Nonlinearity Measures for Distributed Parameter and Descriptor Systems

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Abstract: Control design and state estimation are usually more straightforward for linear than for nonlinear dynamical systems, which has motivated the development of methods for quantifying the extent of nonlinearity in dynamical systems. Although many well-defined methods have been proposed for systems described by ordinary differential equations, such methods are not as well explored for dynamical systems described by PDEs and descriptor systems that represent most chemical processes. This paper reviews, discusses, and compares methods for the definition and computation of nonlinearity measures. The measures are categorized in terms of open- vs. closed-loop control topologies, theoretical vs. numerically computed, state transformation dependency, input scaling dependency, linearization vs. optimized linear modeling vs. average linear modeling, applicability to unstable dynamical systems, and applicability to the right-hand side of the state equation or to input-output relationships. Then extensions of the nonlinearity measures are discussed for hybrid systems and those described by coupled differential, integral, and algebraic equations, often referred to as descriptor/singular systems.

Keywords: nonlinear systems, process control, nonlinear measures, descriptor systems, distributed parameter systems

1. INTRODUCTION

While nearly all real processes are nonlinear, most processes can be described well enough by linear models for the purpose of feedback control design. The greater ease of control design and analyses for linear models (Stengel, 1994; Zhou et al., 1996) has motivated research to quantify the extent to which any particular process is nonlinear (Helbig et al., 2000; Leung, 1999; Du and Johansen, 2017; Li, 2012). The notion is that a control engineer, armed with a suitable nonlinearity measure, would have a quantifiable approach for assessing whether to apply linear control design or whether to invest the added effort needed to design and implement a nonlinear control design.

This review begins with some background on nonlinearity measures for estimation and control design. Section 3 reviews worst-case and stochastic nonlinearity measures that are computable for both types of dynamical systems. Section 4 describes nonlinearity measures that have been defined for specific classes of distributed parameter systems. Section 5 describes extensions of the general nonlinearity measures to handle systems whose mathematical representations also include integral and algebraic equations. Section 6 discusses some practical considerations on nonlinearity measures computation. The article concludes with a summary.

2. BACKGROUND

Nonlinearity measures can be applied in many different contexts. For state-feedback control design, a relevant nonlinearity measure can be applied to quantify the relationship between the input (i.e., manipulated and disturbance variables) to the state, and the state, manipulated, and disturbance variables to the output. For output-feedback control design that does not require the construction of a state estimator, the nonlinearity measure can be applied to the relationship between the input and output variables of the system.

Most nonlinearity measures have ignored the effects of disturbances, with the reasoning that states and outputs will be approximately linear functions of the disturbances provided that the disturbances are small irrespective of whether the relationship between the disturbances and state/output are strictly nonlinear. This implicit assumption is not always correct of course, as many manufacturing processes exist in which a disturbance can cause the operations to cross a bifurcation point, which is a significant nonlinearity. For example, a small disturbance can cause a blown film extruder to transition from locally asymptotically stable operation to highly oscillatory unstable operation (Pirkle, Jr. and Braatz, 2010, 2011).

Nonlinearity measures can also be used to select among methods for the design of a state estimator. Provided that the states are observable, an optimal state estimator for linear dynamical

systems can be designed by applying a Kalman filter (Bucy and Joseph, 1968; Harvey, 1990; Anderson and Moore, 1979) to a linear model. Numerous optimal or nearly optimal state estimation methods have been developed for nonlinear process models, but require much higher computational cost. Nonlinearity measures can also be applied to output estimator design or to select which variables should be measured to achieve an accurate estimate by a linear estimator.

Several nonlinearity measures have been developed over 20+ years, including Helbig et al. (2000), Schweickhardt and Allgöwer (2007), Schweickhardt and Allgöwer (2009), Li (2012), and Du and Johansen (2017). Initial studies analyzed open-loop input-output systems near a steady-state operating point, that is, the nonlinearity measure was some quantification of how close the nonlinear behavior would be to the behavior of a linearization of the nonlinear dynamics about a pre-specified steady-state operating point. Several extensions were considered in subsequent nonlinearity measures, including different types of nonlinear dynamical systems, and quantification based on closed loop rather than open-loop behavior. A motivation for basing nonlinearity measures on closed-loop behavior is that the set of nonlinear dynamical systems that are optimally or nearly optimally controlled by a linear feedback control is much larger than the set of nonlinear dynamical systems whose open-loop dynamics are well described by a linear dynamical system.

