A payoff dynamics model for generalized Nash equilibrium seeking in population games ⋆

Juan Martinez-Piazuelo [a], Nicanor Quijano [b], Carlos Ocampo-Martinez [a]

[a] *Automatic Control Department, Universitat Politècnica de Catalunya, Llorens i Artigas, 4-6, 08028, Barcelona, Spain*

[b] *Departamento de Ingeniería Eléctrica y Electrónica, Universidad de los Andes, Carrera 1 No. 18A-10, Bogotá, Colombia*

Abstract

This paper studies the problem of generalized Nash equilibrium seeking in population games under general affine equality and convex inequality constraints. In particular, we design a novel payoff dynamics model to steer the decision-making agents to a generalized Nash equilibrium of the underlying game, i.e., to a self-enforceable state where the constraints are satisfied and no agent has incentives to unilaterally deviate from her selected strategy. Moreover, using Lyapunov stability theory, we provide sufficient conditions to guarantee the asymptotic stability of the corresponding equilibria set in stable population games. Auxiliary results characterizing the properties of the equilibria set are also provided for general continuous population games. Furthermore, our theoretical developments are numerically validated on a Cournot game considering various market-related and production-related constraints.

Key words: Game theory; Nonlinear models; Evolutionary dynamics models; Payoff dynamics models.

1 Introduction

Population games provide an evolutionary game theoretical framework to study the strategic interaction of a large number of decision-making agents (Hofbauer & Sigmund 1998, Sandholm 2010). The conventional framework consists of two elements: i) a game, which describes the strategic environment where agents interact; and ii) a revision protocol, which provides the strategy selection mechanism that agents use to adapt to the environment. Together, the game and revision protocol define a stochastic evolutionary process for the strategic distribution of agents. Based on such a framework, various works have studied the conditions under which the pair game and revision protocol lead to a Nash equilibrium, i.e., to a self-enforceable state where no agent has incentives to unilaterally change her selected strategy. For instance, Hofbauer & Sandholm (2009) study the so-called class of stable games, and show that the set of Nash equilibria of such games is asymptotically stable under several types of revision protocols; Barreiro-Gomez, Obando & Quijano (2017) provide conditions for the asymptotic stability of Nash equilibria in non well-mixed populations of agents; and Como et al. (2021) study the convergence to Nash equilibria for potential games under imitative revision protocols and community-based structures of interaction. In addition, Park et al. (2019) and Arcak & Martins (2021) study the scenario where the strategies' payoffs (which are usually given as input to the agents' revision protocols) are dynamically provided by a so-called payoff dynamics model (PDM), and characterize the corresponding conditions for convergence to a Nash equilibrium.

On the other hand, besides the theoretical motivation, the formalism of population games has been exploited in several practical applications as well (Quijano et al. 2017). Some examples include wireless networks (Tem-

⋆ Juan Martinez-Piazuelo gratefully acknowledges the Universitat Politècnica de Catalunya and Banco Santander for the financial support of his predoctoral grant FPI-UPC. In addition, the authors would like to thank the project PID2020-115905RB-C21 (L-BEST) funded by MCIN/ AEI /10.13039/501100011033 for supporting this research. Finally, Nicanor Quijano would like to thank the UK-PACT Project (2019-2021), Systemic perspectives on low-carbon cities in Colombia – An integrated urban modelling approach for policy and regulatory analysis. This paper was not presented at any IFAC meeting. Corresponding author Juan Martinez-Piazuelo.

Email addresses: `juan.pablo.martinez.piazuelo@upc.edu` (Juan Martinez-Piazuelo), `nquijano@uniandes.edu.co` (Nicanor Quijano), `carlos.ocampo@upc.edu` (Carlos Ocampo-Martinez).

bine et al. 2010), water distribution systems (Pashaie et al. 2017, Barreiro-Gomez, Ocampo-Martinez & Quijano 2017), demand response problems (Srikantha & Kundur 2017), and electric vehicles (Martinez-Piazuelo et al. 2020), among others. Hence, the study of population games is not only relevant from a game theoretical perspective, but also to the scope of control systems engineering.

Although the problem of Nash equilibrium seeking in population games has been well-studied in the literature, and several practical applications have benefit from it, the broader scenario of generalized Nash equilibrium (GNE) seeking in population games has received limited attention. In the context of population games, a GNE seeking problem refers to the task of reaching a refined Nash equilibrium where joint feasibility constraints define the allowed strategic distributions for the agents. More precisely, a GNE is a self-enforceable state where certain constraints are satisfied and no agent has incentives to unilaterally deviate from her selected strategy. As such, a framework capable of reaching a GNE in not only more motivating from the theoretical perspective, but also enjoys of a richer scope of applications to be explored. Consequently, in this paper we investigate the problem of GNE seeking in population games.

Contributions: Inspired by the ideas on dynamic payoff mechanisms in Park et al. (2019), in this paper we design a novel PDM for GNE seeking in population games under affine equality constraints and (twice continuously differentiable) convex inequality constraints. By considering the class of stable games (Hofbauer & Sandholm 2009) and the family of (locally Lipschitz) impartial pairwise comparison revision protocols (Park et al. 2019), we provide sufficient conditions to guarantee the asymptotic stability of the set of generalized Nash equilibria of the underlying game. Moreover, as an auxiliary result, we prove the non-emptiness and compactness of the equilibria set of the corresponding evolutionary process for any continuous game. Finally, to illustrate the relevance of the developed framework, we provide some numerical examples regarding a Cournot game, which is an abstraction that captures a wide family of practical applications. To the best of our knowledge, this is the first work on GNE seeking in population games at the aforementioned level of generality.

Related work: The problem of GNE seeking in classical multiplayer games has been recently studied from different perspectives. For instance, Tatarenko & Kamgarpour (2019) study the problem of reaching a GNE in potential games with coupled convex inequality constraints; Yi & Pavel (2019*a*) and Yi & Pavel (2019*b*) study the problem of GNE seeking in monotone games under coupled affine constraints; Deng (2021) investigates the GNE seeking problem for nonsmooth aggregative games under coupled affine equality constraints; and Belgioioso et al. (2021) propose a distributed decision-making algorithm for GNE seeking in aggregative games with non-linear aggregation terms and under coupled affine equality constraints.

Although the GNE seeking problem has been studied from various perspectives, limited attention has been given to such a problem in the aforementioned context of population games. Some recent approaches that shed light onto this topic are the ones in Barreiro-Gomez et al. (2016), Barreiro-Gomez & Tembine (2018), and Martinez-Piazuelo et al. (2022). Namely, Barreiro-Gomez et al. (2016) introduce the concept of mass dynamics to consider affine constraints in stable population games under certain imitative revision protocol; Barreiro-Gomez & Tembine (2018) propose some novel revision protocols to consider affine box inequality constraints in stable games; and Martinez-Piazuelo et al. (2022) propose some dynamic payoff mechanism for GNE seeking in potential population games with affine equality constraints and under impartial pairwise comparison revision protocols.

In contrast with the aforementioned previous works, in this paper we consider the problem of GNE seeking in population games under fairly general constraints. More precisely, under affine equality constraints and twice continuously differentiable convex inequality constraints. Clearly, this is a significant contribution with respect to the other population games approaches in Barreiro-Gomez et al. (2016), Barreiro-Gomez & Tembine (2018), and Martinez-Piazuelo et al. (2022), which only consider affine constraints. Our approach is close in nature to the one in Martinez-Piazuelo et al. (2022). However, in contrast with such an approach, we provide complete sufficient conditions for asymptotic stability in continuously differentiable stable games (not only for potential games). Furthermore, the PDM proposed in this paper contemplates the PDM in Martinez-Piazuelo et al. (2022) as a particular case. Thus, the framework of this paper fully generalizes such previous results.

The remainder of this paper is organized as follows. Section 2 introduces the population games framework as well as the evolutionary process that describes the evolution of the strategic distribution of agents over time. Then, Section 3 formally states the GNE seeking problem that is studied throughout the paper. Afterwards, Section 4 presents the proposed dynamic payoff mechanism that is designed for the GNE seeking task, and Section 5 provides our main theoretical results regarding the proposed framework. Finally, Section 6 presents an illustrative example to validate our results, and Section 7 provides some concluding remarks and future directions of research. Additionally, all the proofs are given in Section 8 at the end of the paper.

Notations: Throughout, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathbb{R}^n_{\geq 0}$ and $\mathbb{R}^n_{>0}$ denote the non-negative and positive orthants of $\mathbb{R}^n$, respectively. $\mathbb{Z}_{\geq z}$ is the set of integers greater than or equal to $z \in \mathbb{Z}_{\geq 1}$. Let $\operatorname{col}(\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_N)$ denote the stacked column vector obtained from the collection of column vectors $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_N$. Similarly, $\operatorname{diag}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_N)$ denotes the block diagonal matrix with the matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_N$ in its main diagonal.

2 Population games and evolutionary dynamics

In this section, we introduce some preliminary concepts on population games and evolutionary dynamics (Sandholm 2010).

2.1 Population games

Consider a society of agents partitioned in $N \in \mathbb{Z}_{\geq 1}$ disjoint populations indexed by the set $\mathcal{P} = \{1, 2, \ldots, N\}$. Each population $k \in \mathcal{P}$ is comprised of a large number of strategic agents, and the set of strategies available to the agents of each population $k \in \mathcal{P}$ is $\mathcal{S}^k = \{1, 2, \ldots, n^k\}$, with $n^k \in \mathbb{Z}_{\geq 2}$. Throughout, we let $x_i^k \in \mathbb{R}_{\geq 0}$ be the portion of agents playing the strategy $i \in \mathcal{S}^k$ at population $k \in \mathcal{P}$, for all $i \in \mathcal{S}^k$ and all $k \in \mathcal{P}$. Hence, the strategic distribution of population $k \in \mathcal{P}$ is given by $\mathbf{x}^k = \operatorname{col}\left(x_1^k, x_2^k, \ldots, x_{n^k}^k\right) \in \mathbb{R}_{\geq 0}^{n^k}$, for all $k \in \mathcal{P}$, and the strategic distribution of the entire society is given by $\mathbf{x} = \operatorname{col}\left(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\right) \in \mathbb{R}_{\geq 0}^n$, with $n = \sum_{k \in \mathcal{P}} n^k$. Moreover, the amount of players of population $k \in \mathcal{P}$ is represented by a (constant) mass $m^k \in \mathbb{R}_{>0}$, for all $k \in \mathcal{P}$. Consequently, the set of all possible strategic distributions at population $k \in \mathcal{P}$ is

$$\Delta^k = \left\{ \mathbf{x}^k \in \mathbb{R}_{\geq 0}^{n^k} : \sum_{i \in \mathcal{S}^k} x_i^k = m^k \right\}, \quad \forall k \in \mathcal{P},$$

and the set of all possible strategic distributions at the society level is

$$\Delta = \left\{ \mathbf{x} \in \mathbb{R}_{\geq 0}^n : \begin{array}{c} \mathbf{x} = \operatorname{col}\left(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\right), \\ \text{where } \mathbf{x}^k \in \Delta^k, \forall k \in \mathcal{P} \end{array} \right\}.$$

Under the considered framework, there is a fitness function associated to each strategy at each population. Namely, $f_i^k : \Delta \to \mathbb{R}$ denotes the fitness function of strategy $i \in \mathcal{S}^k$ at population $k \in \mathcal{P}$. For convenience, we stack all the fitness functions in a fitness vector $\mathbf{f} : \Delta \to \mathbb{R}^n$ as $\mathbf{f}(\cdot) = \operatorname{col}\left(\mathbf{f}^1(\cdot), \mathbf{f}^2(\cdot), \ldots, \mathbf{f}^N(\cdot)\right)$, with $\mathbf{f}^k(\cdot) = \operatorname{col}\left(f_1^k(\cdot), f_2^k(\cdot), \ldots, f_{n^k}^k(\cdot)\right) \in \mathbb{R}^{n^k}$, for all $k \in \mathcal{P}$.

In summary, a population game is comprised of a set of populations ($\mathcal{P}$), a set of possible strategic distributions for each population (Δ^k), and a fitness vector ($\mathbf{f}(\cdot)$), i.e., in normal form a population game could be defined as the tuple $\left(\mathcal{P}, \{\Delta^k\}_{k \in \mathcal{P}}, \mathbf{f}(\cdot)\right)$. However, as in Sandholm (2010), throughout this paper we simply use $\mathbf{f}(\cdot)$ to refer to the population game. Moreover, in this paper we pay special attention to the class of stable population games.

Definition 1 $\mathbf{f} : \Delta \to \mathbb{R}^n$ *is a stable game if and only if* $(\mathbf{x} - \mathbf{y})^\top (\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})) \leq 0$, *for all* $\mathbf{x}, \mathbf{y} \in \Delta$. □

2.2 Evolutionary dynamics

Let $t \in \mathbb{R}_{\geq 0}$ denote the continuous-time index, and let $\mathbf{x}(t)$ be the value of $\mathbf{x}$ at time t. Moreover, let $p_i^k(t) \in \mathbb{R}$ be the payoff associated to the strategy $i \in \mathcal{S}^k$ at time t, for all $i \in \mathcal{S}^k$ and all $k \in \mathcal{P}$, and let $\mathbf{p}(t) = \operatorname{col}\left(\mathbf{p}^1(t), \mathbf{p}^2(t), \ldots, \mathbf{p}^N(t)\right) \in \mathbb{R}^n$, where $\mathbf{p}^k(t) = \operatorname{col}\left(p_1^k(t), p_2^k(t), \ldots, p_{n^k}^k(t)\right) \in \mathbb{R}^{n^k}$, for all $k \in \mathcal{P}$. Namely, the payoff vector $\mathbf{p}(t)$ provides the payoff to all society agents at time t. In classical population games, it is typically set that $\mathbf{p}(t) = \mathbf{f}(\mathbf{x}(t))$. However, as in this paper, this is not the only possible choice for $\mathbf{p}(t)$. We provide our choice for $\mathbf{p}(t)$ in Section 4.

