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POINCARÉ SERIES FOR MIXED MULTIPLIER IDEALS

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ABSTRACT. We present a generalization of the Poincaré series to the case of mixed multiplier ideals. For that, we will recall some results about how we can compute the jumping walls associated to a mixed multiplier ideal and introduce some results about the multiplicity of a given point in $\mathbb{R}^r_{\geqslant 0}$.

INTRODUCTION

Let X be a complex surface with at most a rational singularity at a point $O \in X$ (see Artin [3] and Lipman [8] for details) and $\mathfrak{m} = \mathfrak{m}_{X,O}$ be the maximal ideal of the local ring $\mathcal{O}_{X,O}$ at O. Given a tuple of $\mathfrak{m}$-primary ideals $\boldsymbol{\mathfrak{a}} := \{\mathfrak{a}_1, ..., \mathfrak{a}_r\} \subseteq (\mathcal{O}_{X,O})^r$ we will consider a common *log-resolution*, that is a birational morphism $\pi : X' \to X$ such that X' is smooth, $\mathfrak{a}_i \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F_i)$ for some effective Cartier divisors F_i, $i = 1, \dots, r$ and $\sum_{i=1}^r F_i + E$ is a divisor with simple normal crossings where $E = Exc(\pi)$ is the exceptional locus. Actually, the divisors F_i are supported on the exceptional locus since the ideals are $\mathfrak{m}$-primary.

Since the point O has (at worst) a rational singularity, the exceptional locus E is a tree of smooth rational curves $E_1, \dots, E_s$. Moreover, the matrix of intersections $(E_i \cdot E_j)_{1 \leqslant i,j \leqslant s}$ is negative-definite. For any exceptional component E_j, we define the *excess* of $\mathfrak{a}_i$ at E_j as $\rho_{i,j} = -F_i \cdot E_j$. We also recall the following notions:

- A component E_j of E is a *rupture* component if it intersects at least three more components of E (different from E_j).
- We say that E_j is *dicritical* if $\rho_{i,j} > 0$ for some i. They correspond to *Rees valuations* (see [8]).

We define the *mixed multiplier ideal* at a point $\mathbf{c} := (c_1, .., c_r) \in \mathbb{R}^r_{\geqslant 0}$ as [1]

$$\mathcal{J}(\boldsymbol{\mathfrak{a}}^{\mathbf{c}}) := \mathcal{J}(\mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_r^{c_r}) = \pi_* \mathcal{O}_{X'}(\lceil K_\pi - c_1 F_1 - \cdots - c_r F_r \rceil) \tag{1}$$

where $\lceil \cdot \rceil$ denotes the *round-up* and the *relative canonical divisor* $K_\pi = \sum_{i=1}^s k_j E_j$ is a $\mathbb{Q}$-divisor on X' supported on the exceptional locus E which is characterized by the property $(K_\pi + E_i) \cdot E_i = -2$ for every exceptional component E_i, $i = 1, \dots, s$.

Associated to any point $\boldsymbol{c} \in \mathbb{R}^r_{\geqslant 0}$, we consider:

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[1] By an abuse of notation, we will also denote $\mathcal{J}(\boldsymbol{\mathfrak{a}}^{\mathbf{c}})$ its stalk at O so we will omit the word "sheaf" if no confusion arises.

- The *region* of $\boldsymbol{c}$: $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{c}) = \left\{ \boldsymbol{c}' \in \mathbb{R}^r_{\geqslant 0} \;\middle|\; \mathcal{J}\left(\mathfrak{a}^{\boldsymbol{c}'}\right) \supseteq \mathcal{J}\left(\mathfrak{a}^{\boldsymbol{c}}\right) \right\}$
- The *constancy region* of $\boldsymbol{c}$: $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{c}) = \left\{ \boldsymbol{c}' \in \mathbb{R}^r_{\geqslant 0} \;\middle|\; \mathcal{J}\left(\mathfrak{a}^{\boldsymbol{c}'}\right) = \mathcal{J}\left(\mathfrak{a}^{\boldsymbol{c}}\right) \right\}$

The boundary of the region $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{c})$ is what we call the *jumping wall* associated to $\boldsymbol{c}$. One usually refers to the jumping wall of the origin as the *log-canonical wall*. It follows from the definition of mixed multiplier ideals that the jumping walls must lie on *supporting hyperplanes* of the form

$$H_j : \; e_{1,j}z_1 + \cdots + e_{r,j}z_r = \ell + k_j \qquad j = 1, \ldots, s \tag{2}$$

where $\ell \in \mathbb{Z}_{>0}$, and the effective divisors F_i such that $\mathfrak{a}_i \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F_i)$, for $i = 1, \ldots, r$, are of the form $F_i = \sum_{j=1}^{s} e_{i,j}E_j$. Indeed, each hyperplane H_j is associated to an exceptional divisor E_j and the region $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{c})$ is a *rational convex polytope* defined by

$$e_{1,j}z_1 + \cdots + e_{r,j}z_r < \ell + k_j,$$

i.e. the minimal region in the positive orthant $\mathbb{R}^r_{\geqslant 0}$ described by these inequalities. Notice that the facets of the jumping wall of $\boldsymbol{c}$ are also rational convex polytopes. From now on we will denote by $\mathbf{JW}_{\mathfrak{a}}$ the set of jumping walls of $\mathfrak{a}$.

