Configuration space of a textile rectangle

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This work is supported by the European Research Council (ERC) within the European Union Horizon 2020 Programme under grant agreement ERC–2016–ADG–741930 (CLOTHILDE: CLOTH manIpulation Learning from DEmonstrations) and by the Spanish State Research Agency through the María de Maeztu Seal of Excellence to IRI (MDM–2016–0656).

Abstract: Given a rectangular piece of cloth on a planar surface, we aim to characterise its states based on the robot manipulations they would require. Considering the cloth as a set of n points in $\mathbb{R}^2$, we study its configuration space, $Conf_n(\mathbb{R}^2)$. We derive a stratification of $Conf_4(\mathbb{R}^2)$ using that of $\mathscr{Flag}(3)$, and we present some techniques that can be used to determine the adjacencies of $Conf_n(\mathbb{R}^2)$ and some group actions we can define on it.

1 Introduction

Given a rectangular cloth on a planar surface, we could consider it as a surface embedded in $\mathbb{R}^3$ with no self-intersection. Unfortunately considering the different states of such surface and studying their space bears difficulties, as we have to impose, on the already complex space of all possible surfaces with constant area and no self-intersections, constrains such as gravity force and cloth stiffness. In order to simplify, we consider instead the cloth as a set of points on the real plane. Since our aim is to distinguish states based on the types of robot manipulations they permit, we consider the configuration space of n ordered points in $\mathbb{R}^2$, namely $Conf_n(\mathbb{R}^2)$. This space belongs to the far more general family of configuration spaces of points on manifolds,

$$Conf_k(X) = \left\{ \underline{p} = (p_1, \dots, p_k) \in X^k \mid p_i \neq p_j, \text{ for } i \neq j \right\}.$$

Such spaces are interesting topological objects and their (co)homology type has been studied by several authors. In [Arn69] some results regarding the homotopy type of $Conf_n(X)$ are obtained, assuming X is of dimension 2, while the real homotopy type of $Conf_n(X)$, when X is a smooth projective variety, were independently computed by Kriz [Kri94] and Totaro [Tot96]. Assuming $X = \mathbb{R}^n$, Cohen et al. computed the cohomology of $Conf_n(X)$, and in particular, they proved that $Conf_n(\mathbb{R}^n)$ is the classifying space of the *n-strand pure braid* group [CLM76]. The action of $\mathbb{S}_n$ on $Conf_n(\mathbb{R}^n)$ is also studied in [CLM76] and, in particular, the quotient of this action gives the *configuration space of n unordered points*, which is the classifying space of the *n-strand braid* group. Our interest lays mostly on the adjacency relations between the highest dimensional cells of $Conf_n(\mathbb{R}^2)$, when we regard it as a CW-complex. Such cells are "clusters" of similar point configurations and their adjacency information permits navigating between them. A *state* will then be a set of different cells, each one containing configurations of points, that permits similar types of robot manipulations.

Research Perspectives CRM Barcelona, Spring 2019, vol. 10, in Trends in Mathematics
Springer-Birkhäuser, Basel

In Section 2 we consider $n = 4$, for the 4 corner points of the rectangular cloth, and present a stratification of $Conf_4(\mathbb{R}^2)$ using that of $\mathscr{Flag}(3)$. We then move in Section 3 to the general case of $Conf_n(\mathbb{R}^2)$ and show some techniques to derive the adjacency structure of the space together with some group actions that are naturally defined on $Conf_n(\mathbb{R}^2)$.

2 Configuration space of a textile rectangle using 4 points

In order to study the configuration space of the 4 corner points of the rectangular cloth we will make use of the flag manifold of $\mathbb{RP}^2$, $\mathscr{Flag}(3)$. If we consider the configuration $\underline{p} = (p_1, p_2, p_3, p_4)$ with $p_i \in \mathbb{R}^2$, we can embed them in $\mathbb{RP}^2$, mapping a point $p = (x, y)$ to $\widetilde{p} = [x : y : 1]$. The stratification of $\mathscr{Flag}(3)$ induces another on $Conf_4(\mathbb{R}^2)$, see Figure 1.

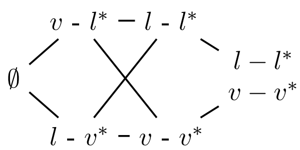

FIGURE 1. We can stratify $\mathscr{Flag}(3)$ with respect to two flags, $V = \{v, l\}$ and $V^* = \{v^*, l^*\}$, using their incidence, indicated by - in the figure, see [Hil82, Mon59].

