LMI-based design of state-feedback controllers for pole clustering of LPV systems in a union of $\mathcal{D}_R$-regions

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ABSTRACT
This paper introduces an approach for the design of a state-feedback controller for LPV systems that achieves pole clustering in a union of $\mathcal{D}_R$-regions. The design conditions, obtained using a partial pole placement theorem, are eventually expressed in terms of linear matrix inequalities, which can be solved efficiently using available solvers. In addition, it is shown that the approach can be modified in a shifting sense, which means that the controller gain is computed such that different values of the varying parameters imply different regions of the complex plane where the closed-loop poles are situated, thus enabling online modification of the closed-loop performance. The effectiveness of the proposed method is demonstrated by means of simulations.

KEYWORDS
Linear parameter varying (LPV) systems, pole placement, union of regions, linear matrix inequalities (LMIs), state-feedback control.

1. Introduction

Linear parameter varying (LPV) systems have received a lot of attention from the control community in the last decades. They were first introduced in Shamma (1988), in order to distinguish such systems from linear time invariant (LTI) and linear time varying (LTV) (Shamma, 2012). The LPV paradigm has proved to be suitable for controlling nonlinear systems by embedding the nonlinearities in the varying parameters, and it has become a standard formalism in systems and control, for analysis, synthesis of controllers and even system identification. In this case, since the varying parameters depend on some endogenous signals, such as states and/or inputs, the system is commonly referred to as quasi-LPV (Rugh & Shamma, 2000). LPV systems are closely related to the Takagi-Sugeno (TS) approach (Takagi & Sugeno, 1985), and some recent works have discussed the existing similarities between the two approaches (López-Estrada, Rotondo, & Valencia-Palomo, 2019; Rotondo, Puig, & Nejjari, 2016; Rotondo, Puig, Nejjari, & Witczak, 2015), stating that the main remarkable difference lies in the use of fuzzy logic by the latter, whereas the former relies on traditional mathematics. More recently, there has been a growing interest in extending these techniques to nonlinear parameter varying systems (NLPV), see e.g. Larimore (2013), Blesa, Jiménez, Rotondo, Nejjari, and Puig (2014), Rotondo and Johansen (2018), R. Yang, Rotondo, and Puig (2019), since in many practical applications there are time-varying nonlinearities that can be dealt with using ad hoc approaches.

In recent years, there has been an important progress in the development of analysis and design techniques for LPV system, and this concept has been further investigated by several researchers, who brought different innovations (Hoffmann & Werner, 2014; Rotondo, 2017). LPV techniques have found application in many fields, such as bicycle (Brizuela Mendoza, Sorcia Vázquez, Guzmán Valdivia, Osorio Sánchez, & Martínez García, 2018), robotics (San Miguel, Puig, & Alenyà, 2019), aerospace (D. Yang, Zong, & Karimi, 2019), ground vehicles (Zhang, Zhang, & Wang, 2016), wind turbines (Pérez-Estrada, Osorio-Gordillo, Alma, Darouach, & Olivares-Peregrino, 2018) and power system (El-Guindy, Schaab, Schürmann, Stursberg, & Althoff, 2017). Remarkable applications can be mentioned as machine learning (Rizvi, Velni, Abbasi, Tóth, & Meskin, 2018), and model predictive control (MPC) (Ding, Dong, & Hu, 2019), and the research is currently undergoing theoretical development (Morato, Normey-Rico, & Sename, 2020).

Among the considered specifications for the design, pole clustering in linear matrix inequality (LMI) regions, also known as $\mathcal{D}$-stability, has received a lot of interest. Initially characterized by Chilali and Gahinet (1996) using a quadratic Lyapunov function with constant matrix, this idea was further developed by Peaucelle, Arzelier, Bachelier, and Bernussou (2000), who considered uncertain systems by means of a parameter dependent Lyapunov function, and is still investigated nowadays, see e.g. the recent improvements in Nguyen, Márquez, Guerra, and Dequidt (2017) and Chesi (2017). However, LMI regions have some limitations, such that they are not able of describing non-convex regions or the union of different regions. For this reason, Peaucelle et al. (2000) proposed a new characterization of LMI regions referred to as $\mathcal{D}_R$-regions and considered uncertain systems by means of a parameter-dependent Lyapunov function. In Peaucelle et al. (2000), $\mathcal{D}_R$-regions were shown to be able to describe non-convex regions but only represented symmetrically, which motivated Bosche, Bachelier, and Mehdi (2005) to extend the concept further to consider non-symmetrical regions. On the other hand, Bachelier and Pradin (1999) developed an approach that allows specifying not only a simple convex region, but also a non-convex region, defined as a union of convex subregions. Then, Maamri, Bachelier, and Mehdi (2006) proposed a technique in order to achieve partial pole placement via aggregation in such regions. This method can influence strongly the performance, in terms of settling time and damping ratio. In Tornil-Sin, Theilliol, Ponsart, and Puig (2010), this method was applied to fault-tolerant control.

Although the concept of the pole is not formally defined for LPV systems, Ghersin and Pena (2002) showed that by including pole clustering specification in LPV design, the performance of the LPV control systems could be improved. Moreover, R. Yang et al. (2019) showed the existence of a relationship between pole placement and the Lyapunov function. In fact, pole placement for gain-scheduled systems has progressed strongly in the last decades, with several results concerning the design of observers (Nejjari, Puig, de Oca, & Sadeghzadeh, 2009), state-feedback controllers (Bouazizi, Kochbati, & Ksouri, 2001; R. Yang et al., 2019), H_∞ controllers (Rotondo, Nejjari, & Puig, 2014; Yu, Chen, & Woo, 2002), and application in many fields, such as aerospace vehicles (Ghersin & Pena, 2002), UAV (López-Estrada, Ponsart, Theilliol, Zhang, &

Astorga-Zaragoza, 2016), missile (Shen, Yu, Luo, & Mei, 2017), power system (Jabali & Kazemi, 2017a), fuel cells (Rotondo, Fernandez-Canti, Tornil-Sin, Blesa, & Puig, 2016) and robotics (Jabali & Kazemi, 2017b).

However, based on the literature review, it seems that the aforementioned partial pole placement technique has not been applied yet to gain-scheduled systems, such as LPV and TS. In fact, such an extension is not trivial, as the design of controller gain through aggregation introduces nonlinearities that destroy the polytopic decomposition usually exploited in the LPV controller design. Motivated by this fact, the main goal of this paper is to consider the problem of designing an LPV state-feedback controller for LPV systems that can guarantee some desired closed-loop poles clustering in a region defined as the union of disjoint and non-symmetric subregions. It is shown that it is possible to exploit the aggregation technique initially proposed for LTI systems in Maamri et al. (2006) to achieve partial pole placement in LPV systems, so that disjoint regions can be assigned for the closed-loop distribution of the poles. The proposed design conditions are formulated through an LMI approach. In order to deal with the nonlinearities introduced by the eigendecomposition required to achieve partial pole placement, new varying parameters are introduced so that a polytopic representation can be recovered.

In addition to the classical pole clustering problem, in this paper, we consider also an extension referred to as *shifting pole placement* (Rotondo, Nejjari, & Puig, 2013, 2015). This approach allows designing the controller gain in such a way that different values of the varying parameters imply different regions where the closed-loop poles are situated. By means of the shifting paradigm, the online modification of the performance can be achieved, as demonstrated for example by Ruiz, Rotondo, and Morcego (2019) and Ruiz, Rotondo, and Morcego (2020), who have applied this concept to saturated system showing that it is possible to schedule the closed-loop performance according to changes in the saturation function.

The main contributions of this paper can be summarized as follows:

- The partial pole placement originally developed in Maamri et al. (2006) for LTI systems is extended to work with LPV systems.
- A procedure for the design of a state-feedback controller which achieves pole clustering in a union of $\mathcal{D}_R$-regions is proposed for LPV systems.
- It is shown that in spite of the nonlinearities introduced by the aggregation technique, it is possible to introduce new sets of varying parameters in terms of which polytopic representations suitable for reducing the number of design LMIs from infinite to finite can be obtained.

The rest of the paper is organized as follows. In Section 2, partial pole placement background information is introduced. Section 3 explains in detail the pole clustering in a union of regions for LPV systems. Section 4 discusses how the approach can be extended according to the shifting paradigm. Section 5 shows the application of the developed technique to a numerical example. Finally, Section 6 summarizes the conclusions and suggests possible future work.

Notation: $\mathbb{R}^{n\times m}$ and $\mathbb{C}^{n\times m}$ denote the set of real and complex matrices with n rows and m columns; A^T, A^* and A^+ denote the transpose, the conjugate and the pseudo-inverse of A, respectively; A^H is the Hermitian matrix defined as $A^H = A + A^*$; the Euclidean norm is indicated by $||A||$; $\otimes$ represents the Kronecker product; in matrix inequalities, negative (semi-)definiteness is indicated by $\prec 0$ ($\preceq 0$), whereas $\succ 0$ ($\succeq 0$) denotes positive (semi-)definiteness.