Many proposed nonlinearity measures compare an optimal linear time-invariant approximation to the full nonlinear dynamical system, through the use of some norm (Helbig et al., 2000; Schweickhardt and Allgöwer, 2009). Additional nonlinearity measures include curvature metrics (Guay et al., 1995) and the gap metric (Du and Johansen, 2017). Challenges that remain include how to best address the effect of model uncertainties and develop closed-loop nonlinearity measures that more accurately capture the dynamics of the closed-loop systems that arise under the type of feedback control that would be designed.

Most of the nonlinearity measures assume that the nonlinear dynamical system is a *lumped parameter system*, that is, is described by a system of ordinary differential equations for continuous-time system representations or discrete-time difference equations. However, in many applications, spanning from pharmaceuticals, chemicals, microelectronics, materials, and biomedicine, processes are described by *distributed parameter systems* (DPS). For such systems, the state variables depend on some other variables such as spatial position, particle size, or cell age. Many of these DPS are described by systems of nonlinear partial differential equations (PDEs),[1] whose spatio-temporal coupling makes nonlinear feedback controller and estimator design much more challenging than for systems described by ordinary differential/difference equations. This increased complexity suggests that nonlinearity measures might be even more useful for DPS than for lumped parameter systems.

Very few nonlinearity measures have been specific to DPS. Both curvature and norm-based nonlinearity measures have been generalized to DPS (Fuxman et al. (2007) and Wu et al. (2017), respectively). For DPS, nonlinearity measures are not yet developed for considering closed-loop dynamics or explicitly taking model uncertainties and actuator, state, and output constraints into account.

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[1] Many DPS are described by integro-partial differential-algebraic equations, which are discussed in Section 5.

The focus of this review is on nonlinearity measures that are applicable to both lumped and DPS. Although such measures are not able to exploit the details on the mathematical structure of DPS, the measures have the advantage of being generally applicable. As discussed in Section 5, the general measures can be applied to more complex classes of nonlinear dynamical systems, such as descriptor/singular systems.

3. GENERAL NONLINEARITY MEASURES

Initial nonlinearity measures for general dynamical systems focus on the quantification of the open-loop nonlinearity (Helbig et al., 2000; Haber, 1985). These nonlinearity measures focused on the input-output behavior of the system. The main focus of Haber (1985) is on data-driven input-output nonlinearity estimates, while Helbig et al. (2000) proposes employing a general nonlinear representation, for example, a nonlinear dynamic operator N that maps admissible input signals and initial conditions into admissible output signals. The process nonlinearity is assessed through comparison with an optimal linear time-invariant system, which can be described by a linear dynamic operator G. In this case, the nonlinearity measure $\Phi_N^{\mathcal{Y}_a}$ is defined as the non-negative number

$$\Phi_N^{\mathcal{Y}_a}(t_f) =$$
$$\inf_{G \in \mathcal{G}} \sup_{(\mathbf{u}, \mathbf{x}_{N,0}) \in \mathcal{S}} \inf_{\mathbf{x}_{G,0} \in \mathcal{X}_{G,0}} \frac{\|G[\mathbf{u}, \mathbf{x}_{G,0}] - N[\mathbf{u}, \mathbf{x}_{N,0}]\|}{\|N[\mathbf{u}, \mathbf{x}_{N,0}]\|}, \quad (1)$$

where

- $\mathbf{u}$, $\mathbf{x}$, and $\mathbf{y}$ are the system inputs, states, and outputs;
- $\mathcal{U}_a$, $\mathcal{X}_{a,0}$, and $\mathcal{Y}_a$ are the spaces of admissible system inputs, initial conditions, and system outputs, respectively;
- $\|\cdot\|$ is a suitable norm in the space of system outputs $\mathcal{Y}$;
- $\mathcal{X}_{G,0}$ is the space of possible initial conditions to the linear dynamic operator $G : \mathcal{U}_a \times \mathcal{X}_{G,0} \to \mathcal{Y}$;
- $N : \mathcal{U}_a \times \mathcal{X}_{a,0} \to \mathcal{Y}$ is the operator for a nonlinear dynamical system with output signals $\mathbf{y}_N \in \mathcal{Y}_a \subseteq \mathcal{Y}$;
- $\mathcal{G}$ is the space of continuous-time linear time-invariant systems;
- $\mathbf{x}_{G,0}$ and $\mathbf{x}_{N,0}$ are the initial values of the state of the linear operator G and nonlinear operator N, respectively; and
- $\mathcal{S} \equiv \{(\mathbf{u}, \mathbf{x}_{N,0}) : \mathbf{u} \in \mathcal{U}_a, \mathbf{x}_{N,0} \in \mathcal{X}_{0,a}, N[\mathbf{u}, \mathbf{x}_{N,0}] \in \mathcal{Y}_a\}$.