If we consider $V = \{p_1, \overline{p_1 p_2}\}$ and $V^* = \{p_3, \overline{p_3 p_4}\}$, then the condition $v - l^*$ corresponds to the alignment of the three points $\{p_1, p_3, p_4\}$. Any alignment of three points p_i, p_j, p_k, with $i < j < k$ can be seen as a pure algebraic condition on the points coordinates, given by the singularity of the determinant $d_{i,j,k} = |\widetilde{p_i}\ \widetilde{p_j}\ \widetilde{p_k}|$. The sign of $d_{i,j,k}$ depends on the clockwise or counter-clockwise position of the ordered triple (p_i, p_j, p_k). As the determinant is a continuous map onto $\mathbb{R}$, if two configurations $\underline{p}$ and $\underline{q}$ differ by one determinant sign, say $d_{i,j,k}$, then we know they belong to different cells. So any continuous path from $\underline{p}$ to $\underline{q}$ has to cross the singularity loci of $d_{i,j,k}$. We identify then a cell σ with the sequence of determinant signs of all triples of points belonging to any configuration $\underline{p}$ in σ. For us, the determinants signs are, in order, of $d_{1,2,3}$, $d_{1,2,4}$, $d_{1,3,4}$ and as last $d_{2,3,4}$. Moreover, an odd number of negative determinants tells us that one point lays inside the triangle spanned by the others. In such cases we call the configuration *internal*, otherwise *external*. One can prove easily that $d_{1,2,3} + d_{1,3,4} = d_{1,2,4} + d_{2,3,4}$, which means that not all sign sequences are admissible, as we can see in Figure 2.

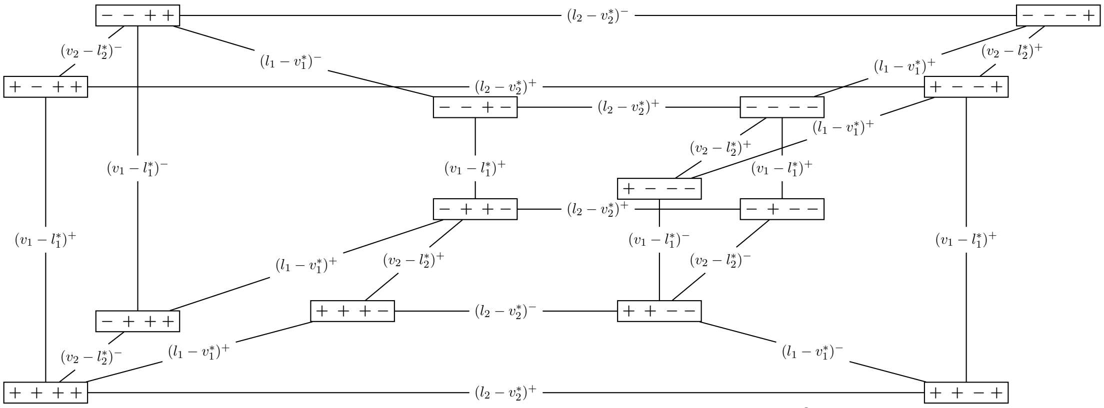

FIGURE 2. We show the adjacency relations of $Conf_4(\mathbb{R}^2)$ using the stratification of the affine flag manifold [**Hil82**] and as flags $V_1 = \{p_1, \overline{p_1, p_2}\}$, $V_1^* = \{p_3, \overline{p_3, p_4}\}$ and $V_2 = \{p_2, \overline{p_2, p_1}\}$, $V_2^* = \{p_4, \overline{p_4, p_3}\}$.

3 Configuration space of an n-points textile rectangle

Regarding $n > 4$, if we want to recover the stratification of $Conf_n(\mathbb{R}^2)$, similarly to Section 2, we would not consider more flags, as they will make less clear the description of the singularity loci. We have that singularities are given by the alignment of three points and again any cell can be identified by a sequence of $\binom{n}{3}$ determinant signs. In the general case, we do not know exactly which sign sequences are admissible, that is, how many cells are in $Conf_n(\mathbb{R}^2)$. Consider the arrangement of lines spanned by pairs of $n-1$ points, we could deduce the cells of $Conf_n(\mathbb{R}^2)$ from the regions they divide $\mathbb{R}^2$ into. Line arrangements, both in the real and projective planes, have been studied extensively in various contexts [**Grü72**] and references therein. Several authors have worked on how to bound the number of regions, triangles or polygons [**Rou86, Str77, Sim73**]. In [**ABH$^+$18**], the authors consider the problem of characterising geometric graphs using the order type of their vertex set. Using the notion of *minimal representation of a graph*, they identify which edges prevent the order type from changing via continuous deformations of the graph. Even if this approach is the closest to ours, to our knowledge in the literature there is not a detailed study of the adjacency relations of $Conf_n(\mathbb{R}^2)$. We present here two theorems that allow us to determine if and how we can move continuously a point (or more if needed) to change only one determinant sign. Due to lack of space and proof technicality, we give here only sketches of the proofs. The following theorem gives us a way to discern when an adjacency cannot exist.

