Addressing the relative degree restriction in nonlinear adaptive observers: A high-gain observer approach

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Abstract

The design of adaptive observers is a common approach for the joint state and parameter-estimation problem. Nonetheless, there are still some obstacles that have to be solved to improve the design of adaptive observers and extend its implementability to a larger class of systems. First, the separation of the state-estimation and the parameter-estimation requires a relative degree one or zero between some known signal and the parameters to be estimated. Second, standard stability proofs for adaptive observers cannot be easily extended to consider the unavoidable presence of sensor noise and unmodelled system uncertainty. Consequently, on the one hand, this work proposed a methodology to relax the relative degree condition through the use of a high-gain observer that will be coupled with the adaptive observer. On the other hand, the stability and performance of the proposed observer scheme will be analysed by the use of a strict Lyapunov function based on the *Mazenc* construction, which allows to have provable convergence and to study the effect of sensor noise and model uncertainty through common Lyapunov theory. Finally, the proposed approach is validated in a compartmental epidemiology model.

Keywords: Robust estimation, Adaptive systems, Nonlinear observer design, parameter-estimation, Compartmental model, strict Lyapunov functions

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1. Introduction and motivation

state-estimation is a central topic for feedback control, system identification and fault diagnosis. The state-estimation problem for linear time-invariant system is a well-understood problem with solutions for generic systems, even in the presence of model uncertainty and sensor noise [1]. Nonetheless, such a generic solution does not exist for the more general case of nonlinear systems, which rely on the system having a specific structure.

For systems with Lipschitz nonlinearities, it is possible to estimate the state through a high-gain observer [2, 3, 4, 5]. Similar results have been obtained for nonlinearities that are bounded [6, 7], that satisfy a bounded Jacobian condition [8], a monotonic condition [9] or that are locally monotonic with finite extrema [10]. The authors in [11] proposed a local transformation that transformed a nonlinear system to a linear one with a measurable nonlinear perturbation that can be exactly cancelled. Naturally, the conditions for the existence of such transformation were very restrictive and difficult to check, being the most restrictive one the necessity of a linear output function. Later works defined a transformation and an observer that allowed a nonlinear output map [12], it has been shown that such transformation exists under mild observability assumptions [13, 14]. In [15], a manifold in which the state-estimation error is stable is designed and then is rendered attractive and invariant by the proper observer design. The shaping of the manifold relied on solving a partial differential equation. This restriction was removed by adding an output filter and a single dynamic scaling parameter [16].

State observers are model-based estimation techniques, consequently, uncertainty and unmodelled disturbances have a direct effect on the state-estimation accuracy. Under certain structural and observability conditions, it is possible to decouple the disturbance and the state in order to have an unbiased estimation [17, 18]. However, in a lot of cases, the observer can only be robustified in an input-to-state (ISS) sense [19] to the disturbances, and, usually, the accuracy can only be improved by increasing the gain of the observer, which generates a well-known trade-off between disturbance rejection and noise sensitivity [20]. Moreover, such approach may give limited information about the disturbances. In this context, higher performance may be achieved if the uncertainty can be modelled as the product of a known vector of functions and a vector of unknown parameters [21]. Then, couple the observer with some parameter adaptation dynamics that estimates the unknown parameters, and decouples the state-estimation from the uncertainty.

This type of observer are commonly referred as *adaptive observers*.

Similar to the non-adaptive case, there is a complete theory on adaptive observers for linear systems [22]. For nonlinear systems, adaptive designs are restricted to certain structures/canonical forms. A common approach is based on finding a Lyapunov function with a parameter error factor. Then, an adequate parameter adaptation is designed to cancel the unknown parameter terms in the derivative of said function [23], which drives the derivative negative semidefinite and allows to prove the state-estimation error convergence to zero through the Barbalat's lemma [23, 24]. In such context, state convergence can be ensured even if the parameter-estimation does not converge to the true value, which is a common thing in the absence of excitation [22].

Nonetheless, the cancellation in the Lyapunov equation derivative can only be achieved in systems that satisfy a strictly positive real condition [23, 25], which is restricted to systems with relative degree one or zero between the measured outputs and the unknown parameter vector. There is a significant amount of systems that do not possess this relative degree property, e.g. fuel cells [26, 27] or DC-converters with unknown constant power loads taking the generated current as the measured output and the constant power load as the parameter to estimate [28, 29]. This fact motivates the design of adaptive observers for higher relative degree systems. The most common approach is to implement a filtered-based coordinate change in the so-called *filtered transformation* [30, 31] or compute a set of auxiliary signals that satisfy the relative degree condition through a set of filter [32, 33, 34, 35]. However, such approach presents some conflicts that should be considered. First, the dynamics depend on the initial conditions of said filter. Therefore, the stability analysis of the observer is trajectory-dependent, in the sense that it pertains only to the trajectory generated for the given filter initial conditions. This fact may have a great impact on the observer performance and the observability study, which is overlooked in some papers. A further discussion in the context of adaptive control can be found in [36]. Second, the stability and accuracy analysis of the observers with filtered transformations in the presence of uncertainty, noise and unknown states is far from trivial and requires further study. Moreover, this analysis needs to consider the initial conditions of the filter as mentioned in the first point. Finally, as the filter depends on the unknown parameters, persistent excitation is needed in order to avoid state-estimation drift. Therefore, state and parameter-estimation are not separated, which was the original objective in the design of an adaptive observer. This fact motivates the design of alternative methodologies to circumvent the relative degree restriction without the use of filters.

A solution was proposed in [37] for reduced-order observers. However, such approach requires solving a PDE which is in general hard to compute. Alternatively, parameter-estimation for higher relative degrees can be achieved by implementing an extended high-gain observer [38, 28]. Nonetheless, this approach is limited to systems where the relative degree between the measured output and the parameters is equal to the dimension of the state vector.

There is a completely different state and parameter-estimation strategy that is based on extending the state with the unknown parameters and then, design a (non-adaptive) state observer to estimate the augmented state [39, 40]. However, again, such approach does not separate state and parameter-estimation. Moreover, the extended system may lose some beneficial structural properties of the original system. Finally, the observability study of the extended system may be significantly more difficult than the analysis of the original system. A similar approach was pursued in [41, 42], in which the observability conditions were relaxed, but relied on a restrictive separability condition between the states and parameters.

Another point to consider is that the stability of most adaptive observer techniques is based on the construction of a weak (weak in the sense that its derivative is just negative semi-definite) Lyapunov function combined with the Barbalat's lemma [10]. Although this approach is sufficient to theoretically prove the stability of the observer in an ideal scenario, it is insufficient for practical applications. First, a weak Lyapunov function cannot be used to analyse the performance of the adaptive observer in presence of uncertainties and sensor noise. Second, a weak Lyapunov function cannot be used to prove the stability of the interconnection between the adaptive observer and another system.

The key-points that are being addressed in this document can be summarized as follows.

- **KP1:** Propose an alternative adaptive observer approach for higher relative degree systems that is not based on the introduction of filters nor augmenting the state vector with the unknown parameters.

- **KP2:** Avoid the standard approach of building a weak Lyapunov function and using the Brabalat's lemma to prove the stability of the proposed observer.

- **KP3:** Analyse the performance of the adaptive technique in presence of sensor noise and unmodelled disturbances.

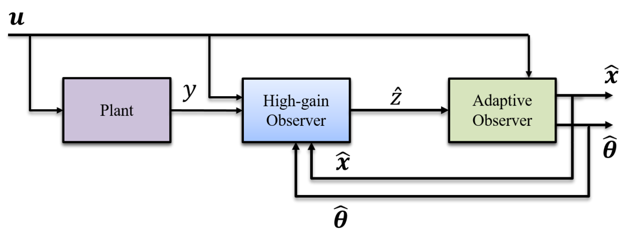

Figure 1: Scheme of the proposed adaptive observer for higher relative degree systems. The high-gain observer is implemented through equation (17). The factor $\hat{z}$ depicts the estimation of the auxiliary signal presented in Section 3, which is used to address the relative degree conflict. Finally, the adaptive observer is computed through equations (11) and (13).

Taking into account the issues presented before, the main contributions of this work can be summarized as follows.