Under the considered framework, the (microscopic) decision making process of the society agents is performed as follows. Each society agent is equipped with a stochastic alarm clock and a revision protocol. The clocks provide strategic revision opportunities to the corresponding agents according to a rate R exponential distribution, and the revision protocols are maps of the form $\rho_{ij}^k : \Delta \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, for all $i, j \in \mathcal{S}^k$ and all $k \in \mathcal{P}$, which provide the conditional switch rate from strategy i to strategy j. At the microscopic level, if at time t an agent playing $i \in \mathcal{S}^k$ receives a revision opportunity, then such an agent either switches to strategy $j \in \mathcal{S}^k \setminus \{i\}$ with probability $\rho_{ij}^k(\mathbf{x}(t), \mathbf{p}(t))/R$, or remains at strategy i with probability $1 - \sum_{j \in \mathcal{S}^k \setminus \{i\}} \rho_{ij}^k(\mathbf{x}(t), \mathbf{p}(t))/R$ (as in (Sandholm 2010, Section 4.1), it is assumed that R is large enough so that the aforementioned probabilities are well defined for all times). Furthermore, following the ideas in (Sandholm 2010, Section 4.2) and given that the number of agents of each population is large, the (macroscopic) dynamics that describe the continuous-time evolution of $x_i^k(t)$ are

$$\dot{x}_i^k(t) = \sum_{j \in \mathcal{S}^k} x_j^k(t) \rho_{ji}^k(\mathbf{x}(t), \mathbf{p}(t)) - x_i^k(t) \rho_{ij}^k(\mathbf{x}(t), \mathbf{p}(t)),$$

for all $i \in \mathcal{S}^k$ and all $k \in \mathcal{P}$. In particular, in this paper we focus on the class of impartial pairwise comparison (IPC) revision protocols (Park et al. 2019).

Definition 2 *A revision protocol* $\rho_{ij}^k : \Delta \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *is an IPC protocol if it has the form* $\rho_{ij}^k(\mathbf{x}(t), \mathbf{p}(t)) = \phi_j^k\left(p_j^k(t) - p_i^k(t)\right)$, *where* $\phi_j^k : \mathbb{R} \to \mathbb{R}_{\geq 0}$ *satisfies that* $\phi_j^k\left(p_j^k(t) - p_i^k(t)\right) > 0$, *if* $p_j^k(t) > p_i^k(t)$, *and* $\phi_j^k\left(p_j^k(t) - p_i^k(t)\right) = 0$, *if* $p_j^k(t) \leq p_i^k(t)$. □

Standing Assumption 1 *For every $i, j \in \mathcal{S}^k$ and every $k \in \mathcal{P}$, the revision protocol $\rho_{ij}^k : \Delta \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an IPC protocol characterized by a locally Lipschitz continuous map $\phi_j^k : \mathbb{R} \to \mathbb{R}_{\geq 0}$.* □

An example of a map satisfying Standing Assumption 1 is $\phi_j^k(\cdot) = \max(\cdot, 0)$, for all $j \in \mathcal{S}^k$ and all $k \in \mathcal{P}$.

Consequently, the evolutionary dynamics model (EDM) considered throughout this paper is

$$\delta_{ij}^k(t) \triangleq p_j^k(t) - p_i^k(t), \tag{1a}$$

$$\dot{x}_i^k(t) = \sum_{j \in \mathcal{S}^k} x_j^k(t)\phi_i^k\left(\delta_{ji}^k(t)\right) - x_i^k(t)\phi_j^k\left(\delta_{ij}^k(t)\right), \tag{1b}$$

for all $i, j \in \mathcal{S}^k$ and all $k \in \mathcal{P}$.

Some well-known properties of the EDM in (1) are characterized in Lemmas 1 and 2.

Lemma 1 *Consider the EDM in* (1)*. If $\mathbf{x}(0) \in \Delta$, then $\mathbf{x}(t) \in \Delta$ for all $t \geq 0$.* □

Lemma 2 *Consider an arbitrary $k \in \mathcal{P}$, the EDM in* (1)*, and let $\dot{\mathbf{x}}^k(t) = \operatorname{col}\left(\dot{x}_1^k(t), \dot{x}_2^k(t), \ldots, \dot{x}_{n^k}^k(t)\right) \in \mathbb{R}^{n^k}$. Then, $\dot{\mathbf{x}}^k(t) = \mathbf{0}$ if and only if it holds that*

$$x_i^k(t) > 0 \Rightarrow p_i^k(t) = \max_{j \in \mathcal{S}^k} p_j^k(t), \quad \forall i \in \mathcal{S}^k. \tag{2}$$

Here, $\mathbf{0}$ is the zero vector of appropriate dimension. □

In particular, Lemma 1 states some invariance properties of the EDM in (1), while Lemma 2 characterizes the equilibria set of the EDM in (1) in terms of the payoff vector $\mathbf{p}(t)$. Based on Lemma 1, we consider the following assumption.

Standing Assumption 2 $\mathbf{x}(0) \in \Delta$. □

Hence, without additional loss of generality, it holds that $\mathbf{x}(t) \in \Delta$, for all $t \geq 0$. The reader should keep this fact in mind for the forthcoming discussions.

3 Problem Statement

In this section, we formally state the problem that is studied throughout the paper.

In classical population games scenarios (with $\mathbf{p}(t) = \mathbf{f}(\mathbf{x}(t))$), the objective is for the society agents to reach a Nash equilibrium of the game $\mathbf{f}(\cdot)$.

Definition 3 *The set of Nash equilibria of the population game $\mathbf{f} : \Delta \to \mathbb{R}^n$ is given by*

$$\operatorname{NE}(\mathbf{f}) = \left\{\mathbf{x} \in \Delta : \mathbf{x} \in \arg\max_{\mathbf{y} \in \Delta} \mathbf{y}^\top \mathbf{f}(\mathbf{x})\right\}.$$

That is, a Nash equilibrium is the best response to itself, where no agent has incentives to unilaterally deviate from her selected strategy. □

As such, the problem of Nash equilibrium seeking in population games has been widely studied in the literature. Notice that for the case when $\mathbf{p}(t) = \mathbf{f}(\mathbf{x}(t))$, the equilibria set of the EDM in (1) coincides with the set of Nash equilibria of the game $\mathbf{f}(\cdot)$ (c.f., Lemma 2 and Definition 3). Hence, under such a scenario, the problem of Nash equilibrium seeking reduces to the stability analysis of the equilibria set of the EDM in (1), which is readily available in Hofbauer & Sandholm (2009) for certain population games.

In contrast with the classical framework, however, in this paper we investigate a generalization of the Nash equilibrium seeking problem in population games. Namely, in this paper the goal is for the society agents to reach a generalized Nash equilibrium of the game $\mathbf{f}(\cdot)$.

Definition 4 *Let $\mathcal{X} \subseteq \mathbb{R}^n$. The set of generalized Nash equilibria of the population game $\mathbf{f} : \Delta \to \mathbb{R}^n$ with respect to $\mathcal{X}$ is given by*

$$\operatorname{GNE}(\mathbf{f}, \mathcal{X}) = \left\{\mathbf{x} \in \Delta \cap \mathcal{X} : \mathbf{x} \in \arg\max_{\mathbf{y} \in \Delta \cap \mathcal{X}} \mathbf{y}^\top \mathbf{f}(\mathbf{x})\right\}.$$

That is, $\mathcal{X}$ represents some constraints to be satisfied by the strategic distribution of the society, i.e., $\mathbf{x}$. □

For the forthcoming analyses, we impose the following conditions on the set $\mathcal{X}$.

Standing Assumption 3 *Let*

$$\mathcal{X} = \left\{\mathbf{x} \in \mathbb{R}^n : \begin{array}{l} h_l(\mathbf{x}) = 0, \ \forall l \in \mathcal{C}_= \\ g_q(\mathbf{x}) \leq 0, \ \forall q \in \mathcal{C}_\leq \end{array}\right\},$$

where $\mathcal{C}_= = \{1, 2, \ldots, C_=\}$, with $C_= \in \mathbb{Z}_{\geq 0}$; $\mathcal{C}_\leq = \{1, 2, \ldots, C_\leq\}$, with $C_\leq \in \mathbb{Z}_{\geq 0}$; $h_l : \mathbb{R}_{\geq 0}^n \to \mathbb{R}$, for all $l \in \mathcal{C}_=$; and $g_q : \mathbb{R}_{\geq 0}^n \to \mathbb{R}$, for all $q \in \mathcal{C}_\leq$. Moreover, the following conditions hold:

(a) *The set $\operatorname{int}(\Delta) \cap \operatorname{int}(\mathcal{X})$ is nonempty. Here, $\operatorname{int}(\Delta) = \Delta \cap \mathbb{R}_{>0}^n$, and*

$$\operatorname{int}(\mathcal{X}) = \{\mathbf{x} \in \mathcal{X} : g_q(\mathbf{x}) < 0, \forall q \in \mathcal{C}_\leq\}.$$

(b) *For all $l \in \mathcal{C}_=$, $h_l(\mathbf{x}) = \mathbf{a}_l^\top \mathbf{x} - b_l$, with $\mathbf{a}_l \in \mathbb{R}^n$ and $b_l \in \mathbb{R}$. Besides, the matrix $\hat{\mathbf{A}} = \left[\mathbf{A}^\top, \mathbf{A}_\Delta^\top\right]^\top \in \mathbb{R}^{(C_= + N) \times n}$ is full row rank (hence, $C_= \leq n - N$). Here, $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{C_=}]^\top \in \mathbb{R}^{C_= \times n}$, and $\mathbf{A}_\Delta \in \mathbb{R}^{N \times n}$ is such that the Δ-related equality constraints $\sum_{i \in \mathcal{S}^k} x_i^k = m^k$, for all $k \in \mathcal{P}$, can be written as $\mathbf{A}_\Delta \mathbf{x} = \mathbf{m}$, with $\mathbf{m} = \operatorname{col}\left(m^1, m^2, \ldots, m^N\right) \in \mathbb{R}_{>0}^N$.*

(c) *For all $q \in \mathcal{C}_{\leq}$, the function $g_q(\cdot)$ is twice continuously differentiable and convex.* □

Remark 1 *Under Standing Assumption 3, it follows that the set $\Delta \cap \mathcal{X}$ is nonempty, convex, and compact. Consequently, as we show in Lemma 6 in Section 5, if $\mathbf{f}(\cdot)$ is continuous, then $\mathrm{GNE}(\mathbf{f}, \mathcal{X})$ is nonempty and compact. Moreover, note that if $\mathcal{X} \supseteq \Delta$, then $\mathrm{GNE}(\mathbf{f}, \mathcal{X}) = \mathrm{NE}(\mathbf{f})$. Hence, the problem of GNE seeking clearly generalizes the problem of NE seeking.* □

The question to be answered then is: how to design a control mechanism to steer the society agents to a GNE of the underlying population game? Observe that according to the considered EDM of Section 2.2, the only influence that one might have over the society agents is through the payoff signal $\mathbf{p}(t)$. Clearly, setting $\mathbf{p}(t) = \mathbf{f}(\mathbf{x}(t))$ is in general not enough for GNE seeking as the society agents would have no information regarding the constraints in $\mathcal{X}$. Therefore, it is necessary to design an appropriate payoff signal $\mathbf{p}(t)$ that incorporates both the game $\mathbf{f}(\cdot)$ and the constraints of $\mathcal{X}$. Next, we present our proposed payoff signal and provide the corresponding theoretical analyses.

4 Proposed approach

As mentioned above, under the considered framework, the only control mechanism to steer the decision-making process of the society agents is through the payoff signal $\mathbf{p}(t)$. Hence, in this section we propose a payoff signal $\mathbf{p}(t)$ that effectively guides the society agents to a GNE of the underlying population game $\mathbf{f}(\cdot)$.

Following the ideas in Park et al. (2019), we let $\mathbf{p}(t)$ be the output of a so-called payoff dynamics model (PDM). More formally, here we propose the PDM given by

$$p_i^k(t) = f_i^k(\mathbf{x}(t)) - \sum_{l \in \mathcal{C}_=} \mu_l(t) \frac{\partial h_l(\mathbf{x}(t))}{\partial x_i^k} - \sum_{q \in \mathcal{C}_{\leq}} \lambda_q(t) \frac{\partial g_q(\mathbf{x}(t))}{\partial x_i^k}, \tag{3}$$

for all $i \in \mathcal{S}^k$ and all $k \in \mathcal{P}$, where $\mu_l(t), \lambda_q(t) \in \mathbb{R}$ are determined by

$$\dot{\mu}_l(t) = \vartheta_l^+(h_l(\mathbf{x}(t))) - \vartheta_l^-(-h_l(\mathbf{x}(t))), \tag{4a}$$

$$\dot{\lambda}_q(t) = \theta_q^+(g_q(\mathbf{x}(t))) - \lambda_q(t)\theta_q^-(-g_q(\mathbf{x}(t))), \tag{4b}$$

for all $l \in \mathcal{C}_=$ and all $q \in \mathcal{C}_{\leq}$. Here, for all $l \in \mathcal{C}_=$ and all $q \in \mathcal{C}_{\leq}$, the functions $\vartheta_l^+ : \mathbb{R} \to \mathbb{R}_{\geq 0}$, $\vartheta_l^- : \mathbb{R} \to \mathbb{R}_{\geq 0}$, $\theta_q^+ : \mathbb{R} \to \mathbb{R}_{\geq 0}$, and $\theta_q^- : \mathbb{R} \to \mathbb{R}_{\geq 0}$, are locally Lipschitz continuous maps satisfying that $z(\alpha) > 0$, if $\alpha > 0$, and $z(\alpha) = 0$, if $\alpha \leq 0$, for all $\alpha \in \mathbb{R}$ and all $z(\cdot) \in \{\vartheta_l^+(\cdot), \vartheta_l^-(\cdot), \theta_q^+(\cdot), \theta_q^-(\cdot)\}$. A simple example of such a map is $z(\cdot) = \tau \max(\cdot, 0)$, where $\tau \in \mathbb{R}_{>0}$ is a positive constant. In (4), the super index $^+$ is used to denote that $\vartheta_l^+(\cdot)$ and $\theta_q^+(\cdot)$ control the growth rate of $\mu_l(t)$ and $\lambda_q(t)$, respectively, while the super index $^-$ is used to highlight that $\vartheta_l^-(\cdot)$ and $\theta_q^-(\cdot)$ control the decay rate of $\mu_l(t)$ and $\lambda_q(t)$, respectively. Moreover, notice that (4a) and (4b) are not symmetric (i.e., while $\dot{\lambda}_q(t)$ depends on $\lambda_q(t)$, $\dot{\mu}_l(t)$ does not depend on $\mu_l(t)$). In fact, the form of (4b) leads to the following invariance property.