All norms and signals are defined over a time interval $[0, t_f)$, where the final time t_f can be either finite or infinite. The outer infimum in (1) defines optimality of the linear dynamic operator G in terms of the norm of the difference between the linear and nonlinear dynamic operators, with optimality defined by the choice of system output norm $\|\cdot\|$. The norm is selected based on the norm that would be used in the linear design method, e.g., an L_2-norm would be appropriate if the linear design method was H_∞-control (Zhou et al., 1996).

The inner infimum in (1) chooses the optimal initial state for the linear operator G in the same manner as for the outer infimum. The supremum in (1) indicates that the worst-case choices of system inputs and initial conditions are considered in the quantification of the extent of nonlinearity.

This nonlinearity measure is computed numerically, as the analytical solution of the optimization (1) usually does not exist. This nonlinearity measure is a relative measure, with value bounded between 0 and 1, with the behavior of the original system N being closer to its optimal linear approximation G as

the nonlinearity measure approaches 0. This nonlinearity measure assesses nonlinearity through a *worst-case* set of system inputs, as described by the optimizations in its definition. The nonlinearity measure can be applied to transient or stationary processes and is a measure of global nonlinearity.

Sometimes, it can be reasonable from the control perspective to demand that the outputs of the operators N and G match exactly at time zero. This requirement motivates the definition of a similar but alternative nonlinearity measure as

$$\hat{\Phi}_N^{\mathcal{Y}_a}(t_f) \equiv$$

$$\inf_{G \in \mathcal{G}} \sup_{(\mathbf{u}, \mathbf{x}_{N,0}) \in \mathcal{S}} \inf_{\mathbf{x}_{G,0} \in \mathcal{X}_{G,0}} \frac{\|G[\mathbf{u}, \mathbf{x}_{G,0}] - N[\mathbf{u}, \mathbf{x}_{N,0}]\|}{\|N[\mathbf{u}, \mathbf{x}_{N,0}]\|}$$

such that

$$\mathbf{0} = G[\mathbf{u}, \mathbf{x}_{G,0}](0) - N[\mathbf{u}, \mathbf{x}_{N,0}](0),$$

with $\mathbf{0}$ being the zero operator.

The above nonlinearity measures that maximize and minimize over various signals are related to a class of controllability measures that have similar optimization formulations (e.g., see Hovd et al. (2003), Ma et al. (2002), and references therein). Unlike the related controllability measures, the above nonlinearity measures include a nonlinear operator in the optimization and include an optimization over a linear operator.

Later publications (Schweickhardt and Allgöwer, 2007; Schweickhardt and Allgöwer, 2009) interpreted some nonlinearity measures as the gain of an error system resulting from the representation of a nonlinear process as the connection between a nominal linear model and an error system. With this approach, different open-loop nonlinearity measures were proposed through the variation of the structure of the aforementioned interconnection between nominal model and error system. For instance, the additive error nonlinearity measure of N for the set of system inputs $\mathcal{U}$ can be defined as

$$\Phi_{AE,N}^{\mathcal{U}} = \inf_{G \in \mathcal{G}} \|\Delta\|^{\mathcal{U}}, \tag{2}$$

$$= \inf_{\substack{G \in \mathcal{G} \\ \mathbf{u} \in \mathcal{U} \\ T > 0}} \sup \frac{\|(N[\mathbf{u}] - G[\mathbf{u}])_T\|}{\|\mathbf{u}_T\|}, \tag{3}$$

where the subscript T means that the associated terms are computed up to time T. This definition corresponds to the additive error interconnection structure shown in Figure 1. Like the previously presented nonlinearity measure, this nonlinearity measure relies on an optimal linear time-invariant model G over the set of possible system inputs $\mathcal{U}$. Nonlinearity measures can be proposed using other error system interconnections (Schweickhardt and Allgöwer, 2007; Schweickhardt and Allgöwer, 2009), including multiplicative output error, multiplicative input error, inverse multiplicative output error, inverse multiplicative input error, and feedback error. All of those nonlinearity measures usually need to be numerically computed, are only applicable if the system N is finite-gain stable, consider a *worst-case* scenario, and measure global nonlinearity. The additive error nonlinearity measure is an absolute measure of nonlinearity that is always nonnegative and bounded by the gain of the system N, while the other aforementioned nonlinearity measures are relative measures, with values that range between 0 and 1. Collectively, these error structures correspond to the norm-bounded model uncertainties that arise in robust control theory (e.g., see Morari and Zafiriou (1989) and references therein).
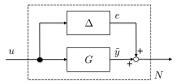