2. Background

The idea of LMI regions was first introduced by Chilali and Gahinet (1996) in order to provide a Lyapunov-based characterization of pole clustering in stable subregions of the complex plane. Their formal definition is given as follows:

Definition 1: *(LMI region)* A subset $\mathcal{D}$ of the complex plane is called an LMI region if there exist a matrix $\alpha = [\alpha_{kl}] \in \mathbb{S}^{m\times m}$ and a matrix $\beta = [\beta_{kl}] \in \mathbb{R}^{m\times m}$ such that:

$$\mathcal{D} = \{s \in \mathbb{C} : f_{\mathcal{D}} \prec 0\} \tag{1}$$

with the *characteristic function* given by:

$$f_{\mathcal{D}}(s) = \alpha + s\beta + s^*\beta^T = [\alpha_{kl} + \beta_{kl}s + \beta_{lk}s^*]_{1\le k,l\le m} \tag{2}$$

In other words, LMI regions are subsets of the complex plane that are represented by an LMI in s and s^*. In addition, a new characterization of regions was proposed in Peaucelle et al. (2000) called $\mathcal{D}_R$-regions.

Definition 2: *($\mathcal{D}_R$-regions)* Let R be a $2d \times 2d$ Hermitian matrix defined as:

$$R = \begin{bmatrix} R_{00} & R_{10} \\ R_{10}^* & R_{11} \end{bmatrix} \in \mathbb{C}^{2d\times 2d} \tag{3}$$

Then, the subset of the complex plane defined according to

$$\mathcal{D}_R = \{s \in \mathbb{C} : R_{00} + (R_{10}s)^H + R_{11}s^*s \prec 0\} \tag{4}$$

is called a $\mathcal{D}_R$-region of degree d.

The class of $\mathcal{D}_R$-regions is a class of open convex subsets of the complex plane that includes (among others) half-planes, disks, conic sectors, vertical and horizontal strips and ellipses, symmetrical or not with respect to the real axis. For example, the vertical left half-plane defined by $Re(s) < \lambda$ is characterized by a matrix R equal to:

$$R = \begin{bmatrix} -2\lambda & 1 \\ 1 & 0 \end{bmatrix} \tag{5}$$

while the interior of a disk with center $c = c_1 + c_2 i$ and radius r is characterized by:

$$R = \begin{bmatrix} c_1^2 + c_2^2 - r^2 & -c_1 + c_2 i \\ -c_1 - c_2 i & 1 \end{bmatrix} \tag{6}$$

In contrast to LMI regions, $\mathcal{D}_R$-regions are able to represent non-convex region. Without any assumption on the matrix R_{11}, $\mathcal{D}_R$-regions are not convex, but with $R_{11} \succ 0$, $\mathcal{D}_R$-regions become a slight modification of the characterization provided by LMI regions (Rotondo, 2017). In Bachelier and Pradin (1999), non-convex regions were considered as unions of convex subregions.

Before introducing the partial pole placement for LPV systems, let us recall the existing approach for LTI system (Maamri et al., 2006).

Let us consider the following system:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{7}$$

Then, the following lemmas can be applied:

Lemma 1 (Peaucelle et al., 2000): *The matrix A is said to be $\mathcal{D}_R$-stable, or in other words all its eigenvalues lie inside the region $\mathcal{D}_R$ if and only if there exists a matrix $P \succ 0$ such that:*

$$R_{00} \otimes P + (R_{10} \otimes (PA))^H + R_{11} \otimes (A^* PA) \prec 0 \tag{8}$$

In Maamri et al. (2006), Lemma 1 has been extended to partial pole placement in $\mathcal{D}_R$, which means that only p eigenvalues are wished to be affected by the feedback. To do so, some matrices are defined as follows:

$$\Lambda = V^{-1}AV = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \quad C = \begin{bmatrix} I_p & 0 \end{bmatrix} V^{-1}$$
$$C^+ = V \begin{bmatrix} I_p \\ 0 \end{bmatrix} \quad \hat{A} = CAC^+ \quad \hat{B} = CB$$

where V is the modal matrix of A, that is, the matrix whose columns are the eigenvectors of A. $\Lambda_1 \in \mathbb{C}^{p \times p}$ is associated to the set of p eigenvalues desired to be affected by the feedback, whereas Λ_2 denotes the remaining eigenvalues.

Then, the following lemma can be applied:

Lemma 2 (Maamri et al., 2006): *There exists a state-feedback gain $\hat{K}$ that assigns p poles in $\mathcal{D}_R$ if and only if there exist an matrix $\hat{X} \succ 0$ and a matrix $\hat{S}$ such that the LMI:*

$$\begin{bmatrix} R_{00} \otimes \hat{X} + (R_{10} \otimes (\hat{A}\hat{X} + \hat{B}\hat{S}))^H & Z^* \otimes (\hat{A}\hat{X} + \hat{B}\hat{S})^* \\ Z \otimes (\hat{A}\hat{X} + \hat{B}\hat{S}) & -I_d \otimes \hat{X} \end{bmatrix} \prec 0 \tag{9}$$

holds, where Z is deduced from the Cholesky factorization $R_{11} = Z^* Z$. In this case, a suitable state-feedback gain is:

$$\hat{K} = \hat{S}\hat{X}^{-1} \tag{10}$$

3. Pole clustering in a union of regions for LPV system

In this section, the aggregation technique is extended to LPV systems to compute a state-feedback controller gain which performs pole clustering in a union of regions for LPV systems.

Let us recall that an LPV system is defined as a finite-dimensional time-varying system whose state equation, although linear, is described by matrices which are function of some varying parameters $\theta(t) \in \Theta \subset \mathbb{R}^{n_\theta}$ (with Θ known closed set), that are assumed to be unknown a priori, but that can be measured or estimated in real-time:

$$\dot{x}(t) = A(\theta(t))x(t) + B(\theta(t))u(t) \tag{11}$$

where $A(\theta(t)) \in \mathbb{R}^{n_x \times n_x}$, $B(\theta(t)) \in \mathbb{R}^{n_x \times n_u}$ are the state and input matrices, $x(t) \in \mathbb{R}^{n_x}$ denotes the system state, $u(t) \in \mathbb{R}^{n_u}$ is the control input. The system (11) is said to be polytopic if it can be represented by state-space matrices $A(\theta(t))$ and $B(\theta(t))$ which range over a convex set:

$$\dot{x} = \sum_{n=1}^{w} \mu_n(\theta(t))\big(A_n x(t) + B_n u(t)\big) \tag{12}$$

where μ_n are the non-negative coefficients of the polytopic decomposition such that:

$$\sum_{n=1}^{w} \mu_n(\theta(t)) = 1 \quad \mu_n(\theta(t)) \geq 0 \quad \forall n = 1, ..., w \quad \forall \theta \in \Theta$$

Let us use a gain-scheduled state-feedback control law given by:

$$u(t) = K(\theta(t))x(t) \tag{13}$$

where $K(\theta(t))$ is the controller gain to be designed.

3.1. Aggregation technique

The main idea of the algorithm employed to achieve pole clustering in a union of regions is to use a structured feedback gain $K(\theta(t))$ that modifies just a subset of the system poles. Consider the Jordan canonical form for the matrix $A(\theta(t))$:

$$\Lambda(\theta(t)) = V(\theta(t))^{-1} A(\theta(t)) V(\theta(t)) \tag{14}$$

where $V(\theta(t))$ is the modal matrix of $A(\theta(t))$, which means that the columns of $V(\theta(t))$ are the eigenvectors of $A(\theta(t))$. Meanwhile $\Lambda(\theta(t))$ can be rearranged in a form such that:

$$\Lambda(\theta(t)) = \begin{bmatrix} \Lambda_1(\theta(t)) & 0 \\ 0 & \Lambda_2(\theta(t)) \end{bmatrix} \tag{15}$$

where $\Lambda_1(\theta(t)) \in \mathbb{C}^{p \times p}$ is associated to the set of p eigenvalues that are wished to be affected by the feedback. Let us define the following matrices:

$$C(\theta(t)) = \begin{bmatrix} I_p & 0 \end{bmatrix} V(\theta(t))^{-1} \tag{16}$$

with pseudo-inverse given by:

$$C(\theta(t))^{+} = V(\theta(t)) \begin{bmatrix} I_p \\ 0 \end{bmatrix} \tag{17}$$

and:

$$\hat{A}(\theta(t)) = C(\theta(t))A(\theta(t))C(\theta(t))^{+} \tag{18}$$

$$\hat{B}(\theta(t)) = \begin{bmatrix} I_p & 0 \end{bmatrix} V(\theta(t))^{-1}B(\theta(t)) \tag{19}$$

$$\bar{B}(\theta(t)) = \begin{bmatrix} 0 & I_{n-p} \end{bmatrix} V(\theta(t))^{-1}B(\theta(t)) \tag{20}$$

and finally, consider a feedback gain defined as:

$$K(\theta(t)) = \hat{K}(\theta(t))C(\theta(t)) \tag{21}$$

The closed-loop matrix then satisfies:

$$\begin{aligned} &A(\theta(t)) + B(\theta(t))\hat{K}(\theta(t))C(\theta(t)) = ... \\ &... = V(\theta(t)) \begin{bmatrix} \Lambda_1(\theta(t)) + \hat{B}(\theta(t))\hat{K}(\theta(t)) & 0 \\ \bar{B}(\theta(t))\hat{K}(\theta(t)) & \Lambda_2(\theta(t)) \end{bmatrix} V(\theta(t))^{-1} \end{aligned} \tag{22}$$

which means that the eigenvalues associated to $\Lambda_1(\theta(t))$ are modified whereas the eigenvalues of $\Lambda_2(\theta(t))$ are invariant with respect to the feedback $K(\theta(t)) = \hat{K}(\theta(t))C(\theta(t))$, i.e. such feedback gain will only modify the p eigenvalues of interest.