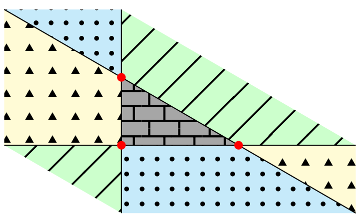

FIGURE 3. Given any triple of points (not aligned), the lines they span divide the plane in 7 regions, that can be seen as three couples of *dual regions*, formed by external and internal configurations, which are coloured identically, and a self-dual internal region.

Theorem 1 *Consider any configuration* $\underline{p} \in Conf_n(\mathbb{R}^2)$ *and a triple* $\{p_i, p_j, p_k\} \subset \underline{p}$. *If there exists either a point* $p_u \in \underline{p}$ *in the self-dual region of* $\{p_i, p_j, p_k\}$, *or two points*

p_s, p_r in two regions not dual w.r.t. $\{p_i, p_j, p_k\}$, then there does not exist a continuous movement of $\underline{p}$ that crosses only the singularity loci $d_{i,j,k} = 0$.

Proof. If $d_{i,j,k}$ is nullified via a continuous map, the 6 outer regions in Figure 3 degenerate into 2 regions, corresponding to a pair of dual regions, depending on the map used, while the other ones degenerate to the line $\overline{p_i, p_j}$. In other words, if a point $p_u \in \underline{p}$ is inside the self-dual region then any continuous map that crosses the singularity loci $d_{i,j,k} = 0$ has to nullify at least one among $d_{i,j,u}, d_{i,k,u}$ and $d_{j,k,u}$. Similarly, if two points $p_s, p_r \in \underline{p}$ are in regions not dual w.r.t. $\{p_i, p_j, p_k\}$, then any continuous map crossing $d_{i,j,k}$ would also cross either $d_{i,j,s}$ or $d_{i,j,r}$. $\square$

The following result tells us when instead it is possible to change sign.

Theorem 2 *Consider any configuration $\underline{p} \in Conf_n(\mathbb{R}^2)$ and a triple of points $\{p_i, p_j, p_k\} \subset \underline{p}$, such that they belong either to the same region or to two distinct and dual regions. If there exists a point $p_u \notin \{p_i, p_j, p_k\}$, such that for any pair $p_s, p_r \notin \{p_i, p_j, p_k, p_u\}$ in the same region, resp. dual regions, and for any pair $p_a, p_b \in \{p_i, p_j, p_k\}$ the configuration of $\{p_a, p_b, p_s, p_r\}$ is external, resp. internal, then the singularity loci can be crossed uniquely at $d_{i,j,k} = 0$.*

Proof. If such point p_u exists, then the line $\overline{p_u\ p_v}$, with $v = i, j, k$, intersects any other line spanned by another two points outside the self-dual region. So we can move p_u along $\overline{p_u\ p_v}$ till $d_{i,j,k}$ changes sign, without crossing any other singularity. $\square$

Note that Theorems 1 and 2 do not cover all cell adjacencies for $n > 6$. If $n \leq 6$ we can compute the exact number of cells. Such number is expected to rise quadratically [**Str77**], thus we want to group cells entailing similar robotic manipulations to form states. We consider also the action of the symmetric group $\mathbb{S}_n$. In terms of our stratification, such action induces an identification between cells whose determinant signs coincide after a permutation of the point labels, $\{1, \ldots, n\}$. For $n = 4, 5$ and 6, we obtain in total $2, 3$ and 6 states, respectively, which are a lot fewer than we would hope for. In other words, such action induces an over-coarsened partition of the configuration space and we prefer to use instead the following refined partition. Let σ be a cell, i.e. a sign sequence, we define

$$\tau_1 \sim_\sigma \tau_2 \iff \exists g \in \mathbb{S}_n,\ g \cdot \tau_1 = \tau_2 \text{ and } d(\sigma, \tau_1) = d(\sigma, \tau_2),$$

where $d(\sigma, \tau_i)$ for $i = 1, 2$ is the number of different signs between cells σ and τ_i. Let Y_σ be the partition of the configuration space induced by the equivalence relation $\sim_\sigma$, which is a refinement of the one obtained via $\mathbb{S}_n$. That is, any equivalence class defined by $\sim_\sigma$ belongs to one and only one $\mathbb{S}_n$-equivalence class. The distance $d(\sigma, \cdot)$ is constant inside each class of Y_σ. We always have a unique state, $-\sigma$, which is $\sim_\sigma$-equivalent only to itself, and that realises the maximum distance from σ. When we consider G_σ, the Hesse diagram of Y_σ induced by the adjacency relation of $Conf_n(\mathbb{R}^2)$, we have that there exists an automorphism of G_σ, that maps σ to $-\sigma$.

In conclusion, given a configuration of n points, we are able to determine in which state τ is and how far it is from another (fixed) state σ. In addition, using G_σ, thanks to Theorems 1 and 2 and the stratification of $Conf_n(\mathbb{R}^2)$, we could be able to plan how to change state from one given state to either σ or $-\sigma$.

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