- **C1:** This work proposes the design of an auxiliary signal, z, that circumvents the relative degree limitation of adaptive observers.

- **C2:** Instead of using filters, the auxiliary signal is estimated through a high-gain observer, which is feedback interconnected with the adaptive observer. A general scheme of the proposed approach is depicted in Fig. 1.

- **C3:** This work proposes a strict (strict in the sense that its derivative is negative definite) Lyapunov function based on the *Mazenc*-construction for the adaptive observer, which allows to prove the stability of the high-gain observer-adaptive observer interconnection through small-gain arguments.

- **C4:** The strict Lyapunov equation is used to analyse the performance of the technique in presence of sensor noise and unknown disturbances.

The rest of the document is organized as follows. Section 2 presents the standard adaptive observer design and the relative degree problematic. Section 3 presents a methodology to design an auxiliary signal which allows to relax the relative degree restriction and introduces a high-gain observer to estimate said auxiliary signal. Section 4 analyses the stability and performance of the adaptive observer and high-gain observer coupling by the use of a strict Lyapunov function. Section 5 validates the observer in a simple synthetic system. Section 6 validates the proposed scheme in a compartmental epidemiology model. Finally, some conclusions are drawn in Section 7.

2. Problem Formulation

Let us consider a multi-input multi-output nonlinear system of the form.

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x},\mathbf{u}) + \mathbf{B}\boldsymbol{\phi}(\mathbf{x},\mathbf{u})\boldsymbol{\theta} + \mathbf{w} \\ \mathbf{y} &= \mathbf{C}(\mathbf{u},\mathbf{y})\mathbf{x} + \mathbf{v} \end{aligned} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ are the system states, $\mathbf{y} = [y_1, ..., y_m]^\intercal \in \mathbb{R}^m$ are the measured outputs, $\mathbf{u} \in \mathbb{R}^q$ are the controlled inputs and $\boldsymbol{\theta} \in \mathbb{R}^p$ is a vector of unknown constant parameters to be estimated. The matrices $\mathbf{B} \in \mathbb{R}^{n\times s}$ and $\mathbf{C}(\cdot,\cdot) \in \mathbb{R}^{m\times n}$ are assumed to be known and bounded. The functions $\mathbf{f}(\cdot,\cdot) \in \mathbf{R}^{n\times 1}$ and $\boldsymbol{\phi}(\cdot,\cdot) \in \mathbb{R}^{s\times p}$ are assumed to be known, bounded and Lipschitz, with L_f and L_ϕ as Lipschitz constant, respectively. The linear regressor factor is assumed to be upper bounded as $\|\boldsymbol{\phi}(\cdot,\cdot)\| \leq \phi_{max}$. The factor $\mathbf{v} \in \mathbb{R}^m$ depicts unknown sensor noise, which is assumed to be upper-bounded by a positive constant ϑ as $\|\mathbf{v}\|_2 \leq \vartheta$. The factor $\mathbf{w} \in \mathbb{R}^{n\times 1}$ depicts unmodelled disturbances or uncertainty, which are also assumed to be upper-bounded as $\|\mathbf{w}\| \leq w_2$. Finally, there exists some sets $\mathbf{X}_0 \subseteq \mathbf{X} \subseteq \mathbb{R}^n$ and $\mathbf{U} \subseteq \mathbb{R}^q$, such that the trajectories of (1), with initial conditions $\mathbf{x}(0)$ in $\mathbf{X}_0$ and input $\mathbf{u}(t)$ belonging to $\mathbf{U}$ for all times, remain in $\mathbf{X}$ for all $t \geq 0$.

In this work, part of the uncertainty is modelled as a linear combination of basis functions, $\mathbf{B}\boldsymbol{\phi}(\mathbf{x},\mathbf{u})\boldsymbol{\theta}$, where the vector $\boldsymbol{\theta}$ is unknown. In some problems, this formulation arise naturally from the first principles of the system. Alternatively, if there is no information of the uncertainty, certain universal approximators can be adapted in this context. For example, neural networks [43, 44] in which only the outer layer is being adapted, uncertainty modelled through fuzzy sets [45, 46] or reproducing kernels [47].

Define $\hat{\mathbf{x}}$ as the estimation of the states and $\hat{\boldsymbol{\theta}}$ as an estimation of the unknown parameters, the dynamics of which will be defined below. It is assumed that $\hat{\mathbf{x}}$ is generated through an observer of the form:

$$\dot{\hat{\mathbf{x}}} = \mathbf{f}(\hat{\mathbf{x}},\mathbf{u}) + \mathbf{g}(\hat{\mathbf{x}},\mathbf{u})\mathbf{e}_y + \mathbf{B}\boldsymbol{\phi}(\hat{\mathbf{x}},\mathbf{u})\hat{\boldsymbol{\theta}}, \tag{2}$$

where $\mathbf{e}_y = \mathbf{y} - \mathbf{C}(\mathbf{u},\mathbf{y})\hat{\mathbf{x}}$ and $\mathbf{g}(\cdot) \in \mathbb{R}^{n\times 1}$ is a bounded function designed such that, in the case where $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$, the state-estimation error, $\mathbf{e}_x = \mathbf{x} - \hat{\mathbf{x}}$, converges to a small bounded value. Specifically, it is assumed that there is a quadratic radially unbounded Lyapunov function,

$$V_x = \frac{1}{2}\mathbf{e}_x^{\intercal}\mathbf{P}(t)\mathbf{e}_x, \tag{3}$$

where $\mathbf{P}(t)$ is a symmetric positive definite matrix, which is (possibly) time-varying and upper and lower bounded by positive constants, such that the following holds

$$\begin{aligned} &\alpha_1\|\mathbf{e}_x\| \le V_x \le \alpha_2\|\mathbf{e}_x\| \\ &\frac{\partial V_x}{\partial t} + \frac{\partial V_x}{\partial \mathbf{e}_x}\dot{\mathbf{e}}_x \le -\alpha_3\|\mathbf{e}_x\|^2 + \alpha_4\|\mathbf{e}_x\|\|\mathbf{w}\| + \alpha_5\|\mathbf{e}_x\|\|\mathbf{v}\|, \end{aligned} \tag{4}$$

where $\alpha_i \quad for \quad i = 1, .., 5$ are positive definite constants.

Remark 2.1. *This document focuses on observers the stability of which can be proven through quadratic Lyapunov functions with a matrix $\boldsymbol{P}$ that can (but is not restricted to) be time-varying. This includes the common observers in [2, 3, 4, 5, 8] for a constant matrix $\boldsymbol{P}$ and [48] for a time-varying matrix.*

In general, the condition $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}$ will not be satisfied. Therefore, the objective is to design some parameter adaptation dynamics, $\dot{\hat{\boldsymbol{\theta}}}$, to reduce the effect of the disturbance term $\mathbf{B}\boldsymbol{\phi}(\mathbf{x},\mathbf{u})\boldsymbol{\theta}$ and recover the performance depicted in (4).

2.1. Adaptation dynamics and the relative degree conflict

An approach to solve the adaptation problem is based on the following observation. Consider the observer (2), the Lyapunov function (3) and the perturbed case where $\boldsymbol{\theta} \neq \hat{\boldsymbol{\theta}}$. Moreover, define the parameter-estimation error as: $\mathbf{e}_\theta = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}$. Then, consider the radially unbounded composite Lyapunov function:

$$V_{x,\theta} = V_x + \frac{1}{2}\mathbf{e}_\theta^{\intercal}\mathbf{e}_\theta. \tag{5}$$

The derivative of function (5) is:

$$\begin{aligned} \frac{\partial V_{x,\theta}}{\partial t} + \frac{\partial V_{x,\theta}}{\partial \mathbf{e}_x}\dot{\mathbf{e}}_x \le& -\alpha_3\|\mathbf{e}_x\|^2 + \mathbf{e}_\theta^{\intercal}\boldsymbol{\phi}(\hat{\mathbf{x}},\mathbf{u})^{\intercal}\mathbf{B}^{\intercal}\mathbf{P}(t)\mathbf{e}_x \\ &- \mathbf{e}_\theta^{\intercal}\dot{\hat{\boldsymbol{\theta}}} + \alpha_4\|\mathbf{e}_x\|\|\mathbf{w}\| + \alpha_5\|\mathbf{e}_x\|\|\mathbf{v}\| \end{aligned} \tag{6}$$

From (6), it is possible to see that the following parameter adaptation:

$$\dot{\hat{\boldsymbol{\theta}}} = \boldsymbol{\phi}(\hat{\mathbf{x}}, \mathbf{u})^\intercal \mathbf{B}^\intercal \mathbf{P}(t) \mathbf{e}_x, \tag{7}$$

would recover the performance depicted in (4).