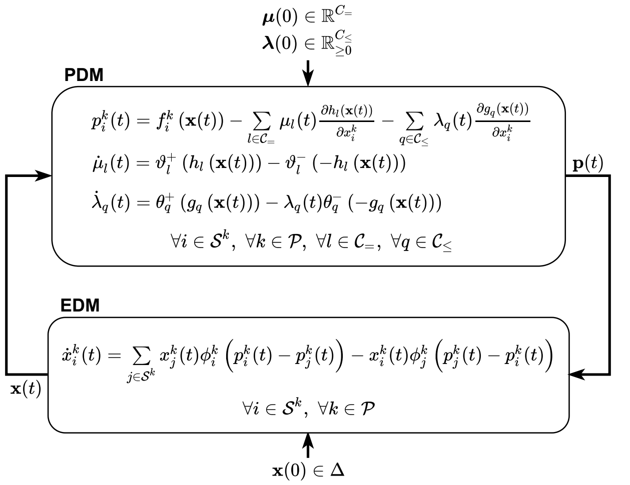

Fig. 1. Considered EDM-PDM system.

Lemma 3 *Consider the dynamics in (4b) and an arbitrary $q \in \mathcal{C}_{\leq}$. If $\lambda_q(0) \geq 0$, then $\lambda_q(t) \geq 0$ for all $t \geq 0$.* □

Based on Lemma 3, we impose the following assumption.

Standing Assumption 4 *Let $\boldsymbol{\mu}(0) \in \mathbb{R}^{C_=}$ and $\boldsymbol{\lambda}(0) \in \mathbb{R}_{\geq 0}^{C_{\leq}}$, where $\boldsymbol{\mu}(t) = \mathrm{col}(\mu_1(t), \mu_2(t), \ldots, \mu_{C_=}(t))$ and $\boldsymbol{\lambda}(t) = \mathrm{col}\left(\lambda_1(t), \lambda_2(t), \ldots, \lambda_{C_{\leq}}(t)\right)$.* □

Hence, without additional loss of generality, through this paper it holds that $\boldsymbol{\lambda}(t) \in \mathbb{R}_{\geq 0}^{C_{\leq}}$, for all $t \geq 0$.

The PDM defined in (3)-(4) provides a causal map from $\mathbf{x}(t)$ to $\mathbf{p}(t)$, and considers both the underlying game $\mathbf{f}(\cdot)$ and the constraints of $\mathcal{X}$. In fact, observe that the EDM in (1) and the PDM in (3)-(4) are interconnected in a positive feedback loop structure as in Fig. 1. That is, based on the society state $\mathbf{x}(t)$, the PDM determines the payoff signal $\mathbf{p}(t)$ and forwards it as an input to the EDM. In Section 5, we formally prove that, for certain population games, such an interconnected EDM-PDM system has an asymptotically stable equilibria set, which coincides with the set $\mathrm{GNE}(\mathbf{f}, \mathcal{X})$. Therefore, the considered EDM-PDM system effectively solves the GNE seeking problem of Section 3.

5 Analysis of the EDM-PDM system

In this section, we provide our main theoretical developments regarding the EDM-PDM system of Section 4. In particular, we characterize the equilibria set of the considered EDM-PDM system and prove the coincidence with $\mathrm{GNE}(\mathbf{f}, \mathcal{X})$ for continuous population games. Moreover, we provide sufficient conditions on the game $\mathbf{f}(\cdot)$ to guarantee the asymptotic stability of the equilibria set of the considered EDM-PDM system. Throughout, whenever we refer to the state vector of the EDM-PDM system of (1) and (3)-(4), we use the tuple notation $(\mathbf{x}(t), \boldsymbol{\mu}(t), \boldsymbol{\lambda}(t)) \in \mathbb{R}^n \times \mathbb{R}^{C_=} \times \mathbb{R}^{C_\leq}$. Besides, due to Lemmas 1 and 3, and Standing Assumptions 2 and 4, it is further considered that $(\mathbf{x}(t), \boldsymbol{\mu}(t), \boldsymbol{\lambda}(t)) \in \Delta \times \mathbb{R}^{C_=} \times \mathbb{R}^{C_\leq}_{\geq 0}$, for all $t \geq 0$.

To characterize the equilibria set of the EDM-PDM system of (1) and (3)-(4), we provide the following auxiliary results.

Lemma 4 *Consider the dynamics in* (4a) *and an arbitrary $l \in \mathcal{C}_=$. It holds that $\dot{\mu}_l(t) = 0$ if and only if $h_l(\mathbf{x}(t)) = 0$.* □

Lemma 5 *Consider the dynamics in* (4b) *and an arbitrary $q \in \mathcal{C}_\leq$. It holds that $\dot{\lambda}_q(t) = 0$ if and only if $g_q(\mathbf{x}(t)) \leq 0$ and $\lambda_q(t) g_q(\mathbf{x}(t)) = 0$.* □

Based on Lemmas 2, 4, and 5, we now characterize the equilibria set of the overall EDM-PDM system.

Theorem 1 *Consider the EDM in* (1) *in conjunction with the PDM in* (3)-(4)*, and let*

$$\mathcal{E} = \left\{ (\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) : \begin{array}{c} \mathbf{x} \in \Delta \cap \mathcal{X},\ \boldsymbol{\mu} \in \mathbb{R}^{C_=},\ \boldsymbol{\lambda} \in \mathbb{R}^{C_\leq}_{\geq 0}, \\ (2) \text{ holds for all } k \in \mathcal{P}, \text{ and} \\ \lambda_q g_q(\mathbf{x}) = 0 \text{ for all } q \in \mathcal{C}_\leq. \end{array} \right\}. \tag{5}$$

Then, $(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*)$ is an equilibirum state of the EDM-PDM system if and only if $(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \in \mathcal{E}$. □

Theorem 1 characterizes the equilibria set of the considered EDM-PDM system. By imposing some conditions on the game $\mathbf{f}(\cdot)$, it is possible to prove certain properties of the sets $\mathrm{GNE}(\mathbf{f}, \mathcal{X})$ and $\mathcal{E}$.

Lemma 6 *Let $\mathbf{f} : \Delta \to \mathbb{R}^n$ be continuous. Then, the set $\mathrm{GNE}(\mathbf{f}, \mathcal{X})$ is nonempty and compact.* □

Theorem 2 *Let $\mathbf{f} : \Delta \to \mathbb{R}^n$ be continuous and consider the set $\mathcal{E}$ in* (5)*. Then, $\mathbf{x}^* \in \mathrm{GNE}(\mathbf{f}, \mathcal{X})$ if and only if $(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \in \mathcal{E}$, for some $\boldsymbol{\mu}^* \in \mathbb{R}^{C_=}$ and $\boldsymbol{\lambda}^* \in \mathbb{R}^{C_\leq}_{\geq 0}$.* □

Lemma 7 *Let $\mathbf{f} : \Delta \to \mathbb{R}^n$ be continuous and consider the set $\mathcal{E}$ in* (5)*. Then, $\mathcal{E}$ is nonempty and compact.* □

Theorem 2 provides sufficient conditions to guarantee the coincidence of the equilibria set of the considered EDM-PDM system with the set of GNE of the underlying game $\mathbf{f}(\cdot)$. Based on such a result, we now proceed to prove that, for certain population games, such an equilibria set is indeed asymptotically stable under the considered EDM-PDM system.

Theorem 3 *Consider the EDM in* (1) *in conjunction with the PDM in* (3)-(4)*, and the equilibria set $\mathcal{E}$ in* (5)*. Moreover, let $\mathbf{f} : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n$ be a continuously differentiable stable game*[1]*. Then, the set $\mathcal{E}$ is asymptotically stable under the considered EDM-PDM system.* □

Theorems 2 and 3 show that the considered EDM-PDM system is in fact suitable for GNE seeking in continuosuly differentiable stable population games. We now proceed to illustrate the application of the proposed EDM-PDM system in some strategic scenario relevant to several practical applications.

6 An illustrative application example

For the following discussions, we let $\mathbf{x} = \mathrm{col}\left(\mathbf{x}^1, \ldots, \mathbf{x}^N\right)$ be equivalently written as $\left(\mathbf{x}^k, \mathbf{x}^{-k}\right)$, for all $k \in \mathcal{P}$, where $\mathbf{x}^{-k} = \mathrm{col}\left(\mathbf{x}^1, \ldots, \mathbf{x}^{k-1}, \mathbf{x}^{k+1}, \ldots, \mathbf{x}^N\right) \in \mathbb{R}^{n-n^k}_{\geq 0}$ is the strategic distribution of all populations other than k. Namely, regardless of k, it always holds that $\left(\mathbf{x}^k, \mathbf{x}^{-k}\right) = \mathrm{col}\left(\mathbf{x}^1, \ldots, \mathbf{x}^N\right) = \mathbf{x}$. That is, the order is preserved regardless of k.

Consider the scenario where each population $k \in \mathcal{P}$ seeks to solve an optimization problem of the form

$$\max_{\mathbf{x}^k \in \mathbb{R}^{n^k}} \psi^k\left(\mathbf{x}^k, \mathbf{x}^{-k}\right), \quad \text{s.t.} \quad \mathbf{x}^k \in \Omega^k\left(\mathbf{x}^{-k}\right), \tag{6}$$

where $\psi^k : \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ is the (concave and differentiable) objective function of population $k \in \mathcal{P}$; and

$$\Omega^k\left(\mathbf{x}^{-k}\right) = \Delta^k \cap \left\{\mathbf{x}^k \in \mathbb{R}^{n^k} : \left(\mathbf{x}^k, \mathbf{x}^{-k}\right) \in \mathcal{X}\right\}$$

is the feasible set of population $k \in \mathcal{P}$ with respect to $\mathbf{x}^{-k}$ and $\mathcal{X}$. Namely, by solving (6), the agents of population $k \in \mathcal{P}$ aim to find the best response strategy to $\mathbf{x}^{-k}$, while satisfying the constraints of $\mathcal{X}$. Now, let

$$\mathbf{f}(\mathbf{x}) = \mathrm{col}\left(\nabla_{\mathbf{x}^1}\psi^1(\mathbf{x}), \nabla_{\mathbf{x}^2}\psi^2(\mathbf{x}), \ldots, \nabla_{\mathbf{x}^N}\psi^N(\mathbf{x})\right), \tag{7}$$

i.e., $\mathbf{f}(\cdot)$ is a pseudo-gradient mapping associated to the functions $\psi^k(\cdot)$, for all $k \in \mathcal{P}$. If $\mathbf{x}^* \in \mathrm{GNE}(\mathbf{f}, \mathcal{X})$, with $\mathbf{x}^* = \mathrm{col}\left(\mathbf{x}^{1*}, \mathbf{x}^{2*}, \ldots, \mathbf{x}^{N*}\right)$, then it follows that every

[1] The domain of $\mathbf{f}(\cdot)$ is assumed to be $\mathbb{R}^n_{\geq 0} \supset \Delta$ to ensure the existence of partial derivatives (Sandholm 2010).

$\mathbf{x}^{k*}$ solves the optimization problem in (6) for the corresponding population $k \in \mathcal{P}$. To see this, observe that $\mathbf{x}^* \in \mathrm{GNE}(\mathbf{f}, \mathcal{X})$ implies that $\mathbf{x}^* \in \Delta \cap \mathcal{X}$ and that

$$(\mathbf{y} - \mathbf{x}^*)^\top \mathbf{f}(\mathbf{x}^*) \leq 0, \quad \forall \mathbf{y} \in \Delta \cap \mathcal{X}.$$

Thus, $\mathbf{x}^* \in \mathrm{GNE}(\mathbf{f}, \mathcal{X})$ implies that $\mathbf{x}^{k*} \in \Omega^k\left(\mathbf{x}^{-k*}\right)$, where $\mathbf{x}^{-k*} = \mathrm{col}\left(\mathbf{x}^{1*}, \ldots, \mathbf{x}^{(k-1)*}, \mathbf{x}^{(k+1)*}, \ldots, \mathbf{x}^{N*}\right)$, and that

$$\left(\mathbf{y}^k - \mathbf{x}^{k*}\right)^\top \nabla_{\mathbf{x}^k} \psi^k(\mathbf{x}^*) \leq 0, \quad \forall \mathbf{y}^k \in \Omega^k\left(\mathbf{x}^{-k*}\right),$$

for all $k \in \mathcal{P}$. Here, $\mathbf{y} = \mathrm{col}\left(\mathbf{y}^1, \mathbf{y}^2, \ldots, \mathbf{y}^N\right)$. Hence, any $\mathbf{x}^* \in \mathrm{GNE}(\mathbf{f}, \mathcal{X})$ satisfies the necessary and sufficient optimality conditions to solve the convex programming problem in (6), for all $k \in \mathcal{P}$. Therefore, by solving the GNE seeking problem regarding the game $\mathbf{f}(\cdot)$ in (7), one solves the problem in (6) simultaneously for all populations.

Convex programming problems of the form in (6) arise in several practical applications. Some examples include charging coordination of plug-in electric vehicles (Grammatico 2017), energy sharing games (Wang et al. 2021), and power control of femtocells (Li et al. 2020), among others. In fact, a wide variety of such practical applications can be abstracted by the framework of Cournot games with market-related and production-related constraints (Yi & Pavel 2019*b*). As such, we consider a Cournot game scenario to validate our results.

Cournot game

Consider N companies (populations) indexed by the set $\mathcal{P}$, and consider $M \in \mathbb{Z}_{\geq 1}$ markets indexed by the set $\mathcal{M} = \{1, 2, \ldots, M\}$. Each company $k \in \mathcal{P}$ decides its strategy in the competition in $n^k \leq M$ markets by delivering $\mathbf{x}^k \in \mathbb{R}^{n^k}_{\geq 0}$ products to the markets it participates in. Namely, $x_i^k \in \mathbb{R}_{\geq 0}$ denotes the amount of product delivered from company k to market $i \in \mathcal{S}^k \subseteq \mathcal{M}$. Moreover, each company has a maximum production capacity denoted m^k. For simplicity, we assume that companies seek to deliver the totality of their production capacity m^k. Note, however, that this fact does not imply any loss of generality as one could always introduce a fictitious market in $\mathcal{M}$ to allocate the surplus of production capacity from all companies. Consequently, the set of possible strategic profiles for company k is given by Δ^k.