3.2. Partial $\mathcal{D}_R$-stability

In order to extend the pole clustering in a union of regions to LPV system, let us consider the $\mathcal{D}_R$-stability of the LPV system (11), in the sense of all the frozen poles of (11) lying in $\mathcal{D}_R$. By applying Lemma 1, $\mathcal{D}_R$-stability holds if there exists a matrix $P \succ 0$ such that $\forall \theta \in \Theta$:

$$R_{00} \otimes P + (R_{10} \otimes (PA(\theta(t))))^H + R_{11} \otimes (A(\theta(t))^* PA(\theta(t))) \prec 0 \tag{23}$$

The above means that for partial pole placement in $\mathcal{D}_R$ being achieved by the state-feedback gain $\hat{K}(\theta(t))$, the gain has to be chosen in such a way that it satisfies:

$$\begin{aligned} &R_{00} \otimes \hat{P} + (R_{10} \otimes (\hat{P}(\hat{A}(\theta(t)) + \hat{B}(\theta(t))\hat{K}(\theta(t)))))^H + ... \\ &... + R_{11} \otimes ((\hat{A}(\theta(t)) + \hat{B}(\theta(t))\hat{K}(\theta(t)))^* \hat{P}(\hat{A}(\theta(t)) + \hat{B}(\theta(t))\hat{K}(\theta(t)))) \prec 0 \end{aligned} \tag{24}$$

for some matrix $\hat{P} \succ 0$.

Deduced from the above, the next theorem states LMI conditions for obtaining a controller gain that achieve partial pole placement.

Theorem 1: There exists a state-feedback gain $\hat{K}(\theta)$ that assigns p eigenvalues of the closed-loop matrix $A(\theta) + B(\theta)\hat{K}(\theta)C(\theta)$ in the region $\mathcal{D}_R$ defined by (4) if there exist matrices $\hat{X} \succ 0$ and $\hat{S}(\theta)$ of appropriate dimensions such that $\forall\theta \in \Theta$:

$$\begin{bmatrix} R_{00} \otimes \hat{X} + (R_{10} \otimes (\hat{A}(\theta)\hat{X} + \hat{B}(\theta)\hat{S}(\theta)))^H & Z^* \otimes (\hat{A}(\theta)\hat{X} + \hat{B}(\theta)\hat{S}(\theta))^* \\ Z \otimes (\hat{A}(\theta)\hat{X} + \hat{B}(\theta)\hat{S}(\theta)) & -I_d \otimes \hat{X} \end{bmatrix} \prec 0 \quad (25)$$

where Z is deduced from the Cholesky factorization $R_{11} = Z^*Z$. Then, the state-feedback gain is given by:

$$\hat{K}(\theta(t)) = \hat{S}(\theta(t))\hat{X}^{-1} \quad (26)$$

Proof: It can be derived by following the steps in the proof of Theorem 2.1 in Maamri et al. (2006). □

Hence, the partial pole placement procedure boils down in obtaining a solution for the LMI (25), and then using (26) to recover the appropriate $\hat{K}(\theta(t))$.

It is necessary to mention that the design condition (25) requires satisfying an infinite number of conditions, which leads to a computational issue. In order to reduce the number of conditions from infinite to finite, the most common way to solve this problem is to use the polytopic assumption. However, the non-linearity introduced by the multiplications in (18) implies that even if a polytopic representation is available for $A(\theta(t))$, it does not necessarily hold that the same polytopic coefficients describe how $\hat{A}(\theta(t))$ varies with respect to $\theta(t)$. However, it is possible to introduce new varying parameters, that are some nonlinear function of the original varying parameters, hereafter denoted by $\hat{\theta}(t)$, and then obtain a polytopic representation for the matrix $\hat{A}$ with coefficients that depend on $\hat{\theta}$. This can be done using available methods in the literature, such as the *bounding box* (Sun & Postlethwaite, 1998) the *singular value decomposition* boxing (Baranyi, 2009) or identification approaches (Fujimori & Ljung, 2005).

Hence the matrices $\hat{A}(\theta(t))$ and $\hat{B}(\theta(t))$ are expressed as polytopic combination of matrices $\hat{A}_i$ and $\hat{B}_i$ as follows:

$$\hat{A}(\theta(t)) = \hat{A}(\hat{\theta}(t)) = \sum_{i=1}^{r} \alpha_i(\hat{\theta}(t))\hat{A}_i \quad (27)$$

$$\hat{B}(\theta(t)) = \hat{B}(\hat{\theta}(t)) = \sum_{i=1}^{r} \alpha_i(\hat{\theta}(t))\hat{B}_i \quad (28)$$

where α_i are the non-negative coefficients of the polytopic decomposition such that:

$$\sum_{i=1}^{r} \alpha_i(\hat{\theta}(t)) = 1 \quad \alpha_i(\hat{\theta}(t)) \geq 0 \quad \forall i = 1, ..., r \quad \forall\hat{\theta} \in \hat{\Theta} \subset \mathbb{R}^{n_{\hat{\theta}}} \quad (29)$$

and the matrix function $\hat{S}(\theta(t))$ is constrained to satisfy:

$$\hat{S}(\theta(t)) = \hat{S}(\hat{\theta}(t)) = \sum_{i=1}^{r} \alpha_i(\hat{\theta}(t))\hat{S}_i \quad (30)$$

Then, the approach proposed by Sala and Ariño Sala and Ariño (2007) to check the definiteness of double polytopic sums (as the ones arising from the terms $\hat{B}(\theta)\hat{S}(\theta)$ in (25)) can be applied by choosing a scalar $s \in \mathbb{N}$ and using the symbols $\mathbb{P}_s$ and $\mathbb{P}_s^+$ to denote the following sets:

$$\mathbb{P}_s = \left\{\vec{p} = [\vec{p}_1, \dots, \vec{p}_s]^T \in \mathbb{N}^s | 1 \leq \vec{p}_k \leq s \, \forall k = 1, \dots, s\right\} \tag{31}$$

$$\mathbb{P}_s^+ = \{\vec{p} \in \mathbb{P}_s | \vec{p}_k \leq \vec{p}_{k+1}, k = 1, \dots, s-1\} \tag{32}$$

whereas $\mathcal{P}(\vec{p}) \subset \mathbb{P}_s$ denotes the set of permutations, with possible repeated elements, of the multi-index $\vec{p}$, thus obtaining the following corollary.

Corollary 1: For any $s \in \mathbb{N}$, with $s \geq 2$, there exist a matrix $\hat{X} \succ 0$ and matrices $\hat{S}_1, \hat{S}_2, \dots, \hat{S}_r$ such that:

$$\sum_{\vec{m} \in \mathcal{P}(\vec{p})} \begin{bmatrix} R_{00} \otimes \hat{X} + (R_{10} \otimes (\hat{A}_{\vec{m}_1}\hat{X} + \hat{B}_{\vec{m}_2}\hat{S}_{\vec{m}_1}))^H & Z^* \otimes (\hat{A}_{\vec{m}_1}\hat{X} + \hat{B}_{\vec{m}_2}\hat{S}_{\vec{m}_1})^* \\ Z \otimes (\hat{A}_{\vec{m}_1}\hat{X} + \hat{B}_{\vec{m}_2}\hat{S}_{\vec{m}_1}) & -I_d \otimes \hat{X} \end{bmatrix} \prec 0 \tag{33}$$

holds $\forall \vec{p} \in \mathbb{P}_s^+$, where Z is obtained from the Cholesky factorization $R_{11} = Z^*Z$, then the state-feedback gain given by (26), with $\hat{S}$ computed using (30), assigns p eigenvalues of the closed-loop matrix $A(\theta) + B(\theta)\hat{K}(\theta)C(\theta)$ in the region $\mathcal{D}_R$ defined by (4).

Proof: Taking into account the definition of $\hat{A}\,(\theta(t))$, $\hat{B}\,(\theta(t))$ and $\hat{S}\,(\theta(t))$ in (27)-(30), the parameter-dependent LMI (25) is equivalent to:

$$\sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i(\hat{\theta})\alpha_j(\hat{\theta}) \begin{bmatrix} R_{00} \otimes \hat{X} + (R_{10} \otimes (\hat{A}_i\hat{X} + \hat{B}_j\hat{S}_i))^H & Z^* \otimes (\hat{A}_i\hat{X} + \hat{B}_j\hat{S}_i)^* \\ Z \otimes (\hat{A}_i\hat{X} + \hat{B}_j\hat{S}_i) & -I_d \otimes \hat{X} \end{bmatrix} \prec 0 \tag{34}$$

which corresponds to the problem of verifying the negativity of a double polytopic sum. By applying Polya's theorem on definite quadratic forms (Sala & Ariño, 2007), (33) is obtained. □

As discussed by Sala and Ariño (2007), the sufficient conditions obtained through the application of Polya's theorem become progressively less conservative when s increases, and actually exact, i.e. necessary and sufficient, for a finite value of s.