This approach allows to exactly cancel the effect of the unknown factor $\mathbf{B}\boldsymbol{\phi}(\mathbf{x}, \mathbf{u})\boldsymbol{\theta}$ without relying on high control gains as it is done in alternative robust observers [2, 6], which results in higher transient performance and better noise sensitivity. Moreover, in the absence of sensor noise and unmodelled uncertainty, $\mathbf{w} = \mathbf{v} = 0$, it can be proved that the parameter-estimation also converges to the true value. See the proof in Appendix A. Notice that this result does not guarantee that the estimation is bounded in the presence of noise and disturbances. This is a direct consequence of V_x being a weak Lyapunov function. For this reason, the stability of the proposed technique will be analysed in with a different approach. This will be the aim of Section 4.

Nonetheless, the adaptation depicted in (7) is not actually computable as it depends on the unknown state-estimation error, $\mathbf{e}_x$. To make this approach feasible, it is required to have some adaptation dynamics that only depends on the known state-estimation, $\hat{\mathbf{x}}$, and the measured output estimation error, $\mathbf{C}(\mathbf{u}, \mathbf{y})\mathbf{e}_x$. A solution is achieved using the following adaptation dynamics:

$$\dot{\hat{\boldsymbol{\theta}}} = \boldsymbol{\phi}(\hat{\mathbf{x}}, \mathbf{u})^\intercal \mathbf{M}(t, \mathbf{u}, \mathbf{y})(\mathbf{y} - \hat{\mathbf{y}}), \tag{8}$$

where $\mathbf{M}(t, \mathbf{u}, \mathbf{y})$ satisfies:

$$\mathbf{M}(t, \mathbf{u}, \mathbf{y})\mathbf{C}(\mathbf{u}, \mathbf{y}) = \mathbf{B}^\intercal \mathbf{P}(t). \tag{9}$$

Although the dynamics in (8) are easy to compute and scalable to high order regressor vectors, the equality in (9) introduces the conflict that this paper is trying to address. Equation (9) can only be solved if each row of $\mathbf{B}^\intercal \mathbf{P}(t)$ lies in span of $\mathbf{C}(\mathbf{u}, \mathbf{y})$, which is equivalent to the condition:

$$rank(\mathbf{C}(\mathbf{u}, \mathbf{y})\mathbf{B}) = rank(\mathbf{B}). \tag{10}$$

Condition (10) establishes that the equality (9) can only be solved if the relative degree between the output, $\mathbf{y}$, and the unknown parameters, $\boldsymbol{\theta}$, is zero or one. As presented in the introduction, there are significant cases where this relative degree condition is not satisfied, which limits the applicability of this adaptive approach and motivates the design of a methodology to circumvent this restriction.

This work proposes implementing an auxiliary signal, $\mathbf{z} \in \mathbb{R}^m$, which is relative degree 1 with respect to the unknown parameters. Next sections will focus on the design of such auxiliary signal, how it can be implemented in to relax the relative degree condition and how it can be estimated through a high-gain observer with provable convergence of the estimation.

3. Main Result

The objective is to find an adequate auxiliary signal, $\mathbf{z} = \mathbf{H}(\mathbf{u}, \mathbf{y})\mathbf{x}$, with a bounded matrix $\|\mathbf{H}(\mathbf{u}, \mathbf{y})\| \leq H_{max}$, to be used in the observer (2) in order to make the adaptation introduced in Section 2 computable. In this work, an auxiliary signal can be depicted as adequate if the following conditions are satisfied:

1. The concerned system is relative degree 1 from the auxiliary signal, $\mathbf{z}$, to the unknown parameter vector $\boldsymbol{\theta}$. Therefore, the function $\mathbf{H}(\mathbf{u}, \mathbf{y})$ is such that
$$rank(\mathbf{H}(\mathbf{u}, \mathbf{y})\mathbf{B}) = rank(\mathbf{B}), \quad \forall \mathbf{u}, \mathbf{y}.$$
2. There exists an observer of the form:
$$\dot{\hat{\mathbf{x}}} = \mathbf{f}(\hat{\mathbf{x}}, \mathbf{u}) + \mathbf{g}_2(\hat{\mathbf{x}}, \mathbf{u})\mathbf{e}_z + \mathbf{B}\boldsymbol{\phi}(\hat{\mathbf{x}}, \mathbf{u})\hat{\boldsymbol{\theta}}, \tag{11}$$
where $\mathbf{e}_z = \mathbf{z} - \mathbf{H}(\mathbf{u}, \mathbf{y})\hat{\mathbf{x}}$, and $\mathbf{g}_2(\cdot) \in \mathbb{R}^{n\times 1}$ is a function bounded as $\|\mathbf{g}_2(\cdot)\| \leq \Xi_{max}$, designed such that the Lyapunov function (3) satisfies (4) with some positive $\alpha_i \ for \ i = 1, ..., 5$.
3. There exists a vector function, $\mathbf{T}(\cdot)$, with a Lipschitz constant L_t independent of $\mathbf{u}$, that allows to reconstruct the auxiliary signal as follows:
$$\mathbf{z} = \mathbf{T}(\mathbf{u}, y_1, \dot{y}_1, ..., y_1^{(r_1-1)}, ..., y_m, ..., y_m^{(r_m-1)}), \tag{12}$$
where r_i is the relative degree index between the i_{th} output, y_i, and the unknown parameter vector, $\boldsymbol{\theta}$.

Definition 3.1. *A system depicted by (1) has a relative degree index r_i from the i_{th} output signal, $\boldsymbol{c}_i(\boldsymbol{u}, \boldsymbol{y})\boldsymbol{x}$, to the unknown parameters, $\boldsymbol{\theta}$, if:*

$$\begin{aligned} &L_{\boldsymbol{B}\boldsymbol{\phi}(\boldsymbol{x},\boldsymbol{u})} L^k_{\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})} \boldsymbol{c}_i(\boldsymbol{u}, \boldsymbol{y})\boldsymbol{x} = 0 \quad \forall k < r_i - 1 \\ &L_{\boldsymbol{B}\boldsymbol{\phi}(\boldsymbol{x},\boldsymbol{u})} L^{r_i-1}_{\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})} \boldsymbol{c}_i(\boldsymbol{u}, \boldsymbol{y})\boldsymbol{x} \neq 0. \end{aligned}$$

where $\boldsymbol{c}_i(\boldsymbol{u}, \boldsymbol{y})$ *is the* i_{th} *row of the matrix* $\boldsymbol{C}(\boldsymbol{u}, \boldsymbol{y})$ *and the factor* $L_{\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})}\boldsymbol{c}_i(\boldsymbol{u},\boldsymbol{y})\boldsymbol{x}$ *operation denotes the Lie derivative of the function* $\boldsymbol{c}_i(\boldsymbol{u}, \boldsymbol{y})\boldsymbol{x}$ *along the vector field* $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$.

The first two points allow to proceed with the adaptive observer design independently of the original relative degree of the system. Specifically, the parameter adaptation can be designed as

$$\dot{\hat{\boldsymbol{\theta}}} = \boldsymbol{\phi}(\hat{\mathbf{x}}, \mathbf{u})^{\intercal}\mathbf{M}(t, \mathbf{u}, \mathbf{y})\mathbf{H}(\mathbf{u}, \mathbf{y})\mathbf{e}_x, \tag{13}$$

where $\mathbf{M}(t, \mathbf{u}, \mathbf{y})$ satisfies:

$$\mathbf{M}(t, \mathbf{u}, \mathbf{y})\mathbf{H}(\mathbf{u}, \mathbf{y}) = \mathbf{B}^{\intercal}\mathbf{P}(t). \tag{14}$$

Equality (14) can always be solved by means of point 1.