One can characterize which hyperplanes define the region of a given point $\boldsymbol{\lambda}$, namely:

Theorem 0.1 (see Theorem 3.3 in [2])**.** *Let $\mathfrak{a} := \{\mathfrak{a}_1, ..., \mathfrak{a}_r\} \subseteq (\mathcal{O}_{X,O})^r$ be a tuple of ideals and let $D_{\boldsymbol{\lambda}} = \sum e_j^{\boldsymbol{\lambda}} E_j$ be the antinef closure of $\lfloor \lambda_1 F_1 + \cdots + \lambda_r F_r - K_\pi \rfloor$ for a given $\boldsymbol{\lambda} \in \mathbb{R}^r_{\geqslant 0}$. Then the region of $\boldsymbol{\lambda}$ is the rational convex polytope determined by the inequalities*

$$e_{1,j}z_1 + \cdots + e_{r,j}z_r < k_j + 1 + e_j^{\boldsymbol{\lambda}},$$

corresponding to either rupture or dicritical divisors E_j.

1. An algorithm to compute mixed multiplier ideals and jumping walls

In [2], the first three authors presented the following algorithm. This algorithm allows us to compute for a given tuple of ideals the associated jumping walls.

Algorithm 1.1 (see Algorithm 3.11 in [2])**.** (Constancy regions and mixed multiplier ideals)

`Input`: A common log-resolution of the tuple of ideals $\mathfrak{a} = \{\mathfrak{a}_1, \ldots, \mathfrak{a}_r\} \subseteq (\mathcal{O}_{X,O})^r$.
`Output`: List of constancy regions of $\mathfrak{a}$ and its corresponding mixed multiplier ideals.

Set $N = \{\boldsymbol{\lambda}_0 = (0, \ldots, 0)\}$ and $D = \emptyset$. From $j = 1$, incrementing by 1

(Step j) :

(j.1) **Choosing a convenient point in the set** N:
- Pick $\boldsymbol{\lambda}_j$ the first point in the set N and compute its region $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_j)$ using Theorem 0.1.
- If there is some $\boldsymbol{\lambda} \in N$ such that $\boldsymbol{\lambda} \in \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_j)$ and $\mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}}) \neq \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}_j})$ then put $\boldsymbol{\lambda}$ first in the list N and repeat this step (j.1). Otherwise continue with step (j.2).

(j.2) **Checking out whether the region has been already computed**:
- If some $\boldsymbol{\lambda} \in D$ satisfies $\mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}}) = \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}_j})$ then go to step (j.4). Otherwise continue with step (j.3).

$(j.3)$ **Picking new points for which we have to compute its region**:

· Compute

$$\mathcal{C}(j) = \mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda}_j) \setminus (\mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda}_1) \cup \cdots \cup \mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda}_{j-1}))\,.$$

· For each connected component of $\mathcal{C}(j)$ compute its outer facets[2].

· Pick one interior point in each outer facet of $\mathcal{C}(j)$ and add them as the last point in N.

$(j.4)$ **Update the sets N and D**:

· Delete $\boldsymbol{\lambda}_j$ from N and add $\boldsymbol{\lambda}_j$ as the last point in D.

2. Multiplicities of jumping points

The goal of this section is to study the Poincaré series associated to a mixed multiplier ideal. For that, we need to begin introducing the notion of multiplicity. Namely, if we consider $\boldsymbol{\mathfrak{a}} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ a tuple of $\mathfrak{m}$-primary ideals. We define the multiplicity attached to a point $\boldsymbol{c} \in \mathbb{R}^r_{\geqslant 0}$ as the codimension of $\mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{\boldsymbol{c}}\right)$ in $\mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{(1-\varepsilon)\boldsymbol{c}}\right)$ for $\varepsilon > 0$ small enough, i.e.

$$m(\boldsymbol{c}) := \dim_{\mathbb{C}} \frac{\mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{(1-\varepsilon)\boldsymbol{c}}\right)}{\mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{\boldsymbol{c}}\right)}\,.$$

Our goal is to compute explicitly these multiplicities. Since we are dealing with any general point, it will be convenient to consider the notion of *maximal jumping divisor*.