To formulate the problem, let $\mathbf{C}^k \in \mathbb{R}^{M \times n^k}$ be a matrix which specifies the markets where company k is involved. Namely, each column of $\mathbf{C}^k$ has exactly one element equal to 1 and the rest equal to 0; each row of $\mathbf{C}^k$ has at most one element equal to 1; and the ℓ-th element of the j-th column of $\mathbf{C}^k$ is equal to 1 if and only if company k participates in market $\ell \in \mathcal{M}$ (c.f., Fig. 2). Hence, with $\mathbf{C} = \left[\mathbf{C}^1, \mathbf{C}^2, \ldots, \mathbf{C}^N\right] \in \mathbb{R}^{M \times n}$ and $\mathbf{x} = \mathrm{col}\left(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\right) \in \mathbb{R}^n$, it follows that $\mathbf{Cx} = \sum_{k \in \mathcal{P}} \mathbf{C}^k \mathbf{x}^k$ is the total supply vector to all markets given the action profile $\mathbf{x}$ of all companies. Furthermore, let $\mathbf{J} : \mathbb{R}^M \to \mathbb{R}^M$ be a map from the total supply vector to the price of product at each market, and let $Q^k : \mathbb{R}^{n^k}_{\geq 0} \to \mathbb{R}$ be the production cost for company k. In particular, following the example in (Yi & Pavel 2019*b*, Section 7), throughout we set $\mathbf{J}(\mathbf{Cx}) = \bar{\mathbf{J}} - \mathbf{DCx}$, and we let $Q^k(\cdot)$ be a continuously differentiable convex function, for all $k \in \mathcal{P}$. Here, $\bar{\mathbf{J}} \in \mathbb{R}^M$ is a vector with positive coefficients, and $\mathbf{D} \in \mathbb{R}^{M \times M}$ is a diagonal matrix with positive diagonal entries. Note that the considered map $\mathbf{J}(\cdot)$ is typically known as a linear inverse demand function in economics. Finally, let $\mathcal{X}$ be characterized by some market-related constraints to be considered in the competition. Without loss of generality, we consider the following constraints: i) the first market (indexed by 1 in $\mathcal{M}$) must receive a fixed amount of supply $d_1 \in \mathbb{R}_{\geq 0}$; and ii) each market has a maximum input capacity $r_\ell \in \mathbb{R}_{\geq 0}$, for all $\ell \in \mathcal{M}$. Hence, $\mathcal{X} = \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{e}_1^\top \mathbf{Cx} = d_1,\ \mathbf{Cx} \preceq \mathbf{r}\right\}$, where $\mathbf{e}_1 \in \mathbb{R}^M$ is the first column of the $M \times M$ identity matrix; and $\mathbf{r} = \mathrm{col}(r_1, r_2, \ldots, r_M) \in \mathbb{R}^M$.

Based on the considered scenario, each company $k \in \mathcal{P}$ seeks to solve an optimization problem of the form in (6) with $\psi^k\left(\mathbf{x}^k, \mathbf{x}^{-k}\right) = (\mathbf{J}(\mathbf{Cx}))^\top \mathbf{C}^k \mathbf{x}^k - Q^k\left(\mathbf{x}^k\right)$. As discussed above, to solve such an optimization problem for all companies simultaneously, it suffices to find some $\mathbf{x}^* \in \mathrm{GNE}(\mathbf{f}, \mathcal{X})$, with $\mathbf{f}(\cdot)$ defined as in (7). Hence, if such an $\mathbf{f}(\cdot)$ is interpreted as a population game, and $\mathbf{f}(\cdot)$ satisfies the conditions of Theorem 3, then the proposed EDM-PDM system can indeed be applied to find a GNE for the considered market competition game.

Now, observe that, under the considered setup, $\psi^k(\mathbf{x}) = \bar{\mathbf{J}}^\top \mathbf{C}^k \mathbf{x}^k - \sum_{z \in \mathcal{P}} \mathbf{x}^{z\top} \mathbf{C}^{z\top} \mathbf{D} \mathbf{C}^k \mathbf{x}^k - Q^k\left(\mathbf{x}^k\right)$, for all $k \in \mathcal{P}$ (here, we have used $\mathbf{Cx} = \sum_{z \in \mathcal{P}} \mathbf{C}^z \mathbf{x}^z$). In consequence, $\nabla_{\mathbf{x}^k} \psi^k(\mathbf{x}) = \mathbf{C}^{k\top} \bar{\mathbf{J}} - 2\mathbf{C}^{k\top} \mathbf{D} \mathbf{C}^k \mathbf{x}^k - \sum_{z \in \mathcal{P} \setminus \{k\}} \mathbf{C}^{k\top} \mathbf{D} \mathbf{C}^z \mathbf{x}^z - \nabla_{\mathbf{x}^k} Q^k\left(\mathbf{x}^k\right)$, for all $k \in \mathcal{P}$. Hence, from (7), $\mathbf{f}(\mathbf{x}) = \mathbf{C}^\top \bar{\mathbf{J}} - \mathbf{Sx} - \nabla_{\mathbf{x}} Q(\mathbf{x})$, where $\nabla_{\mathbf{x}} Q(\cdot) = \mathrm{col}\left(\nabla_{\mathbf{x}^1} Q^1(\cdot), \ldots, \nabla_{\mathbf{x}^N} Q^N(\cdot)\right)$ and $\mathbf{S} = \mathrm{diag}\left(\mathbf{C}^{1\top} \mathbf{D} \mathbf{C}^1, \ldots, \mathbf{C}^{N\top} \mathbf{D} \mathbf{C}^N\right) + \mathbf{T}^\top \mathbf{T}$, with $\mathbf{T} = \left[\sqrt{\mathbf{D}} \mathbf{C}^1, \ldots, \sqrt{\mathbf{D}} \mathbf{C}^N\right]$ and $\mathbf{D} = \sqrt{\mathbf{D}}\sqrt{\mathbf{D}}$. Namely, $\nabla_{\mathbf{x}} Q(\cdot) \in \mathbb{R}^n$, $\mathbf{S} \in \mathbb{R}^{n \times n}$, and $\mathbf{T} \in \mathbb{R}^{M \times n}$. Clearly, $\mathbf{f}(\cdot)$ is continuously differentiable. Moreover, since $Q^k(\cdot)$ is convex, for all $k \in \mathcal{P}$, and $\mathbf{S}$ is symmetric positive semi-definite, it follows that the Jacobian matrix of $\mathbf{f}(\cdot)$ with respect to $\mathbf{x}$ is negative semi-definite. Therefore, from (Sandholm 2010, Theorem 3.3.1), it follows that $\mathbf{f}(\cdot)$ is a continuously differentiable stable game, and, thus, satisfies the conditions of Theorem 3.

As illustration, for our numerical experiments we consider the Cournot game with 10 companies and 7 markets presented in Fig. 2. Moreover, without loss of generality, we set $Q^k\left(\mathbf{x}^k\right) = \sum_{i\in\mathcal{S}^k}\left(\left(\alpha_i^k/2\right)\left(x_i^k\right)^2 + \beta_i^k x_i^k\right)$, for all $k \in \mathcal{P}$, where $\alpha_i^k, \beta_i^k \in \mathbb{R}_{\geq 0}$ are non-negative coefficients, for all $i \in \mathcal{S}^k$ and all $k \in \mathcal{P}$. Furthermore, as in (Yi & Pavel 2019*b*, Section 7), the market capacities r_ℓ are randomly sampled from $[0.5, 1]$, for all $\ell \in \mathcal{M}$; the nonzero elements of $\bar{\mathbf{J}}$ and $\mathbf{D}$ are randomly drawn from $[2, 4]$ and $[0.5, 1]$, respectively; and the coefficients α_i^k and β_i^k are randomly drawn from $[1, 8]$ and $[0.1, 0.6]$, respectively, for all $i \in \mathcal{S}^k$ and all $k \in \mathcal{P}$. Besides, we set $d_1 = 0.1r_1$, and we randomly sample m^k from $[0.1, 1]$, for all $k \in \mathcal{P}$ (it is numerically verified, however, that Standing Assumption 3.(a) holds under the sampled parameters).

To validate our theoretical developments, we consider two different instances of the proposed EDM-PDM system (c.f., Fig. 1). For the first instance (referred to as EDM-PDM A), we set $\phi_i^k(\cdot) = \max(\cdot, 0)$, for all $i \in \mathcal{S}^k$ and all $k \in \mathcal{P}$, and we set $\vartheta_l^+(\cdot) = \vartheta_l^-(\cdot) = \theta_q^+(\cdot) = \theta_q^-(\cdot) = \max(\cdot, 0)$, for all $l \in \mathcal{C}_=$ and all $q \in \mathcal{C}_\leq$. Note that the resulting EDM corresponds to the well-known Smith dynamics (Smith 1984). For the second instance (referred to as EDM-PDM B), we keep the same EDM as in the first instance, but we set $\vartheta_l^+(\cdot) = z(\cdot, \sigma_l^+)$, $\vartheta_l^-(\cdot) = z(\cdot, \sigma_l^-)$, $\theta_q^+(\cdot) = z(\cdot, \eta_q^+)$, and $\theta_q^-(\cdot) = z(\cdot, \eta_q^-)$, where $z(\alpha, \zeta) = e^{\zeta\alpha} - 1$, if $\alpha \geq 0$, and $z(\alpha, \zeta) = 0$ otherwise. Moreover, we randomly sample $\sigma_l^+, \sigma_l^-, \eta_q^+$, and η_q^- from $[2, 4]$, respectively, for all $l \in \mathcal{C}_=$ and all $q \in \mathcal{C}_\leq$ (we use randomized values for $\sigma_l^+, \sigma_l^-, \eta_q^+$, and η_q^-, simply to illustrate an instance of the PDM with heterogeneous functions $\vartheta_l^+(\cdot), \vartheta_l^-(\cdot), \theta_q^+(\cdot)$, and $\theta_q^-(\cdot)$).

In Fig. 3 we present the trajectories of the selected performance index, i.e., $\|\mathbf{x}(t) - \mathbf{x}^*\|_2 / \|\mathbf{x}(0) - \mathbf{x}^*\|_2$ with $\mathbf{x}^* \in \text{GNE}(\mathbf{f}, \mathcal{X})$, for the two considered instances of the EDM-PDM system. Clearly, it is verified that both instances of the EDM-PDM system indeed asymptotically converge to a GNE of the game $\mathbf{f}(\cdot)$. Additionally, for the sake of illustration, Fig. 4 shows the temporal evolution of $\mathbf{x}^5(t)$ under the EDM-PDM B.

To compare the performance of the proposed EDM-PDM system against another related continuous-time dynamical system, we highlight that the EDM-PDM system of Fig. 1 can be regarded as a form of a primal-dual gradient dynamics system (such dynamical systems are fairly popular in the field of convex optimization (Qu & Li 2019)). Namely, the EDM updates the primal variables within the (invariant) set Δ, and the PDM updates the dual variables associated to the constraints of $\mathcal{X}$. Since conventional primal-dual gradient dynamics are usually designed to solve convex programming problems rather than GNE seeking problems, let us consider a simplified version of the aforementioned Cournot game. In particular, let $\mathbf{D}$ be the zero matrix so that the market prices are now fixed at $\bar{\mathbf{J}}$ regardless of the production.

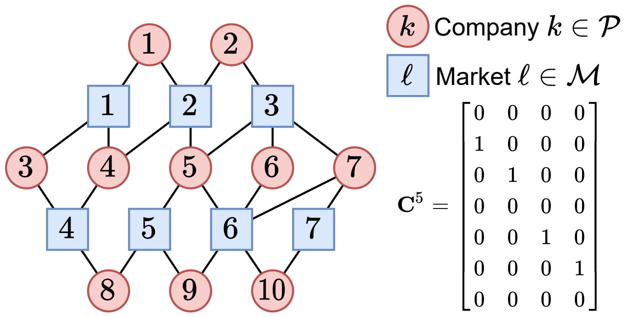

Fig. 2. Cournot game with 10 companies competing in 7 markets, i.e., $N = 10$ and $M = 7$. An edge between company $k \in \mathcal{P}$ and market $\ell \in \mathcal{M}$ means that company k participates in market ℓ. The matrix $\mathbf{C}^5$ is presented as an example.

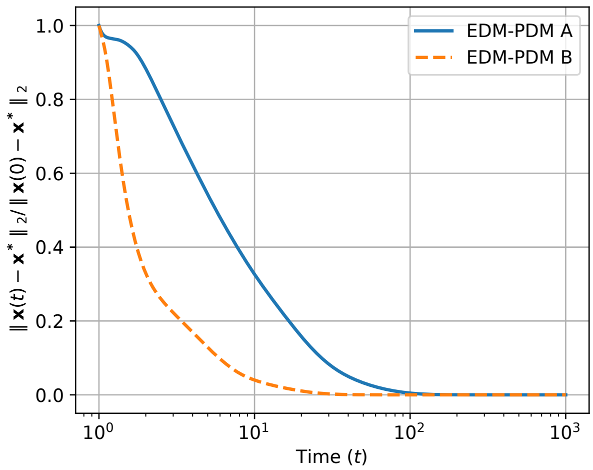

Fig. 3. Trajectories of the selected performance index under the EDM-PDM A and B for the considered Cournot game. In both cases, $x_i^k(0) = m^k/n^k$, $\mu_l(0) = 0$, and $\lambda_q(0) = 0$, for all $i \in \mathcal{S}^k$, $k \in \mathcal{P}$, $l \in \mathcal{C}_=$, and $q \in \mathcal{C}_\leq$.

Under such a simplification, it follows that $\mathbf{f}(\mathbf{x}) = \mathbf{C}^\top \bar{\mathbf{J}} - \nabla_\mathbf{x} Q(\mathbf{x})$, and thus the game $\mathbf{f}(\cdot)$ is now a (full) potential game (Sandholm 2010, Section 3.1.2) with potential function $\varphi(\mathbf{x}) = \sum_{k\in\mathcal{P}} \bar{\mathbf{J}}^\top \mathbf{C}^k \mathbf{x}^k - Q^k\left(\mathbf{x}^k\right)$, i.e., $\mathbf{f}(\mathbf{x}) = \nabla_\mathbf{x}\varphi(\mathbf{x})$. Under such a setup, it is straightforward to verify that $\mathbf{x}^* \in \text{GNE}(\mathbf{f}, \mathcal{X})$ if and only if $\mathbf{x}^* \in \arg\max_{\mathbf{x}\in\Delta\cap\mathcal{X}} \varphi(\mathbf{x})$. As illustration, Fig. 5 depicts the trajectories of the selected performance index under the EDM-PDM systems and the primal-dual gradient dynamics presented in Qu & Li (2019) (with unitary time constants). Clearly, while all dynamical systems indeed converge to a GNE of $\mathbf{f}(\cdot)$, our proposed EDM-PDM systems reach the equilibrium faster.