Remark 3.1. *It should be remarked that the results presented in this work are based on the premise that the system is persistently excited, see Definition Appendix A.1. In the absence of this excitation, the presence of external disturbances may make the parameter-estimation drift to infinity. In the absence of excitation, the parameter drift can be reduced if the parameter-adaptation dynamics (13) are modified to increase its robustness, e.g. through the* σ*-modification or parameter projection [21].*

It is noticeable that the auxiliary signal, $\mathbf{z}$, is not directly computable as it depends on the unknown states. Nonetheless, by means of the point (3), there is a map that relates the auxiliary signal with the input, output and its derivative up to the relative degree. Therefore, it is possible to design a high-gain observer that can achieve an estimation of the auxiliary signal, $\hat{\mathbf{z}}$, which is robust in an ISS sense [19] with respect to the unknown parameters, $\boldsymbol{\theta}$ and states, $\mathbf{x}$. This property is crucial, as it allows the high-gain observer to be coupled with the adaptive observer and have a provable convergence. The next section presents the insights related to the design and performance of said high-gain observer.

3.1. Auxiliary signal estimation through a high-gain observer

This section proposes a reduced-order observer based on the high-gain observer ideas, that is going to be used to estimate the auxiliary signal, $\mathbf{z}$, through expression (12).

Consider a set of coordinates $\boldsymbol{\xi} = [\boldsymbol{\xi}^1, ..., \boldsymbol{\xi}^m]^\intercal$, where $\boldsymbol{\xi}^i \in \mathbb{R}^{r_i \times 1}$ is defined $\forall i = 1, ..., m$ as:

$$\boldsymbol{\xi}^i = \begin{bmatrix} y_i \\ \dot{y}_i \\ \vdots \\ y_i^{(r_i-1)} \end{bmatrix}.$$

The dynamics of the $\boldsymbol{\xi}$ coordinates are depicted by

$$\begin{aligned} \dot{\boldsymbol{\xi}}^i &= \mathbf{A}\boldsymbol{\xi}^i + \boldsymbol{\Psi}^i(\boldsymbol{\xi}^i, \bar{\mathbf{u}}, \mathbf{x}, \boldsymbol{\theta}) + \mathbf{B}_{w,i} d_i, \quad \forall i = 1, ..., m \\ y^i &= \mathbf{C}\boldsymbol{\xi}^i + v_i, \end{aligned} \tag{15}$$

where d_i are unknown disturbances upper-bounded as $\|d_i\| \leq M_i$, v_i is the noise in the i_{th} output and $\mathbf{A}_i \in \mathbb{R}^{r_i \times r_i}$, $\mathbf{B}_{w,i} \in \mathbb{R}^{r_i \times 1}$ and $\mathbf{C}_i \in \mathbb{R}^{1 \times r_i}$ are

$$\mathbf{A}_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}; \quad \mathbf{B}_{w,i} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}; \quad \mathbf{C}_i = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^\intercal$$

$$\boldsymbol{\Psi}^i(\boldsymbol{\xi}^i, \bar{\mathbf{u}}, \mathbf{x}, \boldsymbol{\theta}) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \psi_{r_i}(\boldsymbol{\xi}, \mathbf{x}, \boldsymbol{\theta}, \mathbf{u}, ..., \mathbf{u}^{(r_i - r_{u,i})}) \end{bmatrix} \tag{16}$$

where $r_{u,i}$ is the relative degree between the output, y_i, and the input $\mathbf{u}$ and $\boldsymbol{\Psi}^i$ is Lipschitz.

The expression (15) is a well-known triangular structure in which a high-gain observer can be designed [49]. Specifically,

$$\dot{\hat{\boldsymbol{\xi}}}^i = \mathbf{A}_i \hat{\boldsymbol{\xi}}^i + \boldsymbol{\Psi}^i(\hat{\boldsymbol{\xi}}^i, \bar{\mathbf{u}}_i, \hat{\mathbf{x}}, \hat{\boldsymbol{\theta}}) + \mathbf{E}_i \mathbf{l}_i (y^i - \hat{\xi}_1^i) \tag{17}$$

where $\bar{\mathbf{u}}_i = [\mathbf{u}, ..., \mathbf{u}^{(r_i - r_{u,i})}]^\intercal$, $\mathbf{E}_i \in \mathbb{R}^{r_i \times r_i}$ and $\mathbf{l}_i \in \mathbb{R}^{r_i \times 1}$ are:

$$\mathbf{E}_i = \begin{bmatrix} \frac{1}{\varepsilon} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\varepsilon^{r_i}} \end{bmatrix}; \quad \mathbf{l}_i = \begin{bmatrix} l_{1,i} \\ \vdots \\ l_{r_i,i} \end{bmatrix}.$$

Remark 3.2. *The factor $\boldsymbol{\Psi}_i$ of the observer (17) depends on the state-estimation, $\hat{\boldsymbol{x}}$, and the parameter-estimation, $\hat{\boldsymbol{\theta}}$, of the adaptive observer. Coupling the observers in this way allows to have provable convergence of the state and parameter-estimation, and reduce the feedback gain of the high-gain observer, which improves the transient performance and reduces its noise sensitivity [50].*

Remark 3.3. *The factor $\boldsymbol{\Psi}_i$ depends on the derivative of the inputs, which, in some cases, may be unknown. These derivatives may be robustly estimated through a differentiator [51]. Alternatively, they may be considered as an unknown disturbance and be appended in the factor d_i.*

Observer (17) presents two design elements: the parameters $l_{j,i}\ for\ j = 1, ..., r_i$ and the gain ε_i. First, the factors $l_{j,i}\ for\ i = 1, ..., m$ have to be chosen so the polynomial

$$s^n + l_{1,i}s^{n-1} + \cdots + l_{n-1,i}s + l_{n,i}$$

is Hurwitz. Second, there exist a positive constant ε_1^* such that, for $\varepsilon < min\{1, \varepsilon_1^*\}$, the estimation error, $\boldsymbol{\xi} - \hat{\boldsymbol{\xi}}$, is ISS with respect to the sensor noise, the unknown disturbances and the adaptive observer estimation errors $\mathbf{x} - \hat{\mathbf{x}}$ and $\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}$. Specifically, define the vector $\boldsymbol{\chi} = [\mathbf{x} - \hat{\mathbf{x}}, \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}]^\intercal$, then, it is well-known that for a high-gain observer tuned as specified in this section, the estimation error converges to the following ultimate bound [50]:

$$\|\boldsymbol{\xi} - \hat{\boldsymbol{\xi}}\| \leq \varepsilon k_1 \max\{M_1, ..., M_m\} + \varepsilon k_2 \|\boldsymbol{\chi}\| + \frac{1}{\varepsilon^r} k_3 \vartheta \tag{18}$$

where $k_1, ..., k_3$ are some positive constants.

Theorem 3.1. *Consider the system (15) and the high-gain observer (17). Define the estimation of the auxiliary signal as $\hat{\boldsymbol{z}} = \boldsymbol{T}(\boldsymbol{u}, \hat{\boldsymbol{\xi}})$. Then, the auxiliary signal estimation error is ultimately bounded as*

$$\|\boldsymbol{z} - \hat{\boldsymbol{z}}\| \leq \varepsilon k_1 L_T \max\{M_1, ..., M_m\} + \varepsilon k_2 L_T \|\boldsymbol{\chi}\| + \frac{1}{\varepsilon^r} k_3 L_T \vartheta \tag{19}$$

Proof 3.1. *The function $\boldsymbol{T}(\cdot)$ is assumed to be Lipschitz with a constant L_T. Therefore, the following bound holds*

$$\begin{aligned} \|\boldsymbol{z} - \hat{\boldsymbol{z}}\| \leq L_T \|\boldsymbol{\xi} - \hat{\boldsymbol{\xi}}\| &\leq \varepsilon k_1 L_T \max\{M_1, ..., M_m\} \\ &+ \varepsilon k_2 L_T \|\boldsymbol{\chi}\| + \frac{1}{\varepsilon^r} k_3 L_T \vartheta. \end{aligned}$$

$\square$

4. Performance of the adaptive observer and high-gain observer coupling

Section 2 has presented an adaptive redesign for nonlinear observer that significantly improves the performance of the observer, but, can only be implemented in systems with relative degree zero or one between the output and the unknown parameters. Section 3 has presented a methodology to circumvent this relative degree restriction through the use of an auxiliary signal. Such signal is not directly measurable, however, can be estimated through a high-gain observer. Now it is crucial to analyze under which conditions the adaptive observer and the high-gain observer coupling is stable and which is its performance in the presence of sensor noise and uncertainty.