Definition 2.1. Let $\boldsymbol{\mathfrak{a}} := (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ be a tuple of ideals. Given any point $\boldsymbol{c} \in \mathbb{R}^r_{\geqslant 0}$, we define its *maximal jumping divisor* as the reduced divisor $H_{\boldsymbol{c}} \leqslant \sum_{i=1}^r F_i$ supported on those components E_j such that

$$c_1 e_{1,j} + \cdots + c_r e_{r,j} - k_j \in \mathbb{Z}_{>0}\,.$$

In particular, we have

$$\mathcal{J}(\boldsymbol{\mathfrak{a}}^{(1-\varepsilon)\boldsymbol{c}}) = \pi_* \mathcal{O}_{X'}(\lceil K_\pi - c_1 F_1 - \cdots - c_r F_r \rceil + H_{\boldsymbol{c}})\,,$$

In fact, we can compute the multiplicity using those divisors:

Theorem 2.2. *Let $\boldsymbol{\mathfrak{a}} \subseteq (\mathcal{O}_{X,O})^r$ be a tuple of $\mathfrak{m}$-primary ideals and $H_{\boldsymbol{c}}$ the maximal jumping divisor associated to some $\boldsymbol{c} \in \mathbb{R}^r_{>0}$. Then,*

$$m\left(\boldsymbol{c}\right) = \left(\lceil K_\pi - c_1 F_1 - \cdots - c_r F_r \rceil + H_{\boldsymbol{c}}\right) \cdot H_{\boldsymbol{c}} + \#\left\{\textit{connected components of } H_{\boldsymbol{c}}\right\}.$$

2.1. **Poincaré series of mixed multiplier ideals.** Given a $\mathfrak{m}$-primary ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$, Galindo and Montserrat [6] (see also [1]) introduced its *Poincaré series* as

$$P_{\mathfrak{a}}(t) = \sum_{c \in \mathbb{R}_{>0}} m(c)\, t^c. \tag{3}$$

For a tuple of $\mathfrak{m}$-primary ideals $\boldsymbol{\mathfrak{a}} = \{\mathfrak{a}_1, \ldots, \mathfrak{a}_r\} \subseteq (\mathcal{O}_{X,O})^r$ we are going to give a generalization of this series by considering a sequence of mixed multiplier ideals indexed by

[2] The outer facets of $\mathcal{C}(j)$ are the intersection of the boundary of any connected component of $\mathcal{C}(j)$ with a supporting hyperplane of $\mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda}_j)$.

points in a ray $L : \boldsymbol{c}_0 + \mu\mathbf{u}$ in the positive orthant $\mathbb{R}^r_{>0}$ with a vector $\mathbf{u} = (u_1, \ldots, u_r) \in \mathbb{Z}^r_{\geqslant 0}$, $\mathbf{u} \neq \mathbf{0}$ and $\boldsymbol{c}_0 \in \mathbb{Q}^r_{\geqslant 0}$. Here we are considering, for simplicity, a point $\boldsymbol{c}_0$ belonging to a coordinate hyperplane but not necessarily being the origin and $\mu \in \mathbb{R}_{>0}$. Namely, we consider the sequence of mixed multiplier ideals

$$\mathcal{J}\left(\mathfrak{a}^{\boldsymbol{c}_0}\right) \supsetneq \mathcal{J}\left(\mathfrak{a}^{\boldsymbol{c}_1}\right) \supsetneq \mathcal{J}\left(\mathfrak{a}^{\boldsymbol{c}_2}\right) \supsetneq \cdots \supsetneq \mathcal{J}\left(\mathfrak{a}^{\boldsymbol{c}_i}\right) \supsetneq \cdots$$

where $\{\boldsymbol{c}_i\}_{i>0} = L \cap \mathbf{JW}_{\mathfrak{a}}$ or equivalently $\{\boldsymbol{c}_i\}_{i>0}$ is the set of jumping points of this sequence. Then we define the *Poincaré series of* $\mathfrak{a}$ *alongside the ray* L as

$$P_{\mathfrak{a}}(\underline{t}; L) = \sum_{\boldsymbol{c} \in L} m(\boldsymbol{c})\, \underline{t}^{\boldsymbol{c}}. \tag{4}$$

where $\underline{t}^{\boldsymbol{c}} := t_1^{c_1} \cdots t_r^{c_r}$.

Theorem 2.3. *Let* $\mathfrak{a} = \{\mathfrak{a}_1, \ldots, \mathfrak{a}_r\} \subseteq (\mathcal{O}_{X,O})^r$ *be a tuple of* $\mathfrak{m}$*-primary ideals and* $L : \boldsymbol{c}_0 + \mu\mathbf{u}$ *a ray in the positive orthant* $\mathbb{R}^r_{>0}$*. The Poincaré series of* $\mathfrak{a}$ *alongside* L *can be expressed as*

$$P_{\mathfrak{a}}(\underline{t}; L) = \underline{t}^{\boldsymbol{c}_0} \sum_{\mu \in [0,1]} \left(\frac{m(\boldsymbol{c}_0 + \mu\mathbf{u})}{1 - \underline{t}^{\mathbf{u}}} + \rho_{\boldsymbol{c},\mathbf{u}} \frac{\underline{t}}{(1 - \underline{t}^{\mathbf{u}})^2} \right) \underline{t}^{\mu\mathbf{u}}.$$

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