrix}, D_v = \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 0.02 & 0 \\ 0 & 0 & 0.02 \end{bmatrix}$$

$$D = \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0.1 \end{bmatrix}, F_a = [B \;\; 0_{4\times 3}],$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, F_s = [0_{3\times 2} \;\; I_3].$$



(a) with FD gain $\tilde{G}$

(b) with Kalman gain G^*

Fig. 2. The actuator-FD result.

By satisfying the condition (2a), we consider a pair of matrices T and N as follows:

$$T = \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & -0.0078 & 0 & 0 \\ 0 & 0 & -0.0078 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, N = \begin{bmatrix} 0 & 0 & 0 \\ 1.0078 & 0 & 0 \\ 0 & 1.0078 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

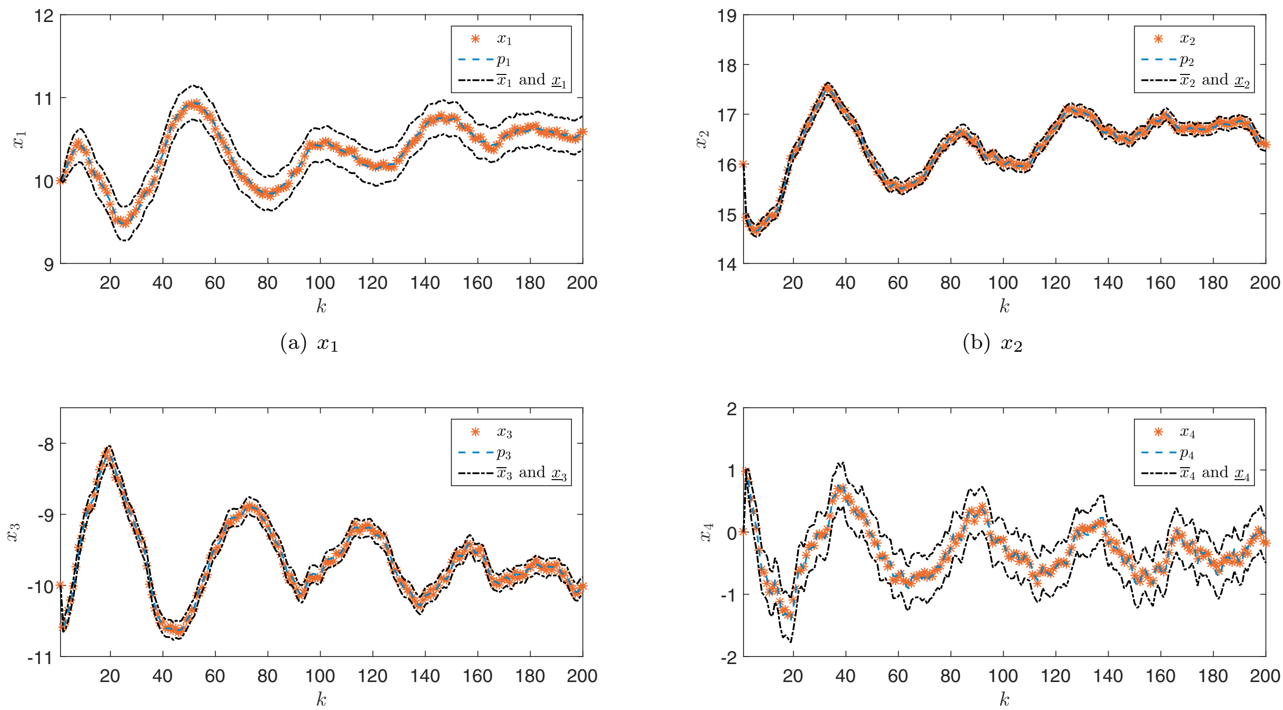
The initial state x_0 is set as $x_0 = [10, 16, -10, 0]^\top$ and assume the initial state zonotope as $\mathcal{X}_0 = \langle p_0, H_0 \rangle$ with $p_0 = x_0$ and $H_0 = 0.1 I_4$. The input signal u is set as $u = [0.4, 0.6]^\top$, $\forall k \in \mathbb{N}$. For the use of $\downarrow_{q,W}(\cdot)$, to reduce the computation time and also taking into account the memory capacity of the computer, we choose $q = 15$ and $W = I$.