7 Concluding remarks

This paper has proposed a novel payoff dynamics model for generalized Nash equilibrium seeking in population games. In particular, we have considered the scenario

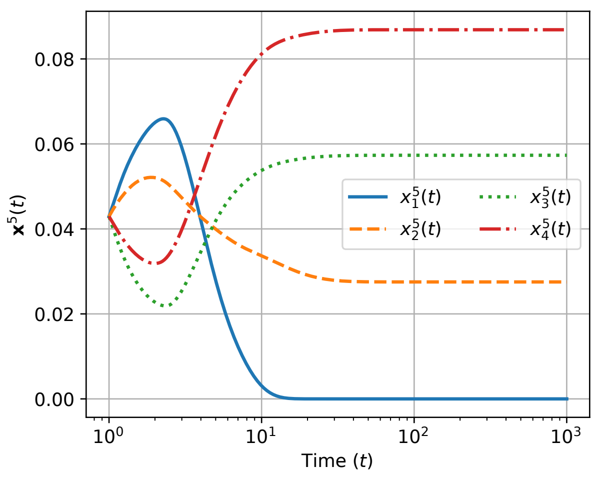

Fig. 4. Trajectory of the strategic distribution of population 5 under the EDM-PDM B in the considered Cournot game.

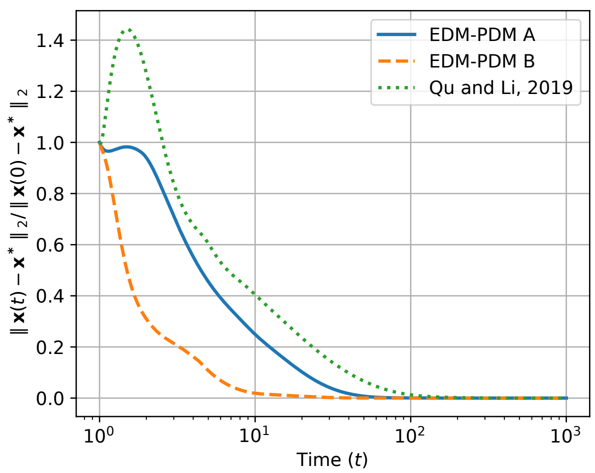

Fig. 5. Trajectories of the selected performance index under the EDM-PDM A and B and the primal-dual gradient dynamics in Qu & Li (2019) for the simplified Cournot game (with $\mathbf{D} = \mathbf{0}$). In all cases, $x_i^k(0) = m^k/n^k$, $\mu_l(0) = 0$, and $\lambda_q(0) = 0$, for all $i \in \mathcal{S}^k$, $k \in \mathcal{P}$, $l \in \mathcal{C}_=$, and $q \in \mathcal{C}_\leq$ (the remaining Lagrange multipliers of the primal-dual gradient dynamics are initialized at zero as well).

where the allowed strategic distributions of agents are subject to some affine equality constraints and some convex inequality constraints, and we have derived sufficient conditions to guarantee the achievement of generalized Nash equilibria in stable population games under impartial pairwise comparison revision protocols.

Future research should focus on the consideration of population games under non-complete interaction structures, and on the extension of the framework to other classes of revision protocols, e.g., imitative revision protocols. In addition, for the sake of generality, future work should attempt to soften the continuous differentiability assumptions on the population game and the convex inequality constraints.

8 Proofs

8.1 Proof of Lemma 1

Note that, for every $k \in \mathcal{P}$,

$$
\begin{aligned}
&\sum_{i\in\mathcal{S}^k} \dot{x}_i^k(t) \\
&= \sum_{i\in\mathcal{S}^k}\sum_{j\in\mathcal{S}^k} x_j^k(t)\phi_i^k\left(\delta_{ji}^k(t)\right) - x_i^k(t)\phi_j^k\left(\delta_{ij}^k(t)\right) \\
&= \sum_{i\in\mathcal{S}^k}\sum_{j\in\mathcal{S}^k} x_j^k(t)\phi_i^k\left(\delta_{ji}^k(t)\right) - \sum_{i\in\mathcal{S}^k}\sum_{j\in\mathcal{S}^k} x_i^k(t)\phi_j^k\left(\delta_{ij}^k(t)\right) \\
&= \sum_{i\in\mathcal{S}^k}\sum_{j\in\mathcal{S}^k} x_j^k(t)\phi_i^k\left(\delta_{ji}^k(t)\right) - \sum_{i\in\mathcal{S}^k}\sum_{j\in\mathcal{S}^k} x_j^k(t)\phi_i^k\left(\delta_{ji}^k(t)\right) \\
&= 0.
\end{aligned}
$$

Hence, if $\sum_{i\in\mathcal{S}^k} x_i^k(0) = m^k$, then $\sum_{i\in\mathcal{S}^k} x_i^k(t) = m^k$ for all $t \geq 0$, for all $k \in \mathcal{P}$. Moreover, from (1) observe that if $x_i^k(t) = 0$, then $\dot{x}_i^k(t) \geq 0$, for all $i \in \mathcal{S}^k$ and all $k \in \mathcal{P}$. Thus, $x_i^k(0) \geq 0$ implies that $x_i^k(t) \geq 0$ for all $t \geq 0$, for all $i \in \mathcal{S}^k$ and all $k \in \mathcal{P}$. Consequently, if $\mathbf{x}^k(0) \in \Delta^k$, then $\mathbf{x}^k(t) \in \Delta^k$ for all $t \geq 0$, for all $k \in \mathcal{P}$, leading to to the desired result. ∎

8.2 Proof of Lemma 2

(Sufficiency) Let condition (2) hold. Since

$$x_i^k(t)\phi_j^k\left(\delta_{ij}^k(t)\right) = x_i^k(t)\phi_j^k\left(p_j^k(t) - p_i^k(t)\right), \quad \forall i,j \in \mathcal{S}^k,$$

it follows from Definition 2 that $x_i^k(t)\phi_j^k\left(\delta_{ij}^k(t)\right) = 0$, for all $i,j \in \mathcal{S}^k$. Hence, $\dot{\mathbf{x}}^k(t) = \mathbf{0}$.

(Necessity) Let $\dot{\mathbf{x}}^k(t) = \mathbf{0}$, but suppose that (2) does not hold. Let $i \in \mathcal{S}^k$ be such that $p_i^k(t) = \max_{j\in\mathcal{S}^k} p_j^k(t)$. Thus, from Definition 2 it follows that $x_i^k(t)\phi_j^k\left(p_j^k(t) - p_i^k(t)\right) = 0$, for all $j \in \mathcal{S}^k$. Hence, $\dot{x}_i^k(t) \geq 0$. Now, given that (2) does not hold, there is some $\ell \in \mathcal{S}^k$ such that $x_\ell^k(t) > 0$ and $p_\ell^k(t) < p_i^k(t)$. Therefore, $x_\ell^k(t)\phi_i^k\left(p_i^k(t) - p_\ell^k(t)\right) > 0$, and $\dot{x}_i^k(t) > 0$. Consequently, $\dot{\mathbf{x}}^k(t) \neq \mathbf{0}$, which is a contradiction. ∎

8.3 Proof of Lemma 3

Note from (4b) that $\lambda_q(t) = 0$ implies that $\dot{\lambda}_q(t) \geq 0$. Hence, if $\lambda_q(0) \geq 0$, then $\lambda_q(t)$ cannot decrease below 0 for any $t \geq 0$. ∎

8.4 Proof of Lemma 4

From (4a), $\dot{\mu}_l(t) = \vartheta_l^+(h_l(\mathbf{x}(t))) - \vartheta_l^-(-h_l(\mathbf{x}(t)))$. Clearly, if $h_l(\mathbf{x}(t)) = 0$, then $\dot{\mu}_l(t) = 0$. In contrast, note that if $h_l(\mathbf{x}(t)) > 0$, then $\vartheta_l^+(h_l(\mathbf{x}(t))) > 0$ and $\vartheta_l^-(-h_l(\mathbf{x}(t))) = 0$. Similarly, if $h_l(\mathbf{x}(t)) < 0$, then $\vartheta_l^+(h_l(\mathbf{x}(t))) = 0$ and $\vartheta_l^-(-h_l(\mathbf{x}(t))) > 0$. Consequently, $h_l(\mathbf{x}(t)) \neq 0$ implies that $\dot{\mu}_l(t) \neq 0$. ■

8.5 Proof of Lemma 5

From (4b), $\dot{\lambda}_q(t) = \theta_q^+(g_q(\mathbf{x}(t))) - \lambda_q(t)\theta_q^-(-g_q(\mathbf{x}(t)))$. Hence,

(a) if $g_q(\mathbf{x}(t)) > 0$, then $\dot{\lambda}_q(t) = \theta_q^+(g_q(\mathbf{x}(t))) > 0$;
(b) if $g_q(\mathbf{x}(t)) = 0$, then $\dot{\lambda}_q(t) = 0$; and
(c) if $g_q(\mathbf{x}(t)) < 0$, then $\dot{\lambda}_q(t) = -\lambda_q(t)\theta_q^-(-g_q(\mathbf{x}(t)))$, with $\theta_q^-(-g_q(\mathbf{x}(t))) > 0$.

(Sufficiency) From (b) we have that $g_q(\mathbf{x}(t)) = 0$ immediately implies that $\dot{\lambda}_q(t) = 0$. Thus, let $g_q(\mathbf{x}) < 0$ and $\lambda_q(t)g_q(\mathbf{x}(t)) = 0$. Clearly, it must hold that $\lambda_q(t) = 0$. Consequently, from (c), $\dot{\lambda}_q(t) = 0$.

(Necessity) From (a) we have that $g_q(\mathbf{x}(t)) > 0$ readily implies that $\dot{\lambda}_q(t) > 0$. Therefore, $g_q(\mathbf{x}(t)) \leq 0$ is clearly a necessary condition for $\dot{\lambda}_q(t) = 0$. Now, suppose that $\dot{\lambda}_q(t) = 0$, but let $g_q(\mathbf{x}(t)) < 0$ and $\lambda_q(t) \neq 0$, so that $\lambda_q(t)g_q(\mathbf{x}(t)) \neq 0$. From (c) it follows that $\dot{\lambda}_q(t) \neq 0$, which leads to a contradiction. ■

8.6 Proof of Theorem 1

From Lemma 2, it follows that $\dot{\mathbf{x}}(t) = \mathbf{0}$ if and only if (2) holds for all $k \in \mathcal{P}$. Moreover, from Lemmas 4 and 5 it follows that $\dot{\boldsymbol{\mu}}(t) = \mathbf{0}$ and $\dot{\boldsymbol{\lambda}}(t) = \mathbf{0}$ if and only if $\mathbf{x}(t) \in \mathcal{X}$ and $\lambda_q(t)g_q(\mathbf{x}(t)) = 0$, for all $q \in \mathcal{C}_\leq$. Putting these facts together with the definition of $\mathcal{E}$ leads to the desired result. ■

8.7 Proof of Lemma 6

Note that $\mathrm{GNE}(\mathbf{f}, \mathcal{X})$ coincides with the solution set of the variational inequality $\mathrm{VI}(\Delta \cap \mathcal{X}, -\mathbf{f})$, which is denoted as $\mathrm{SOL}(\Delta \cap \mathcal{X}, -\mathbf{f})$ and is defined as

$$\begin{aligned}&\mathrm{SOL}(\Delta \cap \mathcal{X}, -\mathbf{f})\\&= \left\{\mathbf{x} \in \Delta \cap \mathcal{X} : (\mathbf{y} - \mathbf{x})^\top(-\mathbf{f}(\mathbf{x})) \geq 0, \forall \mathbf{y} \in \Delta \cap \mathcal{X}\right\}.\end{aligned}$$

Since $-\mathbf{f}(\cdot)$ is continuous, and $\Delta \cap \mathcal{X}$ is nonempty, convex, and compact (c.f., Remark 1), it follows from (Facchinei & Pang 2003, Corollary 2.2.5) that $\mathrm{GNE}(\mathbf{f}, \mathcal{X})$ is nonempty and compact. ■

8.8 Proof of Theorem 2

Let $\gamma_i^k \in \mathbb{R}$ be the Lagrange multiplier associated to the Δ-related constraint $x_i^k \geq 0$, for all $i \in \mathcal{S}^k$ and all $k \in \mathcal{P}$; let $\nu^k \in \mathbb{R}$ be the Lagrange multiplier associated to the Δ-related constraint $\sum_{i\in\mathcal{S}^k} x_i^k = m^k$, for all $k \in \mathcal{P}$; let $\mu_l \in \mathbb{R}$ be the Lagrange multiplier associated to the $\mathcal{X}$-related constraint $h_l(\mathbf{x}) = 0$, for all $l \in \mathcal{C}_=$; and, similarly, let $\lambda_q \in \mathbb{R}$ be the Lagrange multiplier associated to the $\mathcal{X}$-related constraint $g_q(\mathbf{x}) \leq 0$, for all $q \in \mathcal{C}_\leq$. Also, let $\boldsymbol{\gamma} \in \mathbb{R}^n$, $\boldsymbol{\nu} \in \mathbb{R}^N$, $\boldsymbol{\mu} \in \mathbb{R}^{C_=}$, and $\boldsymbol{\lambda} \in \mathbb{R}^{C_\leq}$ be the vectors containing such Lagrange multipliers, respectively. Moreover, consider the conditions

$$\begin{aligned}
x_i^k &\geq 0, \quad \forall i \in \mathcal{S}^k, \forall k \in \mathcal{P}, && \text{(8a)}\\
\sum_{i\in\mathcal{S}^k} x_i^k &= m^k, \quad \forall k \in \mathcal{P}, && \text{(8b)}\\
h_l(\mathbf{x}) &= 0, \quad \forall l \in \mathcal{C}_=, && \text{(8c)}\\
g_q(\mathbf{x}) &\leq 0, \quad \forall q \in \mathcal{C}_\leq, && \text{(8d)}\\
\gamma_i^k &\geq 0, \quad \forall i \in \mathcal{S}^k, \forall k \in \mathcal{P}, && \text{(8e)}\\
\lambda_q &\geq 0, \quad \forall q \in \mathcal{C}_\leq, && \text{(8f)}\\
\gamma_i^k x_i^k &= 0, \quad \forall i \in \mathcal{S}^k, \forall k \in \mathcal{P}, && \text{(8g)}\\
\lambda_q g_q(\mathbf{x}) &= 0, \quad \forall q \in \mathcal{C}_\leq, && \text{(8h)}\\
f_i^k(\mathbf{x}) - \chi_i^k(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) &= \nu^k - \gamma_i^k, \quad \forall i \in \mathcal{S}^k, \forall k \in \mathcal{P}, && \text{(8i)}
\end{aligned}$$

where

$$\chi_i^k(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) = \sum_{l\in\mathcal{C}_=} \mu_l \frac{\partial h_l(\mathbf{x})}{\partial x_i^k} + \sum_{q\in\mathcal{C}_\leq} \lambda_q \frac{\partial g_q(\mathbf{x})}{\partial x_i^k},$$

for all $i \in \mathcal{S}^k$ and all $k \in \mathcal{P}$.