3.3. Pole clustering in a union of regions for LPV systems

Assume now that the region of interest $\mathcal{D}$ is obtained as follows:

$$\mathcal{D} = \bigcup_{k=1}^{q} \mathcal{D}_{R_k} \tag{35}$$

where each subregion $\mathcal{D}_{R_k}$ is a $\mathcal{D}_R$-region defined in (4). Then, the pole clustering in the region $\mathcal{D}$ can be performed by successive partial pole clustering in the subregions $\mathcal{D}_{R_k}$, for $k = 1, ..., q$. This can be achieved according to the following algorithm.

Algorithm

- **Step 1:** Consider the state matrix of the LPV system (11):

$$A(\theta(t)) = \sum_{n=1}^{w} \mu_n(\theta(t)) A_n \tag{36}$$

- **Step 2:** Let $k = 0$ and $A_0(\theta(t)) = A(\theta(t))$.
- **Step 3:** Let $k = k + 1$.
- **Step 4:** Compute $V_{k-1}(\theta(t))$ and $\Lambda_{k-1}(\theta(t))$ such that $A_{k-1}(\theta(t))V_{k-1}(\theta(t)) = V_{k-1}(\theta(t))\Lambda_{k-1}(\theta(t))$; rearrange $\Lambda_{k-1}(\theta(t))$ and $V_{k-1}(\theta(t))$ in the form:

$$\Lambda_{k-1}(\theta(t)) = \begin{bmatrix} \Lambda^1_{k-1}(\theta(t)) & 0 \\ 0 & \Lambda^2_{k-1}(\theta(t)) \end{bmatrix}$$

  where $\Lambda^1_{k-1}(\theta(t))$ contains the p_k eigenvalues to be shifted to $\mathcal{D}_{R_k}$; then calculate $C_{k-1}(\theta(t))$ as:

$$C_{k-1}(\theta(t)) = \begin{bmatrix} I_{p_k} & 0 \end{bmatrix} V_{k-1}(\theta(t))^{-1}$$

- **Step 5:** Compute:

$$\hat{A}_{k-1}(\theta(t)) = C_{k-1}(\theta(t))A_{k-1}(\theta(t))C_{k-1}(\theta(t))^{+} \tag{37}$$

$$\hat{B}_{k-1}(\theta(t)) = C_{k-1}(\theta(t))B_{k-1}(\theta(t)) \tag{38}$$

- **Step 6:** Obtain a polytopic representation for $\hat{A}_{k-1}(\theta(t))$ and $\hat{B}_{k-1}(\theta(t))$ in terms of dependence on a new scheduling vector $\hat{\theta}_{k-1}$:

$$\hat{A}_{k-1}(\theta(t)) = \hat{A}_{k-1}(\hat{\theta}_{k-1}(t)) = \sum_{i=1}^{r} \alpha_{k-1,i}(\hat{\theta}_{k-1}(t))\hat{A}_{k-1,i} \tag{39}$$

$$\hat{B}_{k-1}(\theta(t)) = \hat{B}_{k-1}(\hat{\theta}_{k-1}(t)) = \sum_{i=1}^{r} \alpha_{k-1,i}(\hat{\theta}_{k-1}(t))\hat{B}_{k-1,i} \tag{40}$$

- **Step 7:** Find matrices $\hat{X}_k > 0$ and $\hat{S}_k(\hat{\theta}_{k-1}(t))$ such that Theorem 1/Corollary 1 holds, with $\hat{X} = \hat{X}_k$, $\hat{S}(\theta_{k-1}(t)) = \hat{S}_k(\hat{\theta}_{k-1}(t))$, $\hat{A}(\theta_{k-1}(t)) = \hat{A}_{k-1}(\hat{\theta}_{k-1}(t))$, $\hat{B}(\theta_{k-1}(t)) = \hat{B}_{k-1}(\hat{\theta}_{k-1}(t))$ and $R = R_k$. Then, calculate $\hat{K}_k(\hat{\theta}_{k-1}(t)) = \hat{S}_k(\hat{\theta}_{k-1}(t))\hat{X}_k^{-1}$
- **Step 8:** Once $\hat{K}_k(\hat{\theta}_{k-1}(t))$ has been obtained, compute the state-feedback gain at step k as:

$$K_k(\theta(t), \hat{\theta}_{k-1}(t)) = \hat{K}_k(\hat{\theta}_{k-1}(t))C_{k-1}(\theta(t))$$

  and the closed-loop matrix at step k as:

$$A_k(\theta(t), \hat{\theta}_0(t), .., \hat{\theta}_{k-1}(t)) = A_{k-1}(\theta(t), \hat{\theta}_0(t), .., \hat{\theta}_{k-2}(t)) + B(\theta(t))K_k(\theta(t), \hat{\theta}_{k-1}(t))$$

- **Step 9:** If $k \neq q$ then go to **Step 3**
- **Step 10:** Compute the final state-feedback gain as:

$$K(\theta(t), \hat{\theta}_0(t), ..., \hat{\theta}_{q-1}(t)) = \sum_{k=1}^{q} K_k(\theta(t), \hat{\theta}_{k-1}(t)) \quad (41)$$

  that achieves the desired pole clustering in the specified union of regions.
- **Step 11:** Stop

4. Shifting pole clustering

4.1. $\mathcal{D}_R(\theta(t))$-stability for LPV systems

In this section, a shifting pole clustering approach for LPV system is introduced, which is inspired the ideas in Rotondo et al. (2013). Unlike the approach discussed in Section 3, hereafter we consider the case in which the LMI region is scheduled by the varying parameter, which means that the considered region is some $\mathcal{D}(\theta(t))$, obtained through subsets of the complex plane $\mathcal{D}_R(\theta(t))$ defined according to:

$$\mathcal{D}_R(\theta(t)) = \{s \in \mathbb{C} : R_{00}(\theta(t)) + (R_{10}(\theta(t))s)^H + R_{11}(\theta(t))s^* s \prec 0\} \quad (42)$$

where $R(\theta(t))$ is a $2d \times 2d$ Hermitian matrix given by:

$$R(\theta(t)) = \begin{bmatrix} R_{00}(\theta(t)) & R_{10}(\theta(t)) \\ R_{10}(\theta(t))^* & R_{11}(\theta(t)) \end{bmatrix} \in \mathbb{C}^{2d \times 2d}, \theta \in \Theta \quad (43)$$

If the $\mathcal{D}_R(\theta(t))$-stability of the LPV system (11) is considered, then the inequality (23) becomes:

$$R_{00}(\theta(t)) \otimes P + (R_{10}(\theta(t)) \otimes (PA(\theta(t))))^H + R_{11}(\theta(t)) \otimes (A(\theta(t))^* PA(\theta(t))) \prec 0 \quad (44)$$

and for partial pole placement in $\mathcal{D}_R(\theta(t))$, the aggregated state-feedback gain $\hat{K}(\theta(t))$ has to be chosen in such way that it satisfies:

$$\begin{aligned} &R_{00}(\theta(t)) \otimes \hat{P} + (R_{10}(\theta(t)) \otimes (\hat{P}(\hat{A}(\theta(t)) + \hat{B}(\theta(t))\hat{K}(\theta(t)))))^H + ... \\ &+ R_{11}(\theta(t)) \otimes ((\hat{A}(\theta(t)) + \hat{B}(\theta(t))\hat{K}(\theta(t)))^* \hat{P}(\hat{A}(\theta(t)) + \hat{B}(\theta(t))\hat{K}(\theta(t)))) \prec 0 \end{aligned} \quad (45)$$

for some matrix $\hat{P} \succ 0$.

The following theorem provides the parameter-dependent LMIs required to achieve partial pole placement in the shifting $\mathcal{D}_R(\theta)$-region.

Theorem 2: There exists an aggregated state-feedback gain $\hat{K}(\theta)$ that assigns p eigenvalues in the region $\mathcal{D}_R(\theta)$ defined by (42) if there exist a matrix $\hat{X} \succ 0$ and a matrix $\hat{S}(\theta)$ such that $\forall \theta \in \Theta$:

$$\begin{bmatrix} R_{00}(\theta) \otimes \hat{X} + (R_{10}(\theta) \otimes (\hat{A}(\theta)\hat{X} + \hat{B}(\theta)\hat{S}(\theta)))^H & Z^*(\theta) \otimes (\hat{A}(\theta)\hat{X} + \hat{B}(\theta)\hat{S}(\theta))^* \\ Z(\theta) \otimes (\hat{A}(\theta)\hat{X} + \hat{B}(\theta)\hat{S}(\theta)) & -I_d \otimes \hat{X} \end{bmatrix} \prec 0 \quad (46)$$

where $Z(\theta)$ is given by the parameter-varying Cholesky factorization $R_{11}(\theta) = Z^*(\theta)Z(\theta)$, then the aggregated feedback gain is given by (26).