The Barbalat's lemma argument (see Appendix A) is the common approach to prove the parameter-estimation convergence of adaptive observers [24][31]. Nonetheless, such argument gives no insights of the parameter-estimation performance in presence of sensor noise and/or uncertainty. Consequently, even though the design of adaptive observers has been studied in previous works, very few results relative to the parameter-estimation performance have been provided in the considered case. Moreover, even if we consider the simpler case of noise/uncertainty absence, $\mathbf{v} = \mathbf{w} = 0$, the Barbalat's argument is insufficient to prove the stability of the high-gain observer and adaptive observer coupling.

For this reason, it is convenient to substitute the weak (weak in the sense that its derivative is only negative semidefinite) Lyapunov function (5) for a strict Lyapunov function that can be used to proceed with the analysis. In this work, it is proposed to use the *Mazenc* construction [52] to derive a strict Lyapunov function from (5), which will allows to define the accuracy and convergence rate of the adaptive observer through standard Lyapunov arguments. This approach has proved to be successful in multiple adaptive control problems [53][54].

4.1. Strict Lyapunov function design

This section considers an observer of the form (11), the parameter adaptation (13) and the auxiliary signal estimated through a high-gain observer as presented in Section 3, which has an estimation error defined by (19). The objective is to design a strict Lyapunov function for the mentioned adaptive observer.

First, define the following locally Lipschitz time-varying function

$$\Lambda \triangleq \sqrt{\lambda_{min}(\boldsymbol{\phi}(t)^\intercal \mathbf{B}^\intercal \mathbf{B}\boldsymbol{\phi}(t))}. \tag{20}$$

As the function $\mathbf{B}\boldsymbol{\phi}$ is Lipschitz and bounded, it can be shown that function (20) is also bounded with a bounded derivative. As consequence, there is a value $\bar{\Lambda}$ such that

$$max\{\|\Lambda\|, \|\dot{\Lambda}\|\} \leq \bar{\Lambda}.$$

Moreover, define the following signal:

$$\Upsilon_\Lambda = 1 + 2\bar{\Lambda}T_0 - \frac{2}{T_0}\int_t^{t+T_0}\int_t^m \Lambda(s)^2 ds\ dm. \tag{21}$$

If we assume that the system is persistently excited, as defined in Appendix A, function (21) is bounded as follows [52]

$$1 \leq \Upsilon_\Lambda \leq 1 + 2\bar{\Lambda}T_0.$$

Furthermore, there exists an upper bound for the derivative of (21) [52],

$$\dot{\Upsilon}_\Lambda \leq -\frac{2\mu_2}{T_0} + 2\Lambda^2.$$

Theorem 4.1. *Consider system (1) and the observer (11). Assume that the system is persistently excited as defined in Appendix A. Then, the system admits the following Lyapunov function*

$$V_1 = -\boldsymbol{e}_x^\intercal \boldsymbol{B}\boldsymbol{\phi}(\hat{\boldsymbol{x}}, \boldsymbol{u})\boldsymbol{e}_{\boldsymbol{\theta}} + \frac{1}{2}(\Upsilon_\Lambda + \alpha)(\boldsymbol{e}_x^\intercal \boldsymbol{P}(t)\boldsymbol{e}_x + \boldsymbol{e}_{\boldsymbol{\theta}}^\intercal \boldsymbol{e}_{\boldsymbol{\theta}}), \tag{22}$$

which satisfies the following

$$\begin{aligned}\dot{V}_1 \leq &-\frac{\mu_2}{T_0}\lambda_{min}(\boldsymbol{P}(t))\|\boldsymbol{e}_x\|^2 - \frac{\mu_2}{2T_0}\|\boldsymbol{e}_\theta\|^2 + k_4\|\boldsymbol{e}_\theta\|\|\boldsymbol{z} - \hat{\boldsymbol{z}}\| \\ &+ k_5\|\boldsymbol{e}_x\|\|\boldsymbol{z} - \hat{\boldsymbol{z}}\| + k_6\|\boldsymbol{e}_\theta\|w_2 + k_7\|\boldsymbol{e}_x\|w_2\end{aligned} \tag{23}$$

where k_i for $i = 4, ..., 7$ are some positive constants to be defined, provided that

$$\begin{aligned}\alpha \geq max\{&\alpha_3^{-1}\big(\frac{T_0}{\mu_2}(L_f + \|\boldsymbol{B}\|L_\phi\boldsymbol{\theta} + H_{max})^2\|\boldsymbol{B}^2\|\phi_{max}\|^2 \\ &+ \frac{T_0}{\mu_2}\|\boldsymbol{B}\|^2\|\dot{\boldsymbol{\phi}}(\hat{\boldsymbol{x}}, \boldsymbol{u})\|^2\big), 2\|\boldsymbol{B}\|^2\phi_{max}^2\lambda_{min}(\boldsymbol{P}(t))^{-1}, 1\}.\end{aligned} \tag{24}$$

Proof 4.1. *Let* $\boldsymbol{\chi} = [\boldsymbol{e}_x, \boldsymbol{e}_{\boldsymbol{\theta}}]^{\intercal}$*. Then, in view of (24), the Lyapunov function (22) is positive definite and radially unbounded. Specifically, there exists some constants* $V_{1,min}, V_{1,max} > 0$*, such that*

$$V_{1,min}\|\boldsymbol{\chi}\|^2 \leq V_1 \leq V_{1,max}\|\boldsymbol{\chi}\|^2 \tag{25}$$

where

$$V_{1,min} = min\{\frac{1}{2}(1+\alpha)\lambda_{min}(\boldsymbol{P}(t)) - \|\boldsymbol{B}\|^2\phi_{max}^2, \frac{1}{2}(1+\alpha) - 1\} \tag{26}$$

$$\begin{aligned} V_{1,max} = max\{&\|\boldsymbol{B}\|^2\phi_{max}^2 + (\frac{1}{2} + \bar{\Lambda}T_0 + \frac{\alpha}{2})\lambda_{max}(\boldsymbol{P}(t)), \\ &1 + (\frac{1}{2} + \bar{\Lambda}T_0 + \frac{\alpha}{2})\}. \end{aligned} \tag{27}$$

Notice that, by considering the auxiliary signal, $\boldsymbol{z}$*, the derivative of the function (6) becomes:*

$$\begin{aligned} \frac{\partial V_{x,\theta}}{\partial t} + \frac{\partial V_{x,\theta}}{\partial \boldsymbol{e}_x}\dot{\boldsymbol{e}}_x \leq &-\alpha_3\|\boldsymbol{e}_x\|^2 + \alpha_4\|\boldsymbol{e}_x\|\|\boldsymbol{w}\| \\ &+ \|\boldsymbol{e}_x\|\lambda_{max}(\boldsymbol{P}(t))\Xi_{max}\|\boldsymbol{z} - \hat{\boldsymbol{z}}\| \\ &+ \|\boldsymbol{e}_\theta\|\phi_{max}\|\boldsymbol{M}\|\|\boldsymbol{z} - \hat{\boldsymbol{z}}\|. \end{aligned}$$