Now, observe that $\mathbf{x}^* \in \mathrm{GNE}(\mathbf{f}, \mathcal{X})$ if and only if the conditions in (8) hold at $(\mathbf{x}^*, \boldsymbol{\gamma}^*, \boldsymbol{\nu}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*)$, for some $\mathbf{x}^* \in \mathbb{R}^n$, $\boldsymbol{\gamma}^* \in \mathbb{R}^n$, $\boldsymbol{\nu}^* \in \mathbb{R}^N$, $\boldsymbol{\mu}^* \in \mathbb{R}^{C_=}$, and $\boldsymbol{\lambda}^* \in \mathbb{R}^{C_\leq}$. To see this, note that from Definition 4, it holds that $\mathbf{x}^* \in \mathrm{GNE}(\mathbf{f}, \mathcal{X})$ if and only if $\mathbf{x}^* \in \arg\max_{\mathbf{x}\in\Delta\cap\mathcal{X}} \mathbf{x}^\top \mathbf{f}(\mathbf{x}^*)$. Hence, let $\mathbf{x}^* \in \mathrm{GNE}(\mathbf{f}, \mathcal{X})$ and observe that from the necessity of the Karush-Kuhn-Tucker (KKT) conditions, there must exist some $\boldsymbol{\gamma}^* \in \mathbb{R}^n$, $\boldsymbol{\nu}^* \in \mathbb{R}^N$, $\boldsymbol{\mu}^* \in \mathbb{R}^{C_=}$, and $\boldsymbol{\lambda}^* \in \mathbb{R}^{C_\leq}$, such that the conditions in (8) hold at $(\mathbf{x}^*, \boldsymbol{\gamma}^*, \boldsymbol{\nu}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*)$. Conversely, let the conditions in (8) hold at $\left(\hat{\mathbf{x}}, \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{\nu}}, \hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}}\right)$, for some $\hat{\mathbf{x}} \in \mathbb{R}^n$, $\hat{\boldsymbol{\gamma}} \in \mathbb{R}^n$, $\hat{\boldsymbol{\nu}} \in \mathbb{R}^N$, $\hat{\boldsymbol{\mu}} \in \mathbb{R}^{C_=}$, and $\hat{\boldsymbol{\lambda}} \in \mathbb{R}^{C_\leq}$. From the sufficiency of the KKT conditions, it follows that $\hat{\mathbf{x}} \in \arg\max_{\mathbf{x}\in\Delta\cap\mathcal{X}} \mathbf{x}^\top \mathbf{f}(\hat{\mathbf{x}})$, and, therefore, it holds that $\hat{\mathbf{x}} \in \mathrm{GNE}(\mathbf{f}, \mathcal{X})$. Hence, we conclude that $\mathbf{x}^* \in \mathrm{GNE}(\mathbf{f}, \mathcal{X})$ if and only if $(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \in \mathcal{K}$, for some $\boldsymbol{\mu}^* \in \mathbb{R}^{C_=}$ and some $\boldsymbol{\lambda}^* \in \mathbb{R}^{C_\leq}$, where

$$\mathcal{K} = \left\{ (\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) : \begin{array}{l} \mathbf{x} \in \mathbb{R}^n,\ \boldsymbol{\mu} \in \mathbb{R}^{C_=},\ \boldsymbol{\lambda} \in \mathbb{R}^{C_\leq}, \text{ and} \\ \text{the conditions in (8) hold at} \\ (\mathbf{x}, \boldsymbol{\gamma}, \boldsymbol{\nu}, \boldsymbol{\mu}, \boldsymbol{\lambda}), \text{ for some } \boldsymbol{\gamma} \in \mathbb{R}^n \\ \text{and some } \boldsymbol{\nu} \in \mathbb{R}^N. \end{array} \right\}.$$

Based on the discussion above, to prove the Theorem's result, i.e., that $\mathbf{x}^* \in \mathrm{GNE}(\mathbf{f}, \mathcal{X})$ if and only if $(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \in \mathcal{E}$, for some $\boldsymbol{\mu}^* \in \mathbb{R}^{C_=}$ and $\boldsymbol{\lambda}^* \in \mathbb{R}^{C_\leq}_{\geq 0}$, we should only prove that $(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \in \mathcal{E}$ if and only if $(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \in \mathcal{K}$. We now proceed to prove this claim.

(Sufficiency) Let $(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \in \mathcal{E}$. From (5) it readily follows that conditions (8a)-(8d), (8f), and (8h) hold. Furthermore, since $\mathbf{x}^*$ satisfies (2) for all $k \in \mathcal{P}$, it follows that, for all $i \in \mathcal{S}^k$ and all $k \in \mathcal{P}$,

$$x_i^{k*} > 0 \Rightarrow f_i^k(\mathbf{x}^*) - \chi_i^k(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) = p_{\max}^{k*},$$

with $p_{\max}^{k*} = \max_{j \in \mathcal{S}^k} \left(f_j^k(\mathbf{x}^*) - \chi_j^k(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \right)$, for all $k \in \mathcal{P}$. Hence, conditions (8e), (8g), and (8i), are satisfied by taking $\nu^{k*} = p_{\max}^{k*}$ and $\gamma_i^{k*} = \nu^{k*} - \left(f_i^k(\mathbf{x}^*) - \chi_i^k(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \right)$, for all $i \in \mathcal{S}^k$ and all $k \in \mathcal{P}$. Consequently, $(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \in \mathcal{K}$.

(Necessity) Let $(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \in \mathcal{K}$. Conditions (8a)-(8d) imply that $\mathbf{x}^* \in \Delta \cap \mathcal{X}$; condition (8f) implies that $\boldsymbol{\lambda}^* \in \mathbb{R}^{C_\leq}_{\geq 0}$; and condition (8h) implies that $\lambda_q^* g_q(\mathbf{x}^*) = 0$, for all $q \in \mathcal{C}_\leq$. Moreover, conditions (8g) and (8i) imply that $f_i^k(\mathbf{x}^*) - \chi_i^k(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) = \nu^{k*}$, for all $i \in \mathrm{supp}\left(\mathbf{x}^{k*}\right)$ and all $k \in \mathcal{P}$. Similarly, conditions (8e) and (8i) imply that $f_j^k(\mathbf{x}^*) - \chi_j^k(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) = \nu^{k*} - \gamma_j^{k*} \leq \nu^{k*}$, for all $j \in \mathcal{S}^k$ and all $k \in \mathcal{P}$. Therefore, for all $i \in \mathrm{supp}(\mathbf{x}^*)$ and all $k \in \mathcal{P}$ it follows that $f_i^k(\mathbf{x}^*) - \chi_i^k(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) = \max_{j \in \mathcal{S}^k} \left(f_j^k(\mathbf{x}^*) - \chi_j^k(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \right)$. Thus, the conditions in (2) holds at $(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*)$. Consequently, $(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \in \mathcal{E}$. ■

8.9 *Proof of Lemma 7*

The fact that $\mathcal{E}$ is nonempty is an immediate result from Lemma 6 and Theorem 2. To prove the compactness of $\mathcal{E}$, on the other hand, we proceed to show that $\mathcal{E}$ is closed and bounded.

First, the set $\mathcal{E}$ is closed because it is the preimage of the closed set $\{0\}$ under the continuous map $V(\cdot,\cdot,\cdot)$ defined in (10) in Section 8.10.

To show that $\mathcal{E}$ is bounded, on the other hand, recall the discussion in Section 8.8 on the coincidence between $\mathcal{E}$ and $\mathcal{K}$. Also, recall that $\mathbf{x}^* \in \mathrm{GNE}(\mathbf{f}, \mathcal{X})$ if and only if $\mathbf{x}^*$ solves the convex programming problem given by $\max_{\mathbf{x} \in \Delta \cap \mathcal{X}} \mathbf{x}^\top \mathbf{f}(\mathbf{x}^*)$. Such a convex programming problem can be equivalently written as

$$\max_{\mathbf{x} \in \mathbb{R}^n} \quad \mathbf{x}^\top \mathbf{f}(\mathbf{x}^*) \quad \text{s.t.} \quad \hat{\mathbf{A}}\mathbf{x} = \hat{\mathbf{b}}, \quad \hat{\mathbf{g}}(\mathbf{x}) \preceq \mathbf{0}, \tag{9}$$

where $\hat{\mathbf{A}} \in \mathbb{R}^{(C_= + N) \times n}$ is defined as in Standing Assumption 3.(b); $\hat{\mathbf{b}} = \mathrm{col}(b_1, b_2, \ldots, b_{C_=}, \mathbf{m}) \in \mathbb{R}^{(C_= + N)}$; $\hat{\mathbf{g}}(\mathbf{x}) = \mathrm{col}(\mathbf{g}(\mathbf{x}), -\mathbf{x}) \in \mathbb{R}^{(C_\leq + n)}$; and $(\preceq)$ denotes the element-wise inequality. Additionally, let $\hat{\boldsymbol{\mu}}^* = \mathrm{col}(\boldsymbol{\mu}^*, \boldsymbol{\nu}^*) \in \mathbb{R}^{(C_= + N)}$ be the optimal Lagrange multipliers associated to the equality constraints in (9), and let $\hat{\boldsymbol{\lambda}}^* = \mathrm{col}(\boldsymbol{\lambda}^*, \boldsymbol{\gamma}^*) \in \mathbb{R}^{(C_\leq + n)}_{\geq 0}$ be the optimal Lagrange multipliers associated to the inequality constraints in (9). From strong duality, it follows that, for every $\mathbf{x}^* \in \mathrm{GNE}(\mathbf{f}, \mathcal{X})$ and every $\tilde{\mathbf{x}} \in \mathrm{int}(\Delta) \cap \mathrm{int}(\mathcal{X})$ [c.f., Standing Assumption 3.(a)],

$$\begin{aligned} \mathbf{x}^{*\top} \mathbf{f}(\mathbf{x}^*) &\geq \tilde{\mathbf{x}}^\top \mathbf{f}(\mathbf{x}^*) - \left(\hat{\boldsymbol{\lambda}}^*\right)^\top \hat{\mathbf{g}}(\tilde{\mathbf{x}}) \\ &= \tilde{\mathbf{x}}^\top \mathbf{f}(\mathbf{x}^*) + \left(\hat{\boldsymbol{\lambda}}^*\right)^\top |\hat{\mathbf{g}}(\tilde{\mathbf{x}})| \\ &\geq \tilde{\mathbf{x}}^\top \mathbf{f}(\mathbf{x}^*) + \min\{|\hat{\mathbf{g}}(\tilde{\mathbf{x}})|\} \left(\mathbf{1}^\top \hat{\boldsymbol{\lambda}}^*\right). \end{aligned}$$

Here, $|\cdot|$ denotes the absolute value and is applied element-wise; $\min\{\mathbf{z}\}$ returns the minimum element of the vector $\mathbf{z}$; and $\mathbf{1}$ is the vector of ones of appropriate dimension. In particular, observe that the second equality holds because $\hat{\mathbf{g}}(\tilde{\mathbf{x}}) \prec \mathbf{0}$ (since $\tilde{\mathbf{x}} \in \mathrm{int}(\Delta) \cap \mathrm{int}(\mathcal{X})$). Consequently, since $\min\{|\hat{\mathbf{g}}(\tilde{\mathbf{x}})|\} > 0$, it follows that

$$\mathbf{1}^\top \hat{\boldsymbol{\lambda}}^* \leq \frac{\mathbf{x}^{*\top} \mathbf{f}(\mathbf{x}^*) - \tilde{\mathbf{x}}^\top \mathbf{f}(\mathbf{x}^*)}{\min\{|\hat{\mathbf{g}}(\tilde{\mathbf{x}})|\}} \in [0, \infty),$$

for every $\mathbf{x}^* \in \mathrm{GNE}(\mathbf{f}, \mathcal{X})$ and all $\tilde{\mathbf{x}} \in \mathrm{int}(\Delta) \cap \mathrm{int}(\mathcal{X})$. That is, the set of optimal Lagrange multipliers associated to the inequality constraints in $\Delta \cap \mathcal{X}$ is bounded.