Proof: This theorem is a modification of Theorem 1, in which the matrices describing the $\mathcal{D}_R$-region are allowed to vary according to the varying parameter $\theta(t)$. $\square$

It is necessary to mention that due to the varying nature of $R(\theta(t))$, there would appear triple summations due to terms such as $(R_{10}(\theta(t)) \otimes (\hat{P}(\hat{A}(\theta(t)) + \hat{B}(\theta(t)) \hat{K}(\theta(t)))))^H$. These terms can be handled using Polya's theorem, as done in Corollary 1 for the case of double summations, obtaining sufficient conditions that become progressively less conservative and eventually necessary. The details are omitted. Note that, by looking at (5)-(6), it is possible to constrain $R_{10}(\theta)$ and $R_{11}(\theta)$ to be constant, and still obtain parameter-varying vertical half-planes or disks with a fixed center but a parameter-varying radius, which can be used to change online the closed-loop performance while reducing triple summations into double summations, thus simplifying the application of Polya's theorem.

4.2. Shifting pole clustering in a union of $\mathcal{D}_R(\theta(t))$-regions for LPV system

Assume the region of interest $\mathcal{D}(\theta(t))$ to be defined as follows:

$$\mathcal{D}(\theta(t)) = \bigcup_{k=1}^{q} \mathcal{D}_{R_k}(\theta(t)) \tag{47}$$

where each subregion $\mathcal{D}_{R_k}(\theta(t))$ is a region defined as in (42). Then, the shifting pole clustering in $\mathcal{D}(\theta(t))$ can be performed by successive partial pole placement in the subregions $\mathcal{D}_{R_k}(\theta(t))$, for $k = 1, ..., q$, where $R_k(\theta(t))$ must be constrained to vary polytopically in order to obtain a finite number of conditions which can be solved computationally:

$$R_k(\theta(t)) = \sum_{n=1}^{w} \mu_n(\theta(t)) R_{k,n} \tag{48}$$

The shifting pole clustering in a union of $\mathcal{D}_R(\theta(t))$-regions is achieved through the algorithm introduced in Section 3.3, the main differences are the following:

- At **Step 1**, additionally consider the varying of the regions as (48).
- In **Step 6**, a new polytopic representation for $R_k(\theta(t))$ in terms of the new scheduling vector $\hat{\theta}_{k-1}$:

$$R_k(\theta(t)) = R_k(\hat{\theta}_{k-1}(t)) = \sum_{i=1}^{r} \alpha_{k-1,i}(\hat{\theta}_{k-1}(t)) R_{k,i} \tag{49}$$

  must be sought. An approximate solution can be found by considering the actual values of $R_k(\theta(t))$ for several values of $\theta(t$ as if they were known measurements, while the matrices $R_{k,i}$ are unknown parameters to be estimated using estimation techniques, e.g., *least-squares*. Further details about this point are provided in connection with the example in Section 5.2.
- At **Step 7**, the LMI (46) should hold, with $R(\theta(t))$ replaced by $R_k(\theta(t))$.

5. Numerical example

5.1. $\mathcal{D}_R$-stabilization in a union of $\mathcal{D}_R$-regions

Let us consider a polytopic LPV system as in (12) with matrices given by:

$$A_1 = \begin{bmatrix} 2.05 & -15.93 & 15.32 \\ 40.47 & -326.56 & 298.17 \\ 44.86 & -359.16 & 327.50 \end{bmatrix} \quad A_2 = \begin{bmatrix} 15.86 & -44.81 & 41.76 \\ 53.22 & -361.51 & 335.84 \\ 56.30 & -388.59 & 360.65 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$$

where $A(\theta(t))$ depends on the varying parameters $\theta(t) \in [0, 1]$ as follows:

$$A(\theta(t)) = \mu_1(\theta(t))A_1 + \mu_2(\theta(t))A_2 \tag{50}$$

with:

$$\mu_1(\theta(t)) = \theta(t) \quad \mu_2(\theta(t)) = 1 - \theta(t)$$

The system is clearly open-loop unstable since its frozen poles are all located in the right half-plane, as shown in Fig. 1.

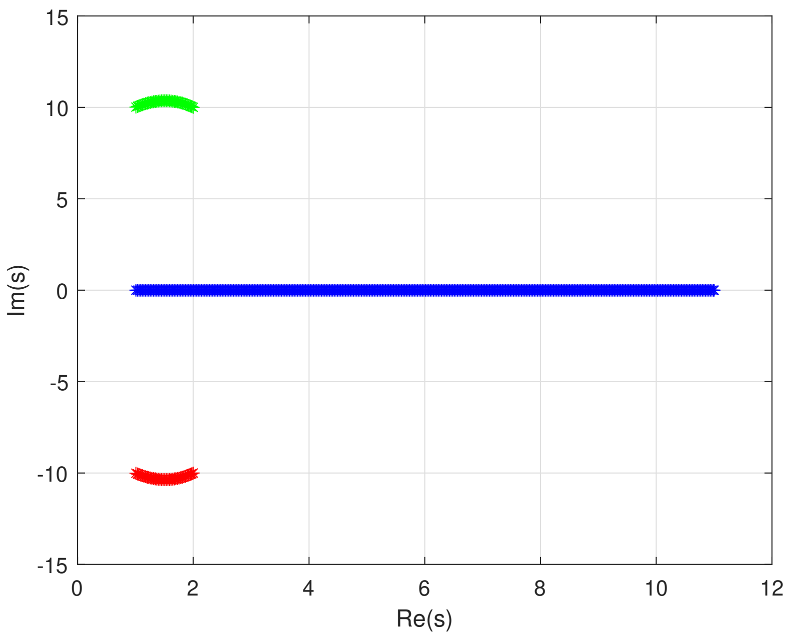
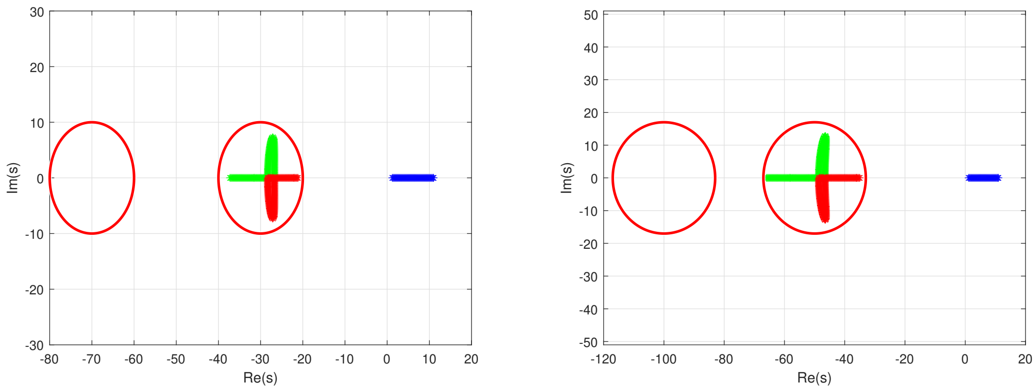


Figure 1. Position of the open-loop frozen poles for $\theta \in [0, 1]$.

Pole clustering is performed as follows: at iteration $k = 1$, the two conjugate poles are moved to $\mathcal{D}_{R_1}$; then, at iteration $k = 2$, the remaining open-loop unstable real pole is moved to $\mathcal{D}_{R_2}$. The mentioned regions $\mathcal{D}_{R_1}$ and $\mathcal{D}_{R_2}$ are selected as disks with predetermined center and radius, as listed in Table 1.

Table 1. Parameter chosen for controller design.

	Center of $\mathcal{D}_{R_1}$	Radius of $\mathcal{D}_{R_1}$	Center of $\mathcal{D}_{R_2}$	Radius of $\mathcal{D}_{R_2}$
$\mathcal{D}_1$	$(-30, 0)$	10	$(-70, 0)$	10
$\mathcal{D}_2$	$(-50, 0)$	17	$(-100, 0)$	17
$\mathcal{D}_3$	$(-50, 0)$	17	$(-150, 0)$	17



(a) Poles position in $\mathcal{D}_1$ at iteration $k = 1$

(b) Poles position in $\mathcal{D}_2$ at iteration $k = 1$

Figure 2. Position of the closed-loop frozen poles after the first iteration $k = 1$.

The design is performed using two different sets of regions, with $\mathcal{D}_1$ located closer to the imaginary axis (hence, corresponding to a slower time response), whereas $\mathcal{D}_2$ is located farther (thus corresponding to a faster response). Note that Table 1 provides also another set of regions, denoted as $\mathcal{D}_3$, which will be considered later, in the next subsection, for the sake of illustrating the shifting pole clustering technique.