The derivative of (22) satisfies the following:

$$\begin{aligned} \dot{V}_1 \leq &-\big[\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) - \boldsymbol{f}(\hat{\boldsymbol{x}}, \boldsymbol{u}) - \boldsymbol{g}_2(\hat{\boldsymbol{x}}, \boldsymbol{u})\boldsymbol{e}_z \\ &\boldsymbol{B}\boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{u})\boldsymbol{\theta} - \boldsymbol{B}\boldsymbol{\phi}(\hat{\boldsymbol{x}}, \boldsymbol{u})\hat{\boldsymbol{\theta}} + \boldsymbol{w}\big]^{\intercal}\boldsymbol{B}\boldsymbol{\phi}(\hat{\boldsymbol{x}}, \boldsymbol{u})\boldsymbol{e}_\theta \\ &- \boldsymbol{e}_x^{\intercal}\boldsymbol{B}\dot{\boldsymbol{\phi}}(\hat{\boldsymbol{x}}, \boldsymbol{u})\boldsymbol{e}_\theta - \boldsymbol{e}_x^{\intercal}\boldsymbol{B}\boldsymbol{\phi}(\hat{\boldsymbol{x}}, \boldsymbol{u})\boldsymbol{\phi}(\hat{\boldsymbol{x}}, \boldsymbol{u})^{\intercal}\boldsymbol{B}^{\intercal}\boldsymbol{P}(t)\boldsymbol{e}_x \\ &+ (\Upsilon_\Lambda + \alpha)\left(\frac{\partial V_{x,\theta}}{\partial t} + \frac{\partial V_{x,\theta}}{\partial \boldsymbol{e}_x}\dot{\boldsymbol{e}}_x\right) + \frac{1}{2}\dot{\Upsilon}_\Lambda(\boldsymbol{e}_x^{\intercal}\boldsymbol{P}(t)\boldsymbol{e}_x + \boldsymbol{e}_{\boldsymbol{\theta}}^{\intercal}\boldsymbol{e}_{\boldsymbol{\theta}}) \\ &\leq \|\boldsymbol{e}_x\|(L_f + \|\boldsymbol{B}\|L_\phi\boldsymbol{\theta} + H_{max})\|\boldsymbol{B}\|\phi_{max}\|\boldsymbol{e}_{\boldsymbol{\theta}}\| \\ &+ (\Xi_{max}\|\boldsymbol{z} - \hat{\boldsymbol{z}}\| + w_2)\|\boldsymbol{B}\|\phi_{max}\|\boldsymbol{e}_{\boldsymbol{\theta}}\| \\ &+ \|\boldsymbol{e}_x\|\|\boldsymbol{B}\|\|\dot{\boldsymbol{\phi}}(\hat{\boldsymbol{x}}, \boldsymbol{u})\|\|\boldsymbol{e}_\theta\| - \alpha_3\alpha\|\boldsymbol{e}_x\|^2 \\ &+ (\alpha + 1 + 2\bar{\Lambda}T_0)\|\boldsymbol{e}_x\|\lambda_{max}(\boldsymbol{P}(t))\Xi_{max}\|\boldsymbol{z} - \hat{\boldsymbol{z}}\| \\ &+ (\alpha + 1 + 2\bar{\Lambda}T_0)\|\boldsymbol{e}_\theta\|\phi_{max}\|\boldsymbol{M}\|\|\boldsymbol{z} - \hat{\boldsymbol{z}}\| \\ &+ (\alpha + 1 + 2\bar{\Lambda}T_0)\|\boldsymbol{e}_x\|\alpha_4 w_2 \\ &- \frac{\mu_2}{T_0}\|\boldsymbol{e}_{\boldsymbol{\theta}}\|^2 - \frac{\mu_2}{T_0}\lambda_{min}(\boldsymbol{P}(t))\|\boldsymbol{e}_x\|^2. \end{aligned} \tag{28}$$

Then, if one applies Young's inequality to (28) as follows:

$$\begin{aligned}&\|\boldsymbol{e}_x\|(L_f + \|\boldsymbol{B}\|L_\phi\boldsymbol{\theta} + H_{max})\|\boldsymbol{B}\|\phi_{max}\|\boldsymbol{e}_{\boldsymbol{\theta}}\|\\ &\leq \frac{\epsilon}{2}(L_f + \|\boldsymbol{B}\|L_\phi\boldsymbol{\theta} + H_{max})^2\|\boldsymbol{B}^2\|\phi_{max}\|^2\|\boldsymbol{e}_x\|^2\\ &+ \frac{1}{2\epsilon}\|\boldsymbol{e}_{\boldsymbol{\theta}}\|^2\end{aligned}$$

$$\begin{aligned}&\|\boldsymbol{e}_x\|\|\boldsymbol{B}\|\|\dot{\boldsymbol{\phi}}(\hat{\boldsymbol{x}}, \boldsymbol{u})\|\|\boldsymbol{e}_\theta\|\\ &\leq \frac{\epsilon}{2}\|\boldsymbol{B}\|^2\|\dot{\boldsymbol{\phi}}(\hat{\boldsymbol{x}}, \boldsymbol{u})\|^2\|\boldsymbol{e}_x\|^2 + \frac{1}{2\epsilon}\|\boldsymbol{e}_{\boldsymbol{\theta}}\|^2,\end{aligned}$$

defines $\epsilon = \dfrac{2T_0}{\mu_2}$ *and considers the relation (24), the bound depicted in (23) can be deduced, where*

$$\begin{aligned}k_4 &= \|\boldsymbol{B}\|\phi_{max}\Xi_{max} + (\alpha + 1 + 2\bar{\Lambda}T_0)\phi_{max}\|\boldsymbol{M}\|\\ k_5 &= (\alpha + 1 + 2\bar{\Lambda}T_0)\lambda_{max}(\boldsymbol{P}(t))\Xi_{max}\\ k_6 &= \|\boldsymbol{B}\|\phi_{max}\\ k_7 &= (\alpha + 1 + 2\bar{\Lambda}T_0)\alpha_4.\end{aligned}$$

□

4.2. Stability conditions for the observer

The aim of this subsection is to develop the conditions in which the high-gain observer and adaptive observer coupling is stable. In this subsection, it will be considered the case without uncertainty/noise, i.e. $\mathbf{w} = \mathbf{v} = 0$. The effect of these disturbances on the stability and performance will be analyzed in the next subsection.

Theorem 4.1 establishes that the proposed adaptive observer with the estimated auxiliary signal is stable in a ISS sense taking the auxiliary signal estimation error, $\|\mathbf{z} - \hat{\mathbf{z}}\|$, as an input. This fact can be formalized through the following theorem.

Theorem 4.2. *Consider the Lyapunov function (22), which satisfies (28). Then, if* $\boldsymbol{w} = \boldsymbol{v} = 0$*, the estimation error* $\|\boldsymbol{\chi}\|$ *is ultimately bounded as*

$$\|\boldsymbol{\chi}\| \leq k_8\|\boldsymbol{z} - \hat{\boldsymbol{z}}\|, \tag{29}$$

where k_8 *is a positive constant.*

Proof 4.2. *Considering* $\boldsymbol{v} = \boldsymbol{w} = 0$*, the bound (23) reduces to:*

$$\begin{aligned}\dot{V}_1 \leq& -\frac{\mu_2}{T_0}\lambda_{min}(\boldsymbol{P}(t))\|\boldsymbol{e}_x\|^2 - \frac{\mu_2}{2T_0}\|\boldsymbol{e}_\theta\|^2 + k_4\|\boldsymbol{e}_\theta\|\|\boldsymbol{z}-\hat{\boldsymbol{z}}\| \\ &+ k_5\|\boldsymbol{e}_x\|\|\boldsymbol{z}-\hat{\boldsymbol{z}}\| \\ &\leq -\frac{\mu_2}{T_0}\min\{\lambda_{min}(\boldsymbol{P}(t)), \frac{1}{2}\}\|\boldsymbol{\chi}\|^2 + \max\{k_4, k_5\}\|\boldsymbol{\chi}\|\|\boldsymbol{z}-\hat{\boldsymbol{z}}\|. \end{aligned} \tag{30}$$

It can be seen that for the region:

$$\|\boldsymbol{\chi}\| \geq 2\frac{\max\{k_4, k_5\}}{\frac{\mu_2}{T_0}\min\{\lambda_{min}(\boldsymbol{P}(t)), \frac{1}{2}\}}\|\boldsymbol{z}-\hat{\boldsymbol{z}}\|.$$

The derivative (30) is bounded as:

$$\dot{V}_1 \leq -\frac{1}{2}\|\boldsymbol{\chi}\|^2.$$

Then, from the comparison lemma and input to state stability theory [19], it is possible to deduce the following ultimate bound for the adaptive observer state and parameter-estimation:

$$\|\boldsymbol{\chi}\| \leq 2\sqrt{\frac{V_{1,max}}{V_{1,min}}}\frac{\max\{k_4, k_5\}}{\frac{\mu_2}{T_0}\min\{\lambda_{min}(\boldsymbol{P}(t)), \frac{1}{2}\}}\|\boldsymbol{z}-\hat{\boldsymbol{z}}\|.$$

□

In parallel, from the bound (19), it can be seen that the auxiliary signal is stable in an ISS sense [19] taking the adaptive estimation error, $\boldsymbol{\chi}$, as an input. The definition of these ultimate bounds (19) and (29) makes the small-gain theorem [19] a convenient method to prove the stability of the observer coupling.