Now, from the KKT stationarity condition, at any $\mathbf{x}^* \in \mathrm{GNE}(\mathbf{f}, \mathcal{X})$ it must hold that $\mathbf{f}(\mathbf{x}^*) - \hat{\mathbf{A}}^\top \hat{\boldsymbol{\mu}}^* - (\mathrm{D}\hat{\mathbf{g}}(\mathbf{x}^*))^\top \hat{\boldsymbol{\lambda}}^* = \mathbf{0}$, where $\mathrm{D}\hat{\mathbf{g}}(\mathbf{x}^*) \in \mathbb{R}^{(C_\leq + n) \times n}$ is the Jacobian matrix of $\hat{\mathbf{g}}(\cdot)$ at $\mathbf{x}^*$. Given that $\hat{\mathbf{A}}$ is full row rank [c.f., Standing Assumption 3.(b)], it follows that $\hat{\boldsymbol{\mu}}^* = \left(\hat{\mathbf{A}}\hat{\mathbf{A}}^\top\right)^{-1} \hat{\mathbf{A}} \left(\mathbf{f}(\mathbf{x}^*) - (\mathrm{D}\hat{\mathbf{g}}(\mathbf{x}^*))^\top \hat{\boldsymbol{\lambda}}^*\right)$. That is, $\hat{\boldsymbol{\mu}}^*$ is the image of a (uniformly continuous) linear map applied to $\mathbf{f}(\mathbf{x}^*) - (\mathrm{D}\hat{\mathbf{g}}(\mathbf{x}^*))^\top \hat{\boldsymbol{\lambda}}^*$, which implies that $\hat{\boldsymbol{\mu}}^*$ is bounded. To see this, note that $\hat{\boldsymbol{\lambda}}^*$ is bounded from the previous discussion; $\mathbf{f}(\mathbf{x}^*)$ is bounded due to the continuity of $\mathbf{f}(\cdot)$ and the compactness of $\mathrm{GNE}(\mathbf{f}, \mathcal{X})$ (c.f., Lemma 6); and $\mathrm{D}\hat{\mathbf{g}}(\mathbf{x}^*)$ is bounded because of the continuity of $\mathrm{D}\hat{\mathbf{g}}(\cdot)$ [c.f., Standing Assumption 3.(c)] and the compactness of $\mathrm{GNE}(\mathbf{f}, \mathcal{X})$. Hence, the set of optimal Lagrange multipliers associated to the equality constraints in $\Delta \cap \mathcal{X}$ is bounded as well.

Based on the discussions above and the coincidence between $\mathcal{E}$ and $\mathcal{K}$ (c.f., Section 8.8), we conclude that for every $\mathbf{x}^* \in \mathrm{GNE}(\mathbf{f}, \mathcal{X})$, the corresponding vectors $\boldsymbol{\mu}^*$ and $\boldsymbol{\lambda}^*$ are bounded. Therefore, the set $\mathcal{E}$ is both closed and bounded, and thus compact. ■

8.10 Proof of Theorem 3

From Lemma 7 we conclude that $\mathcal{E}$ is nonempty and compact. Moreover, from Theorem 1 it follows that $\mathcal{E}$ is positively invariant under the considered EDM-PDM system. Therefore, using an appropriate Lyapunov function, it is possible to investigate the stability properties of $\mathcal{E}$ (Haddad & Chellaboina 2008, Corollary 4.7).

Throughout, let $x_i^k \triangleq x_i^k(t)$, $\mu_l \triangleq \mu_l(t)$, $\lambda_q \triangleq \lambda_q(t)$, $p_i^k \triangleq p_i^k(t)$, $h_l \triangleq h_l(\mathbf{x}(t))$, $g_q \triangleq g_q(\mathbf{x}(t))$, $\mathbf{x} \triangleq \mathbf{x}(t)$, $\boldsymbol{\mu} \triangleq \boldsymbol{\mu}(t)$, and $\boldsymbol{\lambda} \triangleq \boldsymbol{\lambda}(t)$. Moreover, consider the map $V : \mathbb{R}^n_{\geq 0} \times \mathbb{R}^{C_=} \times \mathbb{R}^{C_\leq}_{\geq 0} \to \mathbb{R}_{\geq 0}$ given by

$$V(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) = \sum_{k \in \mathcal{P}} \sum_{j \in \mathcal{S}^k} \sum_{i \in \mathcal{S}^k} x_i^k P_{ij}^k(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) + \sum_{l \in \mathcal{C}_=} H_l(\mathbf{x}) + \sum_{q \in \mathcal{C}_\leq} G_q(\mathbf{x}, \lambda_q), \tag{10}$$

where, for all $i, j \in \mathcal{S}^k$, $k \in \mathcal{P}$, $l \in \mathcal{C}_=$, and $q \in \mathcal{C}_\leq$,

$$P_{ij}^k(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) = \int_0^{p_j^k - p_i^k} \phi_j^k(\sigma)\, d\sigma,$$
$$H_l(\mathbf{x}) = \int_0^{h_l} \vartheta_l^+(\sigma)\, d\sigma + \int_0^{-h_l} \vartheta_l^-(\sigma)\, d\sigma,$$
$$G_q(\mathbf{x}, \lambda_q) = \int_0^{g_q} \theta_q^+(\sigma)\, d\sigma + \lambda_q \int_0^{-g_q} \theta_q^-(\sigma)\, d\sigma.$$

It is straightforward to check that $V(\cdot,\cdot,\cdot)$ is a valid Lyapunov function candidate with respect to $\mathcal{E}$. To see this, note that $V(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) \geq 0$ for all $\mathbf{x} \in \Delta$, $\boldsymbol{\mu} \in \mathbb{R}^{C_=}$, and $\boldsymbol{\lambda} \in \mathbb{R}^{C_\leq}_{\geq 0}$; and $V(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) = 0$ if and only if $(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) \in \mathcal{E}$. To see the latter, observe that $\sum_{k \in \mathcal{P}} \sum_{j \in \mathcal{S}^k} \sum_{i \in \mathcal{S}^k} x_i^k P_{ij}^k(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) = 0$ if and only if the condition in (2) holds for all $k \in \mathcal{P}$ (Sandholm 2010, Theorem 7.2.9); $\sum_{l \in \mathcal{C}_=} H_l(\mathbf{x}) = 0$ if and only if $h_l(\mathbf{x}) = 0$, for all $l \in \mathcal{C}_=$; and $\sum_{q \in \mathcal{C}_\leq} G_q(\mathbf{x}, \boldsymbol{\lambda}) = 0$ if and only if $g_q(\mathbf{x}) \leq 0$ and $\lambda_q g_q(\mathbf{x}) = 0$, for all $q \in \mathcal{C}_\leq$. Consequently, to investigate the stability properties of $\mathcal{E}$, we proceed to analyze the derivatives of $V(\cdot,\cdot,\cdot)$. Letting $V \triangleq V(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$, $P_{ij}^k \triangleq P_{ij}^k(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$, $H_l \triangleq H_l(\mathbf{x})$, and $G_q \triangleq G_q(\mathbf{x}, \lambda_q)$, it follows that

$$\frac{\partial V}{\partial x_s^z} = \sum_{j \in \mathcal{S}^z} P_{sj}^z + \sum_{k \in \mathcal{P}} \sum_{j \in \mathcal{S}^k} \sum_{i \in \mathcal{S}^k} x_i^k \frac{\partial P_{ij}^k}{\partial x_s^z} + \sum_{l \in \mathcal{C}_=} \frac{\partial H_l}{\partial x_s^z} + \sum_{q \in \mathcal{C}_\leq} \frac{\partial G_q}{\partial x_s^z},$$
$$\frac{\partial V}{\partial \mu_v} = \sum_{k \in \mathcal{P}} \sum_{j \in \mathcal{S}^k} \sum_{i \in \mathcal{S}^k} x_i^k \frac{\partial P_{ij}^k}{\partial \mu_v},$$
$$\frac{\partial V}{\partial \lambda_w} = \sum_{k \in \mathcal{P}} \sum_{j \in \mathcal{S}^k} \sum_{i \in \mathcal{S}^k} x_i^k \frac{\partial P_{ij}^k}{\partial \lambda_w} + \int_0^{-g_w} \theta_w^-(\sigma)\, d\sigma,$$

for all $s \in \mathcal{S}^z$, $z \in \mathcal{P}$, $v \in \mathcal{C}_=$, and $w \in \mathcal{C}_\leq$. Here, letting $\delta_{ij}^k \triangleq p_j^k - p_i^k$ one gets

$$\sum_{k \in \mathcal{P}} \sum_{j,i \in \mathcal{S}^k} x_i^k \frac{\partial P_{ij}^k}{\partial x_s^z} = \sum_{k \in \mathcal{P}} \sum_{j,i \in \mathcal{S}^k} x_i^k \phi_j^k(\delta_{ij}^k) \left(\frac{\partial p_j^k}{\partial x_s^z} - \frac{\partial p_i^k}{\partial x_s^z} \right) = \sum_{k \in \mathcal{P}} \sum_{j \in \mathcal{S}^k} \dot{x}_j^k \frac{\partial p_j^k}{\partial x_s^z} \quad \text{[using (1)]}.$$

Similarly,

$$\sum_{l \in \mathcal{C}_=} \frac{\partial H_l}{\partial x_s^z} = \sum_{l \in \mathcal{C}_=} \left(\vartheta_l^+(h_l) \frac{\partial h_l}{\partial x_s^z} - \vartheta_l^-(-h_l) \frac{\partial h_l}{\partial x_s^z} \right) = \sum_{l \in \mathcal{C}_=} \dot{\mu}_l \frac{\partial h_l}{\partial x_s^z} \quad \text{[using (4a)]},$$

and

$$\sum_{q \in \mathcal{C}_\leq} \frac{\partial G_q}{\partial x_s^z} = \sum_{q \in \mathcal{C}_\leq} \left(\theta_q^+(g_q) \frac{\partial g_q}{\partial x_s^z} - \lambda_q \theta_q^-(-g_q) \frac{\partial g_q}{\partial x_s^z} \right) = \sum_{q \in \mathcal{C}_\leq} \dot{\lambda}_q \frac{\partial g_q}{\partial x_s^z} \quad \text{[using (4b)]}.$$

On the other hand,

$$\sum_{k \in \mathcal{P}} \sum_{j,i \in \mathcal{S}^k} x_i^k \frac{\partial P_{ij}^k}{\partial \mu_v} = \sum_{k \in \mathcal{P}} \sum_{j,i \in \mathcal{S}^k} x_i^k \phi_j^k(\delta_{ij}^k) \left(\frac{\partial p_j^k}{\partial \mu_v} - \frac{\partial p_i^k}{\partial \mu_v} \right) = \sum_{k \in \mathcal{P}} \sum_{j \in \mathcal{S}^k} \dot{x}_j^k \frac{\partial p_j^k}{\partial \mu_v} \quad \text{[using (1)]} = -\sum_{k \in \mathcal{P}} \sum_{j \in \mathcal{S}^k} \dot{x}_j^k \frac{\partial h_v}{\partial x_j^k} \quad \text{[using (3)]},$$

and, by symmetry,

$$\sum_{k \in \mathcal{P}} \sum_{j,i \in \mathcal{S}^k} x_i^k \frac{\partial P_{ij}^k}{\partial \lambda_w} = -\sum_{k \in \mathcal{P}} \sum_{j \in \mathcal{S}^k} \dot{x}_j^k \frac{\partial g_w}{\partial x_j^k} \quad \text{[using (1),(3)]}.$$

Now, let $\boldsymbol{\Gamma}_P \triangleq \operatorname{col}\left(\boldsymbol{\Gamma}_P^1, \boldsymbol{\Gamma}_P^2, \ldots, \boldsymbol{\Gamma}_P^N\right) \in \mathbb{R}^n_{\geq 0}$, where

$$\boldsymbol{\Gamma}_P^z \triangleq \operatorname{col}\left(\sum_{j\in\mathcal{S}^z} P_{1j}^z, \sum_{j\in\mathcal{S}^z} P_{2j}^z, \ldots, \sum_{j\in\mathcal{S}^z} P_{n^z j}^z\right) \in \mathbb{R}^{n^z}_{\geq 0},$$

for all $z \in \mathcal{P}$, and let

$$\boldsymbol{\Gamma}_G \triangleq \operatorname{col}\left(\int_0^{-g_1} \theta_1^-(\sigma)d\sigma, \ldots, \int_0^{-g_{C_\leq}} \theta_{C_\leq}^-(\sigma)d\sigma\right),$$

with $\boldsymbol{\Gamma}_G \in \mathbb{R}^{C_\leq}_{\geq 0}$. Also, let $\mathrm{D}\mathbf{p} \in \mathbb{R}^{n\times n}$, be the Jacobian matrix of $\mathbf{p}(t)$ with respect to $\mathbf{x}$ and evaluated at $(\mathbf{x}(t), \boldsymbol{\mu}(t), \boldsymbol{\lambda}(t))$; and let $\mathrm{D}\mathbf{g} \in \mathbb{R}^{C_\leq \times n}$ be the Jacobian matrix of $\mathbf{g}(\cdot) = \operatorname{col}\left(g_1(\cdot), g_2(\cdot), \ldots, g_{C_\leq}(\cdot)\right)$ with respect to $\mathbf{x}$ and evaluated at $\mathbf{x}(t)$. Under these observations, it follows that

$$\begin{aligned}
&\nabla_{\mathbf{x}} V(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) = \boldsymbol{\Gamma}_P + (\mathrm{D}\mathbf{p})^\top \dot{\mathbf{x}} + \mathbf{A}^\top \dot{\boldsymbol{\mu}} + (\mathrm{D}\mathbf{g})^\top \dot{\boldsymbol{\lambda}} \\
&\nabla_{\boldsymbol{\mu}} V(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) = -\mathbf{A}\dot{\mathbf{x}} \\
&\nabla_{\boldsymbol{\lambda}} V(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) = -\mathrm{D}\mathbf{g}\dot{\mathbf{x}} + \boldsymbol{\Gamma}_G,
\end{aligned}$$

where $\mathbf{A}$ is defined as in Standing Assumption 3.(b). Consequently, the gradient of $V(\cdot,\cdot,\cdot)$ along the trajectories of the EDM-PDM system is given by

$$\nabla V^\top \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\boldsymbol{\mu}} \\ \dot{\boldsymbol{\lambda}} \end{bmatrix} = \boldsymbol{\Gamma}_P^\top \dot{\mathbf{x}} + \dot{\mathbf{x}}^\top \mathrm{D}\mathbf{p}\dot{\mathbf{x}} + \boldsymbol{\Gamma}_G^\top \dot{\boldsymbol{\lambda}},$$

where $\nabla V \triangleq \operatorname{col}(\nabla_{\mathbf{x}} V, \nabla_{\boldsymbol{\mu}} V, \nabla_{\boldsymbol{\lambda}} V)$, and we have used the facts that, as scalars, $\dot{\boldsymbol{\mu}}^\top \mathbf{A}\dot{\mathbf{x}} = \dot{\mathbf{x}}^\top \mathbf{A}^\top \dot{\boldsymbol{\mu}}$, and that $\dot{\boldsymbol{\lambda}}^\top \mathrm{D}\mathbf{g}\dot{\mathbf{x}} = \dot{\mathbf{x}}^\top \mathrm{D}\mathbf{g}^\top \dot{\boldsymbol{\lambda}}$. Now, let us analyze the terms $\boldsymbol{\Gamma}_P^\top \dot{\mathbf{x}}$, $\dot{\mathbf{x}}^\top \mathrm{D}\mathbf{p}\dot{\mathbf{x}}$, and $\boldsymbol{\Gamma}_G^\top \dot{\boldsymbol{\lambda}}$ separately.