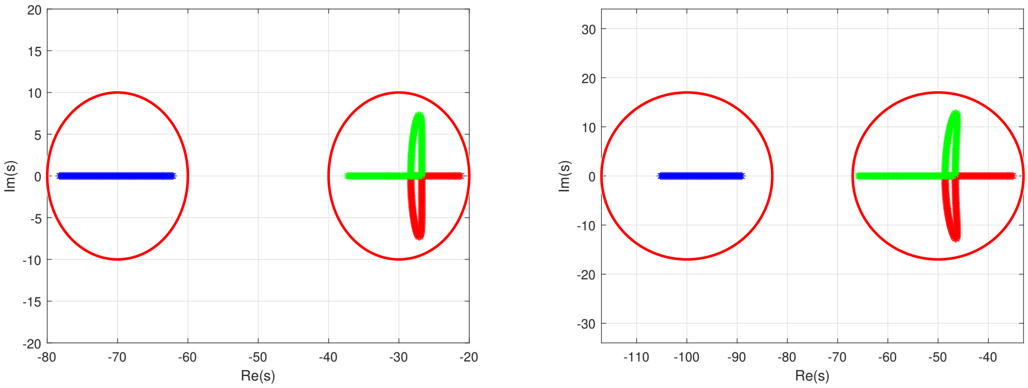
The controller design is performed by applying Algorithm 1. It is worth mentioning that, in Step 6 of the algorithm, after computing $\hat{A}_{k-1}$ and $\hat{B}_{k-1}$, the new scheduling vector $\hat{\theta}_{k-1}$ has been obtained by choosing the elements of $\hat{A}_{k-1}$ and $\hat{B}_{k-1}$ as varying parameters, and the corresponding polytopic representation with matrices $\hat{A}_{k-1,i}$ and $\hat{B}_{k-1,i}$ has been obtained by applying the approach commonly referred to in the literature as *bounding box*, i.e. by considering different combinations of the maximum and minimum values of each element of $\hat{A}_{k-1}$ and $\hat{B}_{k-1}$. The interested reader will find the values of $\hat{A}_{k-1,i}$, $\hat{B}_{k-1,i}$, together with the $\hat{K}_{k,i}$ obtained in step 7 of the algorithm, reported in the Appendix.

Fig. 2 shows that after the iteration $k = 1$ of the algorithm, the two conjugate poles (in the frozen parameter-varying sense) are moved into the first region $\mathcal{D}_{R_1}$, while the third parameter-varying pole is kept at the original location. Fig. 3 shows that at iteration $k = 2$ of the algorithm, the third pole is eventually moved to the second region $\mathcal{D}_{R_2}$.

In order to validate the obtained control law, let us perform simulations starting from the initial condition:

$$x(0) = \begin{bmatrix} 5 & -5 & 5 \end{bmatrix}^T$$

Fig. 4 shows the closed-loop response by applying the designed state-feedback con-



(a) Poles position in $\mathcal{D}_1$ at iteration $k = 2$

(b) Poles position in $\mathcal{D}_2$ at iteration $k = 2$

Figure 3. Position of the closed-loop frozen poles after the second iteration $k = 2$.

trol law. It is clear that if the dominant poles are farther from the imaginary axis, the system converges to zero faster, i.e. the convergence of the states with chosen region $\mathcal{D}_2$ is faster than with region $\mathcal{D}_1$, as it would be predicted by LTI considerations about the pole location.

5.2. Shifting $\mathcal{D}_R(\theta(t))$-stabilization in a union of parameter-varying $\mathcal{D}_R$-regions

In this section, the previously described numerical system is considered for the implementation of the shifting approach described in Section 4. In order to keep the mathematical complexity simple, we consider that only the radius of the disks of interest varies depending on the value of the scheduling parameter $\theta(t)$, which means that:

$$R_k(\theta(t)) = \begin{bmatrix} c_1^2 + c_2^2 - r_k(\theta(t))^2 & -c_1 + c_2 i \\ -c_1 - c_2 i & 1 \end{bmatrix}$$

with $r_k(\theta(t))$ defined as:

$$r_1(\theta(t)) = (1 + 1.2\theta(t))r_{1,0} \tag{51}$$

$$r_2(\theta(t)) = (1 + 3\theta(t))r_{2,0} \tag{52}$$

where $r_{k,0}$ denotes the values of the radii denoted as $\mathcal{D}_3$ in Table 1.

It is worth mentioning that, after obtaining the polytopic matrices $\hat{A}_{k-1,i}$ and $\hat{B}_{k-1,i}$ at step 6 of the algorithm, in terms of the new varying parameter $\hat{\theta}_{k-1}$, it is necessary to describe the variation of the regions of interest $R_k(\theta(t))$ in terms of $\hat{\theta}_{k-1}(t)$ as well as possible. Although an exact solution to this problem, in general, does not exist, an approximate solution can be found by fitting the following equation:

$$r_k(\theta(t))^2 = \sum_{i=1}^{r} \alpha_{k-1,i}(\hat{\theta}_{k-1}(t)) \cdot \beta_{k-1,i}(\hat{\theta}_{k-1}(t)) + \beta_{k-1,0} \tag{53}$$

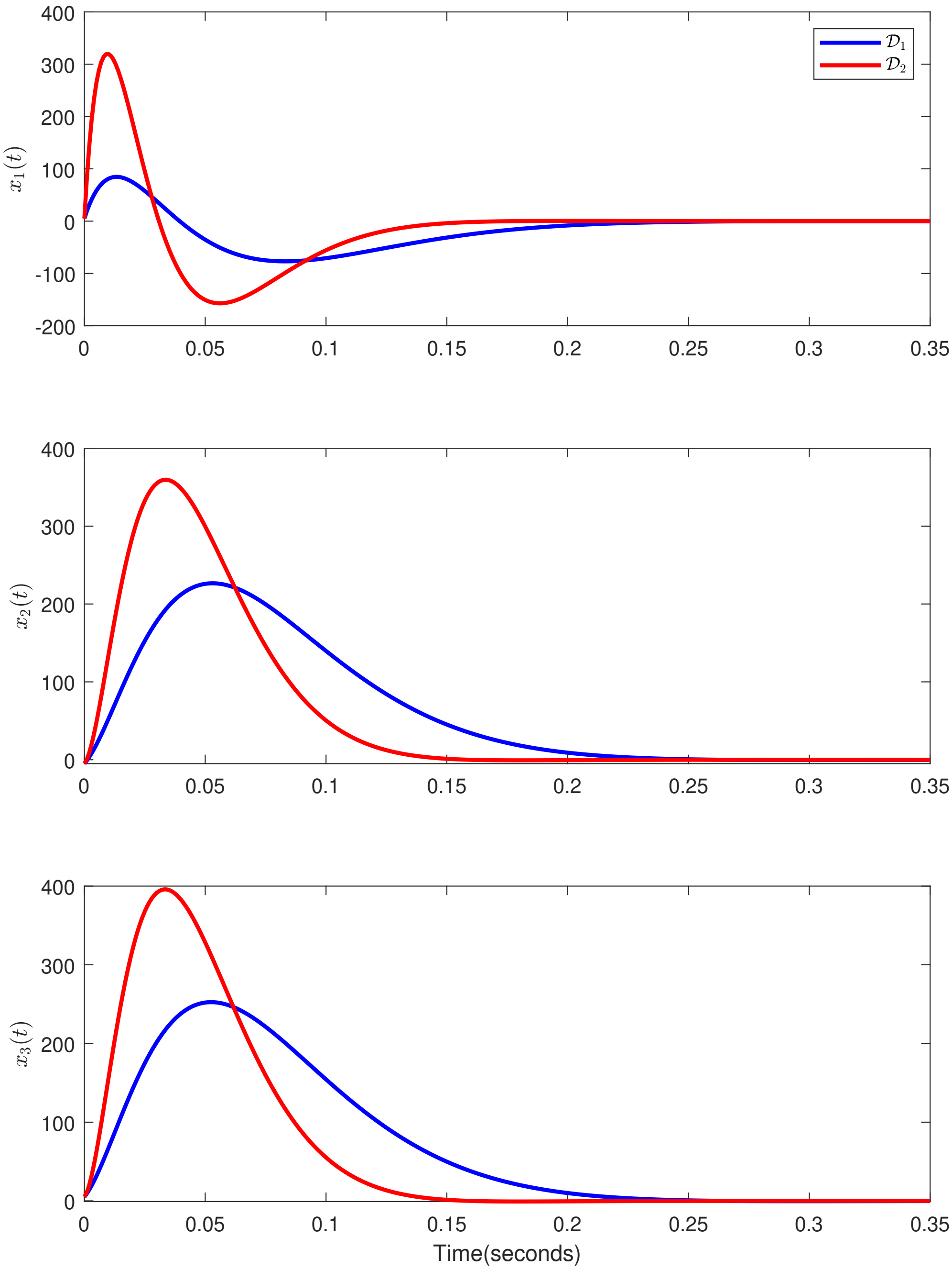
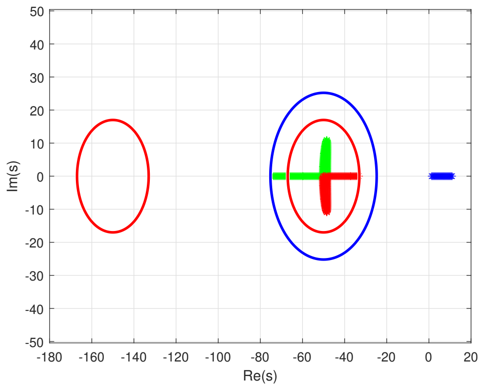


Figure 4. The closed-loop response by pole clustering in $\mathcal{D}_1$ (blue) and $\mathcal{D}_2$ (red).