5.1 State Estimation Results

The first simulation has been carried out by implementing Algorithm 1 for 200 sampling steps. The state estimation results are shown in Fig. 1. The real uncertain states x are plotted using red star points for validation purposes. At each time step, the estimated state results include a value p that is the center of the estimation zonotope and the upper and lower bounds obtained by making the interval hull of the estimation zonotope. From Fig. 1, the bounds of estimation results with optimal Kalman gain are tight and the center p is close to the value of the real uncertain state at each time step.

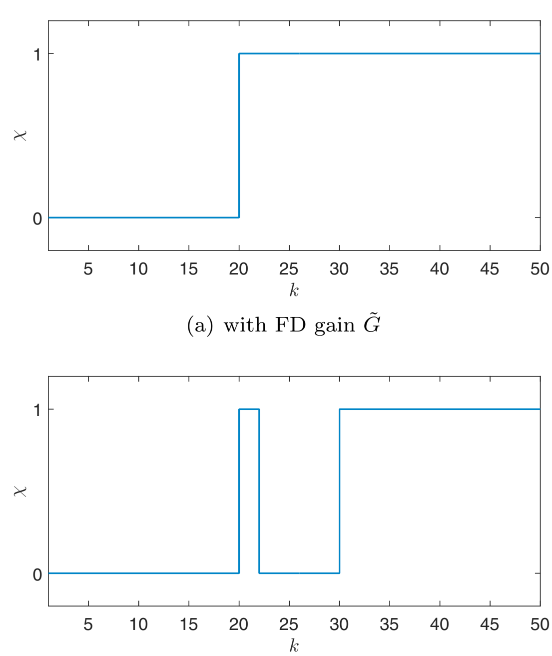
5.2 Robust Fault Detection Results

The second simulation has been carried out by implementing Algorithm 2, where the proposed optimization problems are solved in MATLAB with the YALMIP toolbox [Löfberg, 2004] and the MOSEK solver [MOSEK ApS, 2015]. In the simulation, we consider two additive fault



(a) x_1 (b) x_2 (c) x_3 (d) x_4

Fig. 1. The state estimation results in a deterministic set.



(a) with FD gain $\tilde{G}$

(b) with Kalman gain G^*

Fig. 3. The sensor-FD result.

signals $f = \left[f_a^\top,\ 0\right]^\top$ for actuator faults and $f = \left[0,\ f_s^\top\right]^\top$ for sensor faults, where f_a and f_s are set by step signals:

$$f_a = \begin{cases} \left[0,\ 0\right]^\top & k < 20 \\ \left[0.25,\ 0\right]^\top & 20 \le k < 40 \\ \left[0.25,\ 0.27\right]^\top & k \ge 40 \end{cases}$$

$$f_s = \begin{cases} \left[0,\ 0,\ 0\right]^\top & k < 20 \\ \left[0.4,\ 0,\ 0\right]^\top & 20 \le k < 30 \\ \left[0.4,\ 0.3,\ 0\right]^\top & 30 \le k < 40 \\ \left[0,\ 0,\ 0.5\right]^\top & k \ge 40 \end{cases}$$

In order to show the effectiveness of the FD observer gain, we also run the simulation with the optimal Kalman gain G^* obtained by (8) for comparison. The simulation results of robust FD are shown in Fig. 2 and 3. From these results, it is shown that with the Kalman gain G^*, some wrong FD alarms appear at some steps since the observer gain is designed to only minimize the effects of system uncertainties and with FD gain $\tilde{G}$, the zonotopic observer is able to detect those step faults. Besides, for actuator faults are detected with one step delay and there is no delay for detecting sensor faults.

6. CONCLUSION

In this paper, we have proposed a framework of zonotopic UIO of discrete-time descriptor systems. The considered descriptor systems are affected by uncertainties including unknown-but-bounded system disturbances and measurement noise as well as unknown inputs. Under this framework, we then apply the zonotopic UIO for state estimation and robust FD. With different objectives, an optimal Kalman gain and FD gain are designed. Finally, through the simulation results, we have shown the effectiveness of the proposed algorithms.

As future work, we will improve the stability assumption of the FD observer and link with a fault isolation strategy for discrete-time descriptor systems. The robust stability and convergence can be improved based on a single LMI condition. Besides, the $\mathcal{H}_-$ fault sensitivity will be considered in a set-based framework as discussed in Wang et al. [2017c].

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