($\boldsymbol{\Gamma}_P^\top \dot{\mathbf{x}}$) Following a similar analysis as in the proofs of (Sandholm 2010, Theorem 7.2.9) or (Hofbauer & Sandholm 2009, Theorem 7.1), it is straightforward to show that $\boldsymbol{\Gamma}_P^\top \dot{\mathbf{x}} \leq 0$ for all times, and that $\boldsymbol{\Gamma}_P^\top \dot{\mathbf{x}} = 0$ if and only if (2) holds for all $k \in \mathcal{P}$.

($\dot{\mathbf{x}}^\top \mathrm{D}\mathbf{p}\dot{\mathbf{x}}$) From (3), and the fact that $h_l(\mathbf{x}) = \mathbf{a}_l^\top \mathbf{x} - b_l$, for all $l \in \mathcal{C}_=$, it follows that $\mathrm{D}\mathbf{p} = \mathrm{D}\mathbf{f} - \sum_{q\in\mathcal{C}_\leq} \lambda_q \mathrm{D}^2 g_q$, where $\mathrm{D}\mathbf{f} \in \mathbb{R}^{n\times n}$ is the Jacobian of $\mathbf{f}$ with respect to $\mathbf{x}$ and is evaluated at $\mathbf{x}(t)$, and $\mathrm{D}^2 g_q \in \mathbb{R}^{n\times n}$ denotes the Hessian of $g_q(\cdot)$ with respect to $\mathbf{x}$ and is evaluated at $\mathbf{x}(t)$. Hence, $\dot{\mathbf{x}}^\top \mathrm{D}\mathbf{p}\dot{\mathbf{x}} = \dot{\mathbf{x}}^\top \mathrm{D}\mathbf{f}\dot{\mathbf{x}} - \sum_{q\in\mathcal{C}_\leq} \lambda_q \dot{\mathbf{x}}^\top \mathrm{D}^2 g_q \dot{\mathbf{x}}$. Since $\mathbf{f}(\cdot)$ is a continuously differentiable stable game, we conclude from (Sandholm 2010, Theorem 3.3.1) that $\dot{\mathbf{x}}^\top \mathrm{D}\mathbf{f}\dot{\mathbf{x}} \leq 0$ for all times. Similarly, from Standing Assumption 3.(c) and Lemma 3, we conclude that $\sum_{q\in\mathcal{C}_\leq} \lambda_q \dot{\mathbf{x}}^\top \mathrm{D}^2 g_q \dot{\mathbf{x}} \geq 0$ for all times. Consequently, $\dot{\mathbf{x}}^\top \mathrm{D}\mathbf{p}\dot{\mathbf{x}} \leq 0$ for all times. Also, is clear that $\dot{\mathbf{x}}^\top \mathrm{D}\mathbf{p}\dot{\mathbf{x}} = 0$ whenever $\dot{\mathbf{x}} = \mathbf{0}$. Thus, from Lemma 2, $\dot{\mathbf{x}}^\top \mathrm{D}\mathbf{p}\dot{\mathbf{x}} = 0$ if (2) holds for all $k \in \mathcal{P}$.

($\boldsymbol{\Gamma}_G^\top \dot{\boldsymbol{\lambda}}$) Observe that $\boldsymbol{\Gamma}_G^\top \dot{\boldsymbol{\lambda}} = \sum_{q\in\mathcal{C}_\leq} \dot{\lambda}_q \int_0^{-g_q} \theta_q^-(\sigma)d\sigma$. Clearly, if $g_q \geq 0$, then $\int_0^{-g_q} \theta_q^-(\sigma)d\sigma = 0$. On the other hand, if $g_q < 0$, then $\int_0^{-g_q} \theta_q^-(\sigma)d\sigma > 0$ and $\dot{\lambda}_q \leq 0$ [c.f., (4b)]. Therefore, $\boldsymbol{\Gamma}_G^\top \dot{\boldsymbol{\lambda}} \leq 0$ for all times. Moreover, observe that $\boldsymbol{\Gamma}_G^\top \dot{\boldsymbol{\lambda}} = 0$ if and only if it holds that

$$g_q < 0 \Rightarrow \lambda_q g_q = 0, \quad \forall q \in \mathcal{C}_\leq. \tag{11}$$

Based on the three separate analyses above, we conclude that $\boldsymbol{\Gamma}_P^\top \dot{\mathbf{x}} + \dot{\mathbf{x}}^\top \mathrm{D}\mathbf{p}\dot{\mathbf{x}} + \boldsymbol{\Gamma}_G^\top \dot{\boldsymbol{\lambda}} \leq 0$ for all times, and, consequently, the set $\mathcal{E}$ is stable in the sense of Lyapunov.

To prove the asymptotic stability of $\mathcal{E}$, on the other hand, we rely on LaSalle's Theorem (Haddad & Chellaboina 2008, Theorem 3.3). In particular, based on the three separate analyses above, note that $\boldsymbol{\Gamma}_P^\top \dot{\mathbf{x}} + \dot{\mathbf{x}}^\top \mathrm{D}\mathbf{p}\dot{\mathbf{x}} + \boldsymbol{\Gamma}_G^\top \dot{\boldsymbol{\lambda}} = 0$ if and only if $(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) \in \mathcal{R}$, with

$$\mathcal{R} = \left\{ (\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) : \begin{array}{c} \mathbf{x} \in \Delta,\ \boldsymbol{\mu} \in \mathbb{R}^{C_=},\ \boldsymbol{\lambda} \in \mathbb{R}^{C_\leq}_{\geq 0}, \text{ and} \\ \text{(2) and (11) hold, for all } k \in \mathcal{P} \end{array} \right\}.$$

Clearly, $\mathcal{R} \supseteq \mathcal{E}$. In fact, $\mathcal{E}$ is the subset of $\mathcal{R}$ where $\mathbf{x} \in \mathcal{X}$. From LaSalle's Theorem, it follows that if $\mathcal{E}$ is shown to be the largest invariant set of the EDM-PDM system within $\mathcal{R}$, then $\mathcal{E}$ is asymptotically stable under the EDM-PDM system. We now proceed to prove such a claim by contradiction (a similar argument can be found in the proof of (Martinez-Piazuelo et al. 2022, Theorem 1)).

First, recall that the Lyapunov stability of $\mathcal{E}$ means that the trajectories of the EDM-PDM system can be bounded in an arbitrary open neighborhood of $\mathcal{E}$ within $\Delta \times \mathbb{R}^{C_=} \times \mathbb{R}^{C_\leq}_{\geq 0}$ (Haddad & Chellaboina 2008, Definition 4.10). Second, let $\mathcal{I} \subseteq \mathcal{R}$ be the largest invariant set of the EDM-PDM system within $\mathcal{R}$. From Theorem 1, it follows that $\mathcal{E} \subseteq \mathcal{I}$. Now, let $\mathcal{T} = \mathcal{I} \setminus \mathcal{E}$, suppose that $\mathcal{T} \neq \emptyset$, and let the state of the EDM-PDM system at a time $\tau \geq 0$ be an arbitrary point in $\mathcal{T}$, i.e., $(\mathbf{x}(\tau), \boldsymbol{\mu}(\tau), \boldsymbol{\lambda}(\tau)) \in \mathcal{T}$. Since $(\mathbf{x}(\tau), \boldsymbol{\mu}(\tau), \boldsymbol{\lambda}(\tau)) \in \mathcal{T}$ and $\mathcal{T} \subset \mathcal{I} \subseteq \mathcal{R}$, it follows that $\mathbf{x}(t) = \mathbf{x}(\tau)$ for all $t \geq \tau$ (c.f., Lemma 2). Moreover, since $\mathbf{x}(\tau)$ is fixed and $\mathbf{x}(\tau) \notin \mathcal{X}$ (as $(\mathbf{x}(\tau), \boldsymbol{\mu}(\tau), \boldsymbol{\lambda}(\tau)) \notin \mathcal{E}$), it holds that either $\|\boldsymbol{\mu}(t)\| \to \infty$ as $t \to \infty$, or $\|\boldsymbol{\lambda}(t)\| \to \infty$ as $t \to \infty$ [c.f., (4)]. Here, $\|\cdot\|$ is any p-norm. Consequently, the assumption that $\mathcal{T} \neq \emptyset$ leads to a contradiction with the Lyapunov stability of $\mathcal{E}$. Hence, $\mathcal{T} = \emptyset$, $\mathcal{I} = \mathcal{E}$, and $\mathcal{E}$ is the largest invariant set of the EDM-PDM system within $\mathcal{R}$. This completes the proof. ■

References

Arcak, M. & Martins, N. C. (2021), 'Dissipativity tools for convergence to Nash equilibria in population games', *IEEE Transactions on Control of Network Systems* **8**(1), 39–50.

Barreiro-Gomez, J., Obando, G. & Quijano, N. (2017), 'Distributed population dynamics: Optimization and control applications', *IEEE Transactions on Systems, Man, and Cybernetics: Systems* **47**(2), 304–314.

Barreiro-Gomez, J., Ocampo-Martinez, C. & Quijano, N. (2017), 'Dynamical tuning for mpc using population games: A water supply network application', *ISA Transactions* **69**, 175–186.

Barreiro-Gomez, J., Quijano, N. & Ocampo-Martinez, C. (2016), 'Constrained distributed optimization: A population dynamics approach', *Automatica* **69**, 101–116.

Barreiro-Gomez, J. & Tembine, H. (2018), 'Constrained evolutionary games by using a mixture of imitation dynamics', *Automatica* **97**, 254 – 262.

Belgioioso, G., Nedić, A. & Grammatico, S. (2021), 'Distributed generalized Nash equilibrium seeking in aggregative games on time-varying networks', *IEEE Transactions on Automatic Control* **66**(5), 2061–2075.

Como, G., Fagnani, F. & Zino, L. (2021), 'Imitation dynamics in population games on community networks', *IEEE Transactions on Control of Network Systems* **8**(1), 65–76.

Deng, Z. (2021), 'Distributed generalized Nash equilibrium seeking algorithm for nonsmooth aggregative games', *Automatica* **132**, 109794.

Facchinei, F. & Pang, J.-S. (2003), *Finite-dimensional variational inequalities and complementarity problems*, Vol. 1, Springer.

Grammatico, S. (2017), 'Dynamic control of agents playing aggregative games with coupling constraints', *IEEE Transactions on Automatic Control* **62**(9), 4537–4548.

Haddad, W. M. & Chellaboina, V. (2008), *Nonlinear dynamical systems and control: a Lyapunov-based approach*, Princeton University Press.

Hofbauer, J. & Sandholm, W. H. (2009), 'Stable games and their dynamics', *Journal of Economic Theory* **144**(4), 1665–1693.

Hofbauer, J. & Sigmund, K. (1998), *Evolutionary games and population dynamics*, Cambridge University Press.

Li, Z., Li, Z. & Ding, Z. (2020), 'Distributed generalized Nash equilibrium seeking and its application to femtocell networks', *IEEE Transactions on Cybernetics* pp. 1–13.

Martinez-Piazuelo, J., Quijano, N. & Ocampo-Martinez, C. (2020), Decentralized charging coordination of electric vehicles using multi-population games, *in* 'Proceedings of the 59th IEEE Conference on Decision and Control (CDC)', pp. 506–511.

Martinez-Piazuelo, J., Quijano, N. & Ocampo-Martinez, C. (2022), 'A payoff dynamics model for equality-constrained population games', *IEEE Control Systems Letters* **6**, 530–535.

Park, S., Martins, N. C. & Shamma, J. S. (2019), From population games to payoff dynamics models: A passivity-based approach, *in* 'Proceedings of the 58th IEEE Conference on Decision and Control (CDC)', pp. 6584–6601.

Pashaie, A., Pavel, L. & Damaren, C. J. (2017), 'A population game approach for dynamic resource allocation problems', *International Journal of Control* **90**(9), 1957–1972.

Qu, G. & Li, N. (2019), 'On the exponential stability of primal-dual gradient dynamics', *IEEE Control Systems Letters* **3**(1), 43–48.

Quijano, N., Ocampo-Martinez, C., Barreiro-Gomez, J., Obando, G., Pantoja, A. & Mojica-Nava, E. (2017), 'The role of population games and evolutionary dynamics in distributed control systems: The advantages of evolutionary game theory', *IEEE Control Systems Magazine* **37**(1), 70–97.

Sandholm, W. H. (2010), *Population games and evolutionary dynamics*, MIT Press.

Smith, M. J. (1984), 'The stability of a dynamic model of traffic assignment: an application of a method of Lyapunov', *Transportation Science* **18**(3), 245–252.

Srikantha, P. & Kundur, D. (2017), 'Resilient distributed real-time demand response via population games', *IEEE Transactions on Smart Grid* **8**(6), 2532–2543.

Tatarenko, T. & Kamgarpour, M. (2019), 'Learning generalized Nash equilibria in a class of convex games', *IEEE Transactions on Automatic Control* **64**(4), 1426–1439.

Tembine, H., Altman, E., El-Azouzi, R. & Hayel, Y. (2010), 'Evolutionary games in wireless networks', *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)* **40**(3), 634–646.

Wang, Z., Liu, F., Ma, Z., Chen, Y., Jia, M., Wei, W. & Wu, Q. (2021), 'Distributed generalized Nash equilibrium seeking for energy sharing games in prosumers', *IEEE Transactions on Power Systems* **36**(5), 3973–3986.

Yi, P. & Pavel, L. (2019*a*), 'Distributed generalized Nash equilibria computation of monotone games via double-layer preconditioned proximal-point algorithms', *IEEE Transactions on Control of Network Systems* **6**(1), 299–311.

Yi, P. & Pavel, L. (2019*b*), 'An operator splitting approach for distributed generalized Nash equilibria computation', *Automatica* **102**, 111–121.