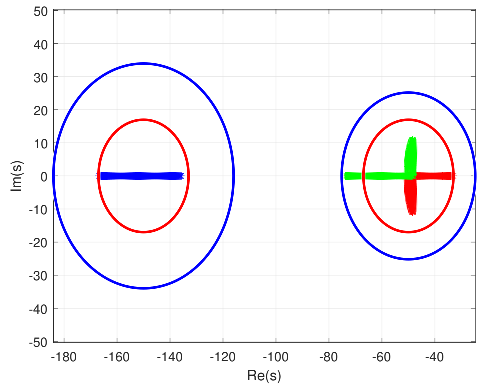
in the least-squares sense, using as input data $(\alpha_{k-1,i}(\hat{\theta}_{k-1}(t)); r_k(\theta(t))^2)$ where $\alpha_{k-1,i}(\hat{\theta}(t))$ are the known coefficients of the polytopic decomposition obtained beforehand and $r_k(\theta(t))^2$ is the radius of the circular region $R_k(\theta(t))$. The coefficient vector $\beta_{k-1} = [\ \beta_{k-1,0} \quad \beta_{k-1,1} \quad \cdots \quad \beta_{k-1,r}\ ]^T$ is then computed as:

$$\beta_{k-1} = \begin{bmatrix} I & \alpha_{k-1} \end{bmatrix}^+ r_k(\theta(t))^2 \tag{54}$$

Fig. 5 shows the frozen-parameter poles position after shifting pole clustering in $\mathcal{D}_3$, where the circles in red correspond to $R_k(\theta(t))$ when $\theta = 0$ while those in blue to $R_k(\theta(t))$ when $\theta = 1$. Moreover, Fig.6 shows that the desired regions and the actual positions of the closed-loop poles vary according to the value of θ.



(a) Poles position in $\mathcal{D}_3$ at iteration $k = 1$



(b) Poles position in $\mathcal{D}_3$ at iteration $k = 2$

Figure 5. Position of the closed-loop frozen poles using the shifting pole clustering approach.

Fig.7 shows that the behavior of the closed-loop state variables depends upon the trajectory of θ. These responses have been obtained starting from the initial state $x(0) = \begin{bmatrix} -5.6 & 5.4 & 6.1 \end{bmatrix}^T$ in four different cases, three of which corresponding to constant values of the scheduling parameter $\theta = 0.2$, $\theta = 0.6$ and $\theta = 1$ (red, green and blue line, respectively), and the remaining case corresponding to a time-varying scheduling parameter trajectory as follows: $\theta(t) = 0.6 - 0.4\cos(5\pi t)$ (purple line). It can be seen from the figure that the closed-loop system behaves as expected, in the sense that for the value of θ which corresponds to poles located farther from the imaginary axis, a faster dynamics of the closed-loop system is obtained. The purple curve shows that at the beginning of the simulation, the system behaves in an underdamped way due to the value of the varying parameter being approximately equal to $\theta = 0.2$ (exactly equal at $t = 0$); then, as the value of $\theta(t)$ increases with time, the closed-loop frozen pole get closer to the real axis, which increases the damping and causes the oscillations to fade away.

6. Conclusions

This paper has provided a procedure for pole clustering in a union of $\mathcal{D}_R$-regions for LPV systems. From this technique, a method to compute a state-feedback gain that achieves the desired closed-loop pole location has been deduced. Furthermore,

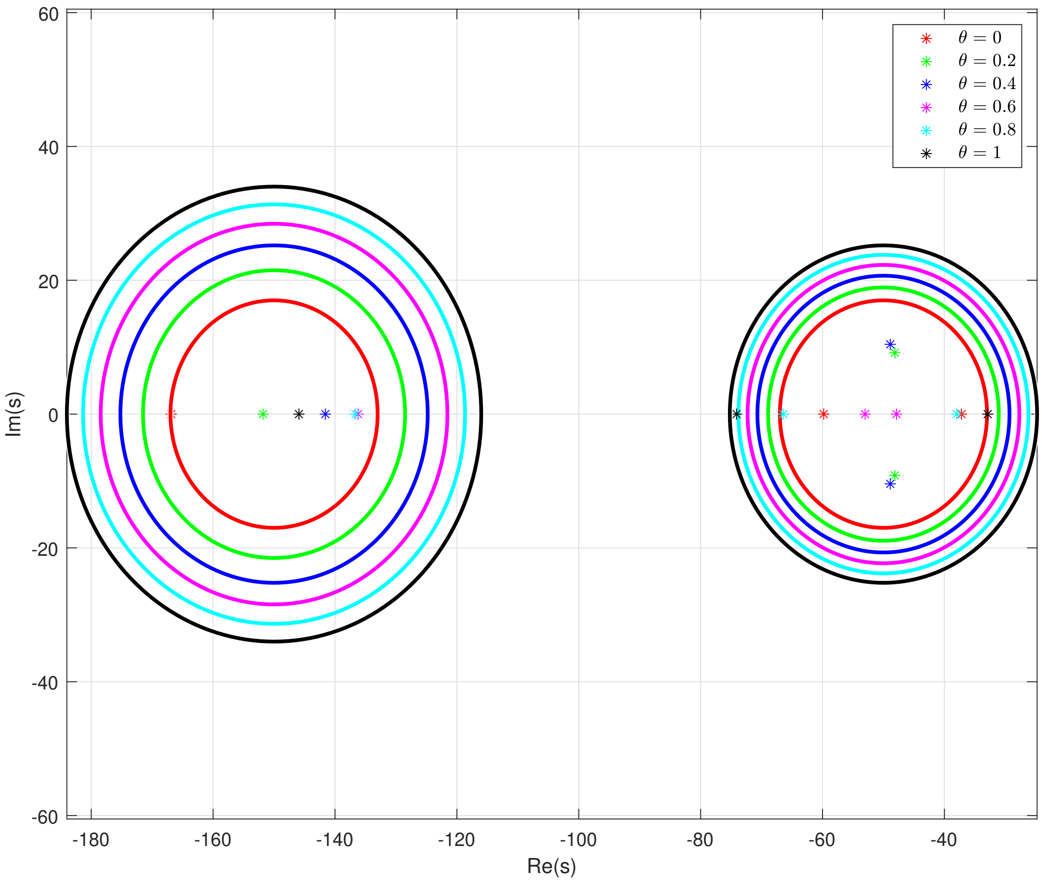


Figure 6. Position of the closed-loop poles for different values of θ using the shifting approach.

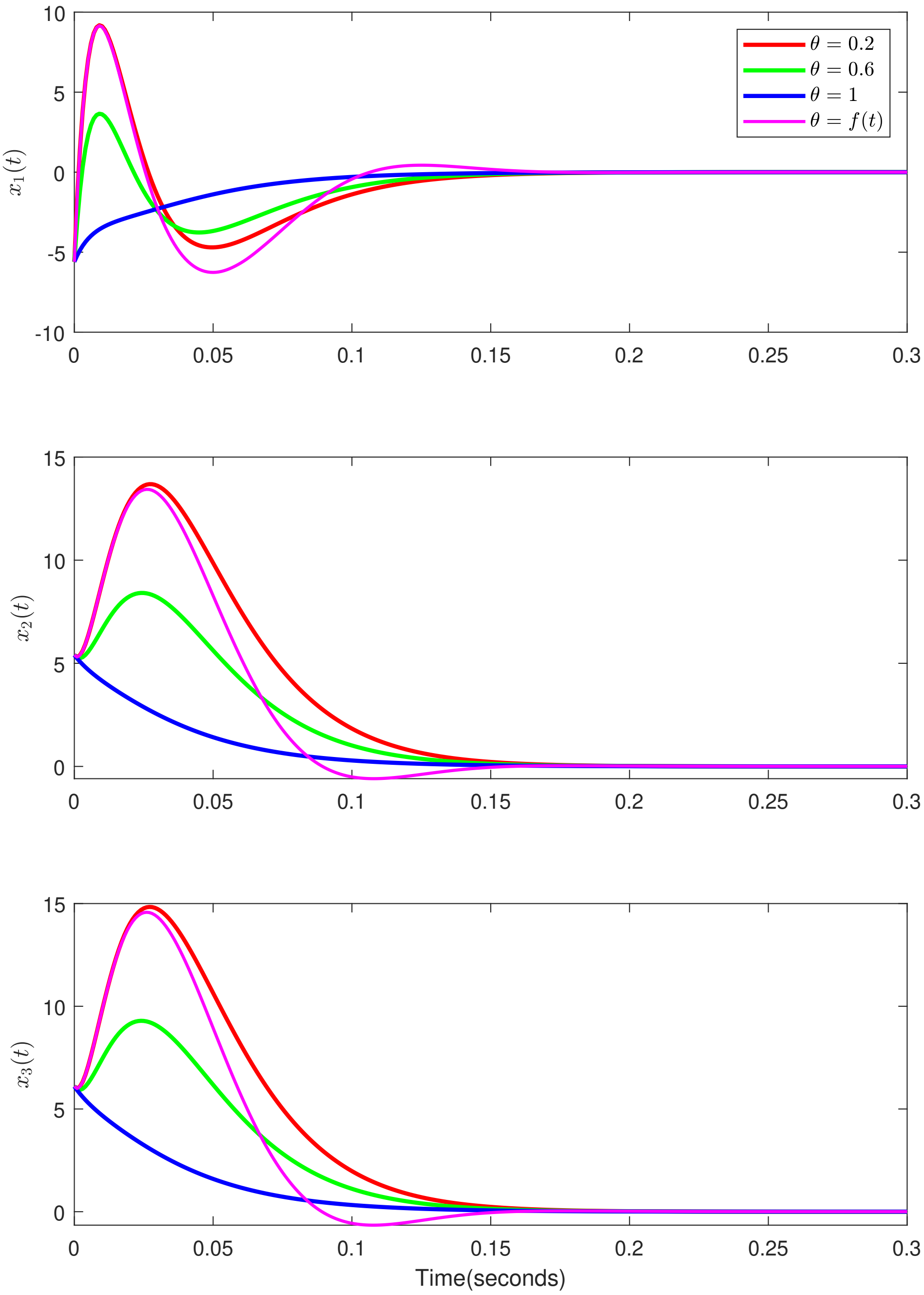


Figure 7. System response for different values/trajectories of $\theta(t)$ using the shifting approach.

this technique has been extended to the shifting case, thus allowing pole clustering in parameter-dependent varying regions which enables online modification of the transient performance. The proposed technique has been validated using simulations on a numerical system, demonstrating the main characteristics and the effectiveness of the developed control strategy. Future work will be devoted to developing equivalent techniques for LPV systems with residual nonlinearities, such as parameter-varying Lipschitz or quadratic terms.