Fig. 1. Additive error interconnection structure

Li (2012) proposed a nonlinearity measure for stochastic systems. This measure was based on the deviation between a nonlinear function and the subspace of linear functions. The best linear approximation was then obtained by stochastic optimization. The closeness between a nonlinear function n_k and the set of all linear functions $\mathcal{L}$ can be defined as

$$J_k = \inf_{L_k \in \mathcal{L}} \left(\mathrm{E}[\|L_k(x) - n_k(x)\|_2^2] \right)^{1/2}, \tag{4}$$

where $L_k \in \mathcal{L}$ is a linear function, and E is the expectation with respect to the random variable x_k. This nonlinearity measure can be normalized as

$$\nu_k = \frac{J_k}{[\mathrm{tr}(C_{g_k})]^{1/2}}, \tag{5}$$

where C_{g_k} is the covariance matrix of g_k and $\mathrm{tr}(A)$ is the trace of matrix A. This nonlinearity measure is neutral in the sense of depending on all the system inputs rather than only on worst-case inputs, and can be preferable for stochastic systems analysis. The stochastic measure quantifies global nonlinearity, but can be adapted to quantify nonlinearity that is more localized by restricting the set of random inputs. This nonlinearity measure is nonnegative and absolute rather than relative. This open-loop nonlinearity measure is invariant under invertible affine transformations of the independent variable. The stochastic nonlinearity measure also nearly always needs to be computed numerically, due to the challenge in finding analytical solutions for such optimizations.

A closed-loop control-relevant nonlinearity measure defined based on the gap metric (Du and Johansen, 2017) has been reported to be comparatively easier and simpler to compute and apply. The measure can be used to provide some guidance in feedback controller design and serves as a criterion to assess the closed-loop performance of the controller, being defined as

$$\mathrm{NM}_1 = \frac{\delta_{\max}(P^*)}{b_{\mathrm{opt}}(P^*)}, \tag{6}$$

with

$$\delta_{\max}(P^*) = \max_{1 \le i \le n} \delta(P^*, P_i), \tag{7a}$$

$$b_{\mathrm{opt}}(P_i) = \sqrt{1 - \left\| [\tilde{N}_i] \tilde{M}_i \right\|_H^2}, \tag{7b}$$

where

- $P_i = N_i M_i^{-1}$ is the normalized right coprime factorization of the linear system P_i,
- $P_i (i = 1, \ldots, n)$ is one of n linearized models, with one for each equilibrium point of the system,
- P^* is the best local linear model, and
- $\delta(P_i, P_j)$ is the gap metric between linear time-invariant systems P_i and P_j.

The gap metric is bounded between 0 and 1, and is defined as the maximum of two directed gaps

$$\delta(P_i, P_j) \equiv \max \left\{ \boldsymbol{\delta}(P_i, P_j), \boldsymbol{\delta}(P_j, P_i) \right\}, \tag{8}$$

where

$$\boldsymbol{\delta}(P_i, P_j) \equiv \inf_{Q \in H_\infty} \left\| \begin{bmatrix} M_i \\ N_i \end{bmatrix} - \begin{bmatrix} M_j \\ N_j \end{bmatrix} Q \right\|_\infty, \qquad (9)$$

and the maximum gap metric stability margin of P_i, which is an intrinsic property of P_i, is defined by

$$b_{\text{opt}}(P_i) \equiv \inf_{\text{stabilizing } K} b_{P,K}$$

$$= \inf_{\text{stabilizing } K} \left\| \begin{bmatrix} I \\ K \end{bmatrix} (I + P_i K)^{-1} [I \ \ P_i] \right\|_\infty^{-1}$$

$$= \sqrt{1 - \left\| [\tilde{N}_i \ \ \tilde{M}_i] \right\|_H^2} < 1, \qquad (10)$$

where $P_i = \tilde{M}_i^{-1} \tilde{N}_i$ is the normalized left coprime factorization of P_i, K is a stabilizing feedback controller for P_i, and $\|\cdot\|_H$ is the Hankel norm.