Theorem 4.3. *Consider the system (15), the high-gain observer depicted in (17) tuned to ensure the bound (19) and the adaptive observer (11) and (13), which satisfies the bound (29). Assume that* $\boldsymbol{v} = \boldsymbol{w} = 0$*, then, the auxiliary signal estimation error,* $\|\boldsymbol{z}-\hat{\boldsymbol{z}}\|$*, and the adaptive observer estimation error,* $\|\boldsymbol{\chi}\|$*, converges to zero provided that*

$$\varepsilon k_2 L_T k_8 < 1. \tag{31}$$

Proof 4.3. *If condition (31) is satisfied, the ultimate bounds (19) and (29) define a contraction. Therefore, by the small-gain theorem [19] it can be shown that* $\|\boldsymbol{z}-\hat{\boldsymbol{z}}\|$*, and the adaptive observer estimation error,* $\|\boldsymbol{\chi}\|$ *converges to zero.* □

Remark 4.1. *Notice that condition (31) can be rearranged as:*

$$\varepsilon < \frac{1}{k_2 L_T k_8} \triangleq \varepsilon_2^*.$$

Therefore, in the conditions where (19) and (29) are satisfied, the stability of the observer coupling can be ensured by reducing enough the high-gain observer parameter ε*.*

Remark 4.2. *As it will be seen in the next subsection, it may be convenient to increase* ε_2^* *in order to reduce the noise sensitivity of the observer scheme. This can be achieved by reducing either* k_2 *or* k_8*. The factor* k_2 *can be reduced by the proper tuning of the parameters* $\boldsymbol{l}_i$ for $i = 1, ..., m$*. The factor* k_8 *can be reduced by increasing the excitation of the system, which implies an increment of the factor* $\dfrac{\mu_2}{T_0}$ *and a reduction of the constants* k_4 *and* k_5*.*

4.3. Performance of the observer under sensor noise and model uncertainty

Last section has established, in the absence of uncertainty/noise, $\mathbf{w} = \mathbf{v} = 0$, the conditions in which the high-gain observer and adaptive observer coupling estimation converges to the states and parameters true value. This subsection will extend these results to the case where $\mathbf{w} \neq 0$ and $\mathbf{v} \neq 0$ and will show that the estimation converges to a bounded error in the presence of this uncertainty.

Theorem 4.4. *Consider the system (15), the high-gain observer depicted in (17) tuned to ensure the bound (19) and the adaptive observer (11) and (13) which satisfies the bound (29). Assume that the condition (31) is satisfied. Then, the adaptive observer estimation,* $\|\boldsymbol{\chi}\|$*, converges to the following ultimate bound:*

$$\|\boldsymbol{\chi}\| \leq max\{\varepsilon k_8 \max\{M_1, ..., M_m\}, \frac{1}{\varepsilon^r} k_9 \vartheta, k_{10} w_2\} \tag{32}$$

where $k_8, ..., k_{10}$ *are some positive constant to be defined.*

Proof 4.4. *By substituting the bound (19) in (28), one obtains:*

$$\begin{aligned}\dot{V}_1 \leq & -Q\|\boldsymbol{\chi}\|^2 \\ & + \varepsilon \max\{k_4, k_5\}\|\boldsymbol{\chi}\| k_1 L_T \max\{M_1, ..., M_m\} \\ & + \frac{1}{\varepsilon^r} \max\{k_4, k_5\}\|\boldsymbol{\chi}\| k_3 L_T \vartheta \\ & + max\{k_6, k_7\}\|\boldsymbol{\chi}\| w_2. \end{aligned} \tag{33}$$

where Q is a positive constant defined as

$$Q = \big(\frac{\mu_2}{T_0} \min\{\lambda_{min}(\boldsymbol{P}(t)), \frac{1}{2}\} - \varepsilon k_2 L_T max\{k_4, k_5\}\big).$$

The factor Q is positive by means of (31).

It can be shown that in the region

$$\|\boldsymbol{\chi}\| \geq \max\{\varepsilon C1 \max\{M_1, ..., M_m\}, \frac{1}{\varepsilon^r} C2\vartheta, C3 w_2\}$$

where

$$\begin{aligned} C1 &= 2\frac{1}{Q} \max\{k_4, k_5\} k_1 L_T \max\{M_1, ..., M_m\} \\ C2 &= 2\frac{1}{Q} \max\{k_4, k_5\} k_3 L_T \vartheta \\ C3 &= 2\frac{1}{Q} max\{k_6, k_7\} w_2, \end{aligned}$$

the derivative (30) is bounded as:

$$\dot{V}_1 \leq -\frac{1}{2}\|\boldsymbol{\chi}\|^2.$$

Then, from the comparison lemma and input to state stability theory [19], it is possible to deduce the ultimate bound (32) with

$$\begin{aligned} k_8 &= 2\sqrt{\frac{V_{1,max}}{V_{1,min}}} \frac{1}{Q} \max\{k_4, k_5\} k_1 L_T \\ k_9 &= 2\sqrt{\frac{V_{1,max}}{V_{1,min}}} \frac{1}{Q} \max\{k_4, k_5\} k_3 L_T \\ k_{10} &= 2\sqrt{\frac{V_{1,max}}{V_{1,min}} \frac{1}{Q}} max\{k_6, k_7\} \end{aligned}$$

$\square$

Naturally, Theorem 4.4 depicts that the presence of sensor noise and model uncertainty introduces a bias in the state and parameter-estimation of the adaptive observer. Nonetheless, now it is possible to present some insights on how the observer parameter tuning can reduce the effect of noise/uncertainty on the accuracy of the estimation.

It is clear that the first term on the left-hand side in (32) can be arbitrarily reduced by decreasing the high-gain observer parameter, ε. Nonetheless, the reduction of ε increases the effect of the sensor noise, second term on the left-hand side in (32), which limits the value of ε. This property is a consequence of the well-known trade-off between noise sensitivity and disturbance rejection of high-gain observers [50] and observers in general [20]. From (32), it is possible to show that there is an optimal ε value, in terms of maximizing disturbance rejection and reducing noise sensitivity, achieved in:

$$\varepsilon = \sqrt[r-1]{\frac{k_9 \vartheta}{k_8 \max\{M_1, ..., M_m\}}}. \tag{34}$$

Further estimation error reduction can be achieved by increasing the excitation of the system, which implies an increment of the factor $\dfrac{\mu_2}{T_0}$ and a reduction of the constants k_8 and k_9. Moreover, the increase in the excitation also increases the constant Q, which reduces all the factors in left-hand side of (32). Furthermore, the constants k_8 and k_9 can also be reduced by decreasing the constants k_1 and k_3, which can be achieved by the proper tuning of the parameters $\mathbf{l}_i \ for \ i = 1, ..., m$ of the high-gain observer.

Remark 4.3. *In the case of time-varying parameters, i.e.* $\dot{\boldsymbol{\theta}} \neq 0$*. There will be a factor dependent on* $\dot{\boldsymbol{\theta}}$ *in (23). This factor can be interpreted as an unmodelled disturbance and be appended in* $\boldsymbol{w}$*, which does not modify the conclusions drawn in this section. This fact shows how the proposed Lyapunov function can be used to analyze the performance of the observer with "slow" time-varying parameters.*

This section has presented the mathematical formalisms that allows to prove the performance and stability of the proposed observer scheme. Now, it is interesting to validate this results in a practical example. Next sections will apply the proposed technique in a synthetic system and in a compartmental epidemiology model. Before introducing the concerning systems, it is convenient to establish the following result.