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Appendix A. Numerical values for pole clustering in $\mathcal{D}_1$

The matrices $\hat{A}_{k-1,i}$ and $\hat{B}_{k-1,i}$ in (39)-(40) during iteration of Algorithm 1 $k = 1$ are given as follows:

$$\hat{A}_{0,1} = \begin{bmatrix} 2 & -10 \\ 10.36 & 2 \end{bmatrix} \quad \hat{A}_{0,2} = \begin{bmatrix} 2 & -10 \\ 10.36 & 1 \end{bmatrix} \quad \hat{A}_{0,3} = \begin{bmatrix} 2 & -10 \\ 10 & 2 \end{bmatrix}$$
$$\hat{A}_{0,4} = \begin{bmatrix} 2 & -10 \\ 10 & 1 \end{bmatrix} \quad \hat{A}_{0,5} = \begin{bmatrix} 2 & -10.36 \\ 10.36 & 2 \end{bmatrix} \quad \hat{A}_{0,6} = \begin{bmatrix} 2 & -10.36 \\ 10.36 & 1 \end{bmatrix}$$
$$\hat{A}_{0,7} = \begin{bmatrix} 2 & -10.36 \\ 10 & 2 \end{bmatrix} \quad \hat{A}_{0,8} = \begin{bmatrix} 2 & -10.36 \\ 10 & 1 \end{bmatrix} \quad \hat{A}_{0,9} = \begin{bmatrix} 1 & -10 \\ 10.36 & 2 \end{bmatrix}$$
$$\hat{A}_{0,10} = \begin{bmatrix} 1 & -10 \\ 10.36 & 1 \end{bmatrix} \quad \hat{A}_{0,11} = \begin{bmatrix} 1 & -10 \\ 10 & 2 \end{bmatrix} \quad \hat{A}_{0,12} = \begin{bmatrix} 1 & -10 \\ 10 & 1 \end{bmatrix}$$
$$\hat{A}_{0,13} = \begin{bmatrix} 1 & -10.36 \\ 10.36 & 2 \end{bmatrix} \quad \hat{A}_{0,14} = \begin{bmatrix} 1 & -10.36 \\ 10.36 & 1 \end{bmatrix} \quad \hat{A}_{0,15} = \begin{bmatrix} 1 & -10.36 \\ 10 & 2 \end{bmatrix}$$
$$\hat{A}_{0,16} = \begin{bmatrix} 1 & -10.36 \\ 10 & 1 \end{bmatrix}$$

$$\hat{B}_{0,1} = \begin{bmatrix} -0.43 \\ -0.10 \end{bmatrix} \quad \hat{B}_{0,2} = \begin{bmatrix} -0.45 \\ -0.11 \end{bmatrix} \quad \hat{B}_{0,3} = \begin{bmatrix} -0.43 \\ -0.11 \end{bmatrix} \quad \hat{B}_{0,4} = \begin{bmatrix} -0.45 \\ -0.10 \end{bmatrix}$$

then, the obtained controller gains are $\hat{K}_{k,i}$ are:

$$\hat{K}_{1,1} = \begin{bmatrix} 91.68 & 192.69 \end{bmatrix} \quad \hat{K}_{1,2} = \begin{bmatrix} 91.97 & 187.00 \end{bmatrix} \quad \hat{K}_{1,3} = \begin{bmatrix} 88.38 & 187.32 \end{bmatrix}$$
$$\hat{K}_{1,4} = \begin{bmatrix} 89.11 & 185.33 \end{bmatrix} \quad \hat{K}_{1,5} = \begin{bmatrix} 91.44 & 191.78 \end{bmatrix} \quad \hat{K}_{1,6} = \begin{bmatrix} 91.78 & 186.36 \end{bmatrix}$$
$$\hat{K}_{1,7} = \begin{bmatrix} 88.18 & 185.52 \end{bmatrix} \quad \hat{K}_{1,8} = \begin{bmatrix} 89.03 & 184.72 \end{bmatrix} \quad \hat{K}_{1,9} = \begin{bmatrix} 90.88 & 193.00 \end{bmatrix}$$
$$\hat{K}_{1,10} = \begin{bmatrix} 90.87 & 186.08 \end{bmatrix} \quad \hat{K}_{1,11} = \begin{bmatrix} 88.27 & 191.49 \end{bmatrix} \quad \hat{K}_{1,12} = \begin{bmatrix} 88.50 & 186.24 \end{bmatrix}$$
$$\hat{K}_{1,13} = \begin{bmatrix} 91.03 & 192.73 \end{bmatrix} \quad \hat{K}_{1,14} = \begin{bmatrix} 90.89 & 185.82 \end{bmatrix} \quad \hat{K}_{1,15} = \begin{bmatrix} 87.92 & 190.19 \end{bmatrix}$$
$$\hat{K}_{1,16} = \begin{bmatrix} 88.58 & 186.09 \end{bmatrix}$$

During the iteration $k = 2$, the matrices $\hat{A}_{k-1,i}$, $\hat{B}_{k-1,i}$ and $\hat{K}_{k,i}$ are obtained as follows:

$$\hat{A}_{1,1} = 11 \quad \hat{A}_{1,2} = 1 \quad \hat{B}_{1,1} = 0.18 \quad \hat{B}_{1,2} = 0.16$$

$$\hat{K}_{2,1} = -405.98 \quad \hat{K}_{2,2} = -343.60$$

Appendix B. Numerical information of pole placement in $\mathcal{D}_2$

When the region of interest is the one denoted as $\mathcal{D}_2$ in Table 1, at iteration $k = 1$ of Algorithm 1, the matrices $\hat{A}_{k-1,i}$ and $\hat{B}_{k-1,i}$ have the same values as the ones previously shown for the case of $\mathcal{D}_1$. Then, the obtained controller gains $\hat{K}_{k,i}$ are as follows:

$$\hat{K}_{1,1} = \begin{bmatrix}102.96 & 549.39\end{bmatrix} \quad \hat{K}_{1,2} = \begin{bmatrix}102.32 & 539.21\end{bmatrix} \quad \hat{K}_{1,3} = \begin{bmatrix}99.73 & 539.80\end{bmatrix}$$
$$\hat{K}_{1,4} = \begin{bmatrix}99.44 & 533.60\end{bmatrix} \quad \hat{K}_{1,5} = \begin{bmatrix}102.48 & 550.38\end{bmatrix} \quad \hat{K}_{1,6} = \begin{bmatrix}102.98 & 538.35\end{bmatrix}$$
$$\hat{K}_{1,7} = \begin{bmatrix}100.40 & 541.86\end{bmatrix} \quad \hat{K}_{1,8} = \begin{bmatrix}98.61 & 531.05\end{bmatrix} \quad \hat{K}_{1,9} = \begin{bmatrix}103.57 & 550.09\end{bmatrix}$$
$$\hat{K}_{1,10} = \begin{bmatrix}104.04 & 536.49\end{bmatrix} \quad \hat{K}_{1,11} = \begin{bmatrix}100.49 & 547.46\end{bmatrix} \quad \hat{K}_{1,12} = \begin{bmatrix}99.79 & 537.14\end{bmatrix}$$
$$\hat{K}_{1,13} = \begin{bmatrix}104.01 & 548.72\end{bmatrix} \quad \hat{K}_{1,14} = \begin{bmatrix}104.29 & 535.83\end{bmatrix} \quad \hat{K}_{1,15} = \begin{bmatrix}99.43 & 547.46\end{bmatrix}$$
$$\hat{K}_{1,16} = \begin{bmatrix}99.24 & 536.97\end{bmatrix}$$

At iteration $k = 2$, the matrices $\hat{A}_{k-1,i}$ and $\hat{B}_{k-1,i}$ are obtained as follows:

$$\hat{A}_{1,1} = 11 \quad \hat{A}_{1,2} = 1 \quad \hat{B}_{1,1} = 0.08 \quad \hat{B}_{1,2} = 0.06$$

and the controller gains $\hat{K}_{k,i}$ are:

$$\hat{K}_{2,1} = -1611.80 \quad \hat{K}_{2,2} = -1465.50$$