If the gap metric nonlinearity measure $\text{NM}_1 < 1$, there exists a linear controller K that can, in theory, stabilize the nonlinear dynamical system over the entire operating space. In this case, the system is said to *weakly nonlinear* under the maximum stability criterion, and *strongly nonlinear* otherwise. As the value of $b_{\text{opt}}(P^*)$ does not depend on the controller used, the gap metric nonlinearity measure NM_1 is a universal measure, in the sense of having no dependency on control strategies, and can be computed before the controller is designed.

A control-relevant gap metric nonlinearity measure is introduced in (Du and Johansen, 2017) as

$$\text{NM}_2 = \frac{\delta_{\max}(P^*)}{b_{P^*,K}}, \qquad (11)$$

where K is a linear stabilizing controller designed based on P^*. If this nonlinearity measure $\text{NM}_2 < 1$ and the controller K satisfies the desired closed-loop performance requirements, then the original system is said to be *closed-loop linear*. Otherwise, a nonlinear control method will be necessary for the system to be stabilized and satisfy the closed-loop performance requirements. This nonlinearity measure cannot be computed before the linear time-invariant controller is designed, and so depends on both the system and the controller.

4. NONLINEARITY MEASURES FOR DISTRIBUTED PARAMETER SYSTEMS

Unlike the above nonlinearily measures, which are applicable to both lumped and DPS, a few papers have proposed nonlinearity measures specific to DPS. Fuxman et al. (2007) proposed a curvature-based measure of steady-state nonlinearity for a subclass of hyperbolic DPS which are described by a system of first-order PDEs of the form

$$\sum_{j=1}^n b_{ij} \frac{\partial x_j}{\partial t} + \sum_{j=1}^n a_{ij} \frac{\partial x_j}{\partial z} = c_i, i = 1, 2, \ldots, n, \qquad (12)$$

where t and z are the only two independent variables, $x_j(z, t)$ are n distributed state variables, and a_{ij}, b_{ij}, and c_i are scalar functions that can depend on z, x_j, and input variables u_k. This class of hyperbolic PDEs appear in simplified models of many chemical process applications including packed-bed reactors and tubular heat exchangers, for which t is the time and z is the axial distance from the process inlet. These types of models assume that any diffusive phenomena (aka Brownian motion) have a negligible effect on the states x_j, and that the states have negligible variation in the radial direction. This class of models also arise in simplified models of particulate processes, in which case t is usually time and z is some characteristic of

an individual particle, such as length, mass, or age (Ramkrishna and Singh, 2014).

For this class of DPS, the distribution of the acceleration matrix along the spatial coordinate z, i.e.,

$$A_c \equiv \frac{\partial}{\partial \mathbf{u}'} \left(\frac{\partial x}{\partial \mathbf{u}} \right),$$

is computed from the steady-state locus defined from the system of PDEs by

$$\sum_{j=1}^n a_{ij} \frac{\partial x_j}{\partial z} - c_i = g\left(x, \frac{\partial x}{\partial z}, \mathbf{u} \right) = 0. \qquad (13)$$

Taking partial derivatives of the steady-state map g with respect to the inputs and through differential manipulations, a system of equations is derived whose solution can be evaluated at different points to obtain the acceleration matrix A_c. The elements of the acceleration matrix quantify the nonlinearity of the process, with larger elements indicating larger nonlinearity. The acceleration matrix can be decomposed into a normal and a tangential curvatures, each of which providing different information about the nonlinearity of the process. This approach generalizes the curvature approach of Guay et al. (1995) to an industrially important class of DPS.

Wu et al. (2017) proposed a nonlinearity measure for a different class of DPS expressed as follows:

$$\frac{\partial \mathbf{x}}{\partial t} = \frac{\partial}{\partial z} \left(D(\mathbf{x}) \frac{\partial \mathbf{x}}{\partial z} \right) - \mathbf{v}(\mathbf{x}) \frac{\partial \mathbf{x}}{\partial z} + \mathbf{f}(\mathbf{x}) + \mathbf{u}(z, t), \qquad (14)$$

where $\mathbf{x}$ is a vector of distributed states that are function of two scalar independent variables z and t, D is the diffusivity, $\mathbf{v}$ is the velocity in the z direction which is a scalar equal to dz/dt, $\mathbf{f}$ is a vector of state-dependent forcing functions, and

$$\mathbf{u} \equiv \sum_{i=1}^p u_i(t) \mathbf{h}_i(z) \qquad (15)$$

is a vector of process inputs that only depend on the independent variables, u_i are scalar process units, and $\mathbf{h}_i$ are vectors in which each element is a function of the spatial variable z. This representation treats the process input as a vector of lumped variables (u_i, $\forall i = 1, \ldots, p$) and the process output as a vector of distributed states $\mathbf{x}$. The spatial dimension z is assumed to belong to a bounded domain Ω, and the distributed states are required to satisfy a number of initial and boundary conditions.

For positive diffusivity D, this set of PDEs is parabolic, and is a generalization of (12) in the sense of including both diffusion and advection. The convection term in (14) has more restrictive dependencies on variables than (12), so neither set of PDEs is a subset of the other. The above class of DPS also appears in simplified models of many chemical processes including packed-bed reactors and tubular heat exchangers, for which t is the time and z is the axial distance from the inlet. The more restricted form of the velocity $\mathbf{v}(\mathbf{x})$ occurs for packed-bed reactors of constant cross-sectional area, with the nontrivial dependency on the distributed state occurring for operations in which the fluid density is a function of the species concentrations $\mathbf{x}_i$. Such density changes are significant for gas-phase chemical reactions in which the number of gaseous reactants in some of the reactions is not equal to the number of gaseous products, e.g., Torchio et al. (2016). This class of models also applies to some particulate processes, in which the diffusive term models growth dispersion (Ramkrishna and Singh, 2014).

Wu et al. (2017) proposed the nonlinearity measure

$$\delta_N = \inf_{G \,\in\, \mathcal{G}} \; \sup_{\substack{\mathbf{u} \,\in\, \mathcal{U} \\ \|N[\mathbf{u}]\| \,\neq\, 0}} \; \frac{\|G[\mathbf{u}] - N[\mathbf{u}]\|}{\|N[\mathbf{u}]\|}, \qquad (16)$$

where $\mathbf{u}(z, t)$ is a spatiotemporal input signal and the norm in the spatiotemporal domain was defined as

$$\|\mathbf{x}(z, t)\| = \sqrt{\int_\Omega \int_0^\infty |\mathbf{x}(z, t)|^2 \, dt \, dz}, \qquad (17)$$

being $|\mathbf{x}|$ the Euclidean norm of $\mathbf{x}$.

This worst-case nonlinearity measure is similar to the general measure (1), including being a relative measure with value between zero and one. As in both nonlinearity measures, optimization over the linear operator G cannot produce a larger value than the value of the norm for any particular value of G. The choice of $G \in \mathbf{G}$ thus results in an upper bound on the solution of the optimization (the inner optimizations fall away because they do not affect the objective function when $G = \mathbf{0}$). The nonlinearity behavior is completely nonlinear when the nonlinearity measure is equal to one, since then no linear operator G exists to approximate operator N that is more optimal than the zero operator $\mathbf{0}$.

The measure (16) is not general for several reasons. First, the set $\mathcal{U}$ was restricted by Wu et al. (2017) to be of the form (15), which restricts the process input $\mathbf{u}$ to be a function of two independent variables and to be restricted to have a particular bilinear form of the dependency on t and z. Second, the DPS for which the nonlinearity measure was defined was restricted to have the form in (14). Both of these restrictions were exploited by Wu et al. (2017) to enable the derivation of an efficient numerical algorithm for computing the nonlinearity measure (16) based on several steps that include proper orthogonal decomposition.

5. DESCRIPTOR/SINGULAR SYSTEMS

The extensions of the nonlinearity measures for ordinary differential equations conceptually apply to descriptor (aka singular) systems, that is, dynamical systems described by coupled algebraic and ordinary differential equations.

The nonlinearity measure (1) applies directly, as long as a well-posed system norm is defined for the class of descriptor/singular systems. An important consideration when numerically computing such measures is whether the norm of the difference between the linear and nonlinear operators is a smooth function of the system inputs. Descriptor/singular systems can have dynamics in which either the state or the output can show Dirac delta-type behavior even when the system inputs are bounded (Ascher and Petzold, 1998; Brenan et al., 1996).

Whether such behavior can occur is characterized in terms of the *index* of the system of differential-algebraic equations (Ascher and Petzold, 1998; Brenan et al., 1996). Such systems with an index equal to zero or one will have states and output that are continuous functions of the system inputs, in which case the nonlinearity measure (1) can be computed numerically.

Defining and computing a useful nonlinearity measure is more challenging for differential-algebraic systems that have an index of two or higher due to the potential for discontinuities in the states and output. Although the numerical simulation of such systems requires some care, nowadays many techniques have been developed to produce numerically stable and reliable simulation results. For example, the original set of differential-algebraic equations can be reformulated to create an equivalent set of differential-algebraic equations that has an index of one (Mattsson and Söderlind, 1993). The wide availability of such techniques to handle high-index systems means numerical simulation by itself does not pose any limitation to defining and computing a meaningful nonlinearity measure.

Whether the nonlinearity measure for a system of differential-algebraic equations of higher index is more challenging to compute numerically concerns whether any discontinuities occur in the relevant input-output relationship of the system. Even if discontinuities occur for some region in the space of allowable system inputs, whether the nonlinearity measure is bounded can depend on the choice of norm. For example, a Dirac delta-type behavior in the process output may not be a concern provided that the system norm remains bounded for such behavior. [2]

Nonlinearity measures for PDEs conceptually apply to processes described by systems of integro-partial differential-algebraic equations, which commonly arise in the advanced manufacturing processes (e.g., see Paulson et al. (2018) and references therein). The theoretical and numerical algorithm extensions to handle integrals are straightforward, and the algebraic equations are manageable provided that the system has a differential index of zero or one (e.g., see Martinson and Barton (2016) and references therein). Integro-partial differential-algebraic equations of higher differential index can also normally be addressed by mathematical reformulation.

The definition of nonlinearity measures can be straightforwardly extended to hybrid dynamical systems including both continuous and discrete behaviors, provided that the system state or output of interest is a continuous function of the system input. For general hybrid system representations, an extension to (1) is to search for a linear time-variant system for each subsystem that can be reached by the logic that switches between subsystems. That extension is quite computationally expensive. A subset of hybrid systems that describe many practical manufacturing processes can be modeled by nonsmooth differential-algebraic equations, which can be numerically solved efficiently for large-scale systems (Stechlinski and Barton, 2016). For this class of hybrid systems, the nonlinearity measure definition (1) can be applied directly.

6. COMPUTATIONAL CONSIDERATIONS

Most nonlinearity measures are formulated in terms of optimizations – usually nested – that need to be solved numerically. Consequently, the computational cost must be considered, especially for large-scale systems. These optimization problems are nearly always too complicated to solve by branch-and-bound global optimization algorithms, which motivates the use of Monte Carlo sampling-based methods. While the practical cost is alleviated by the fact that the calculations are off-line, and are easily run in parallel, the cost is still high. Each nesting increases computational cost, as each extra optimization level results in another loop of sampling-based optimization. Depending on the complexity of the operators and the level of accuracy needed, it may be useful to reduce the computational

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[2] As a simple algebraic example, the 1-norm of the Dirac delta function is equal to one, whereas its ∞-norm is unbounded.

cost by removing one of the levels (e.g., an optimization over initial conditions), in exchange for computing a weaker nonlinearity measure. The dependencies of convergence rates of sampling-based methods on the dimensionality and the properties of the underlying operators have been well investigated, and readers are refereed to a recent survey (Homem-de-Mello and Bayraksan, 2014).

7. CONCLUDING REMARKS

This paper reviews nonlinearity measures for lumped and distributed parameter systems, and reviews in more detail those measures that are applicable to DPS. Various considerations are discussed, including whether the system inputs are treated as worst-case or described by a stochastic distribution, whether a measure quantifies the extent of nonlinearity for the open-loop or closed-loop dynamics, or whether a measure applies to both lumped and distributed parameter systems, or to only a subclass of distributed parameter systems. Then extensions to nonlinear dynamical systems with mathematical representations that include integral and algebraic equations in addition to ordinary and/or partial differential equations are discussed. Then nonsmooth differential-algebraic equations and computational cost are considered. The mathematical formulation of nonlinearity measures that are both practically useful and computational tractable remains an open field of research. Some future directions include extensions to more general classes of nonlinear dynamical systems, such as determining whether the numerical method proposed for the efficient computation of nonlinearity measures from those reviewed into this paper can be either improved or extended to more general distributed parameter and/or descriptor systems.

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