Theorem 4.5. *Consider a system of the form*

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{B}\boldsymbol{\phi}(\boldsymbol{x}, t)\boldsymbol{\theta} + \boldsymbol{w} \\ z &= \boldsymbol{h}^{\intercal}(t)\boldsymbol{x} + \boldsymbol{v},\end{aligned} \tag{35}$$

and assume that there are no unknown parameters, i.e. $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}$. *Moreover, consider the following observer:*

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}(t)\hat{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{\phi}(\hat{\boldsymbol{x}}, t)\hat{\boldsymbol{\theta}} + \boldsymbol{K}(t)(z - \boldsymbol{h}^{\intercal}(t)\hat{\boldsymbol{x}}) \tag{36}$$

$$\dot{\boldsymbol{P}}(t) = -\sigma\boldsymbol{P}(t) - \boldsymbol{A}(t)^{\intercal}\boldsymbol{P}(t) - \boldsymbol{P}(t)\boldsymbol{A}(t) + \boldsymbol{h}(t)\boldsymbol{h}^{\intercal}(t) \tag{37}$$

where $\boldsymbol{P}(0) = \boldsymbol{P}(0)^{\intercal} > 0$ *and* $\boldsymbol{K}(t)$ *is a time-varying matrix defined as follows*

$$\boldsymbol{K}(t) = \boldsymbol{P}(t)^{-1}\boldsymbol{h}(t). \tag{38}$$

Consider the Lyapunov function (3), with the matrix $\boldsymbol{P}(t)$ *computed through (37). Then, the inequalities in (4) are satisfied if the parameter* σ *is designed such that* $\sigma > max\{2|\lambda_{max}(\boldsymbol{A})|, \sigma^*\}$, *where* σ^* *is a positive constant to be defined and the pair* $\Big(\boldsymbol{A}(t), \boldsymbol{h}(t)\Big)$ *is uniform completely observable, as defined in [55].*

Proof 4.5. *The proof has been included in Appendix B.*

5. Numerical validation in a synthetic example

Consider a second order system, $\mathbf{x} = [x_1, x_2]^{\intercal}$, with two unknown parameters, $\boldsymbol{\theta} = [\theta_1, \theta_2]^{\intercal}$, and the form:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}(u)\mathbf{x} + \mathbf{b}\boldsymbol{\phi}(\mathbf{x})\boldsymbol{\theta} + \mathbf{b}\mathbf{w} \\ z &= \mathbf{c}\mathbf{x} + \mathbf{v},\end{aligned}$$

with:

$$\mathbf{A}(u) = \begin{bmatrix} -0.3 & u \\ 0 & 0 \end{bmatrix}; \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad \mathbf{c} = \begin{bmatrix} 1 & 0 \end{bmatrix};$$

$$\boldsymbol{\phi}(\mathbf{x}) = \begin{bmatrix} -x_1 & (1 - x_1^2)x_2 \end{bmatrix}. \tag{39}$$

The input is defined as a time-dependant signal of the form $u = 0.2\sin\left(\frac{t}{5}\right) + 0.5$. The system is disturbed with an unmodelled factor $\mathbf{w} = 0.01\sin(t)$. The output signal is corrupted with some zero-mean high-frequency noise, $\mathbf{v}$, with variance 0.01. The value of the unknown parameters are summarized in Table 1.

The objective is to design an observer for the joint state and parameter problem. This problem is of interest for multiple reasons. First, the system presents singular inputs [55], e.g. the condition $u = 0$ drives the system unobservable, which prevents the transformation of the system to the standard observer canonical form [2]. Second, the relative degree between the measured output and the unknown parameters is larger than one, which prevents the adaptive modification introduced in Section 2. To see this fact, notice that $\mathbf{cb} = 0$, therefore, the rank condition in (10) is not satisfied.

Firstly, it is required to design an auxiliary signal, z, that satisfies the points presented in Section 3. It can be seen that the signal $z = x_1 + ux_2 = \mathbf{h}^\intercal(t)\mathbf{x}$, where $\mathbf{h}(t) = [1, u]$, is relative degree 1 with respect to the unknown parameters, $\boldsymbol{\theta}$. This is a result of the equality $\mathbf{h}(t)\mathbf{b} = ux_2$, which satisfies the rank condition in (10) for all $u \neq 0$. Furthermore, the pair $\Big(\mathbf{A}(t), \mathbf{h}(t)\Big)$ is uniform completely observable, thus, the states can be estimated through the observer (36)-(37). Finally, this signal can be reconstructed as $z = \dot{y}$, thus, it satisfies condition (12) and the parameters can be adapted through (13) that satisfies (14). Specifically, the parameter adaptation takes the form (13) with

$$\mathbf{M}(t) = \mathbf{b}^\intercal(\mathbf{K}(t)^\intercal)^\dagger \tag{40}$$

where $(\mathbf{K}(t)^\intercal)^\dagger$ is the left Moore-Penrose pseudo-inverse computed as

$$(\mathbf{K}(t)^\intercal)^\dagger = (\mathbf{K}(t)\mathbf{K}(t)^\intercal)^{-1}\mathbf{K}(t). \tag{41}$$

To better understand this design of the matrix $\mathbf{M}$, notice that from (38) it can be deduced that

$$\mathbf{P}(t) = (\mathbf{K}(t)^\intercal)^\dagger\mathbf{h}^\top(t).$$

Then, it is direct to see that (40) solves the equation in (9).

In the concerned system, the constant σ^* is lower than $2|\lambda_{max}(\mathbf{A})| = 0.6$, thus, the adaptive observer parameter has been tuned as $\sigma = 1.1 > max\{\sigma^*, 0.6\}$. Applying the points deduced in Section 2, the high-gain observer has been tuned to have adequate convergence rate and satisfy the condition (31), while presenting adequate noise performance and disturbance rejection. Specifically, the observer parameters are summarized in Table 1.

Table 1: True model parameters and observer design parameters

Parameter	Value
True Parameters	
θ_1	0.3
θ_2	1
Observer Parameters	
σ	1.1
α_1	3
α_2	2
ε	0.81

It should be remarked that the considered system satisfies the persistent excitation condition defined in Appendix 1. Therefore, according to the theory presented in this work, both, state and parameter-estimation converges to a bounded error.

One of the key-points of this work is to address the relative degree limitation of adaptive observers without relying on filters. For this reason, it is convenient to compare the performance of the proposed approach with an already existing technique that does use filters. Due to the structure of the proposed example, it would be reasonable to use the standard filter-based adaptive observer proposed in [35]. Therefore, the proposed technique will be compared with the technique in [35]. To ease the readability of the section, the details of the design of this adaptive observer will be obviated.

The evolution of the state-estimation error of both observers can be seen in Fig. 2. Furthermore, the evolution of the parameter-estimation error can be observed in Fig. 3. As it can be observed, the estimation of both techniques converge to a relatively similar bounded error. Nonetheless, it is appreciable that the convergence rate of estimation error in the proposed approach is significantly faster in the proposed approach. The slow convergence rate is a consequence of the introduction of filters, which reduces the signals excitation levels and induces a slow parameter-estimation convergence. It should be remarked, that faster convergence rate could be achieved by increasing the gain of the observer. Nonetheless, this would significantly increase the sensitivity to sensor noise. This result exemplifies the motivation of avoiding filters in order to solve the relative degree restriction in adaptive observers.

Finally, in relation to the proposed approach, the estimation error converges to a relative error below the 2%. This result validates the performance of the proposed scheme under measurement noise and unmodelled disturbances.

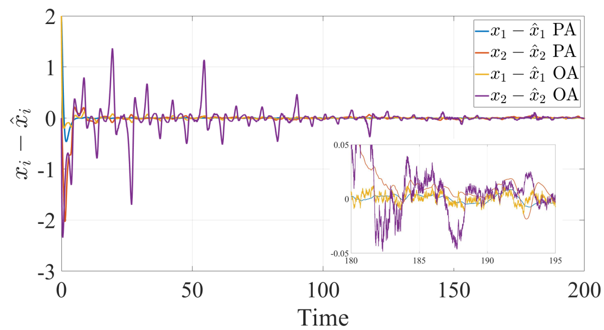

Figure 2: Evolution of the state-estimation error. Blue and orange lines depict the state-estimation error of the proposed approach (PA). Yellow and purple lines depict the state-estimation error of the approach in [35] (OA).

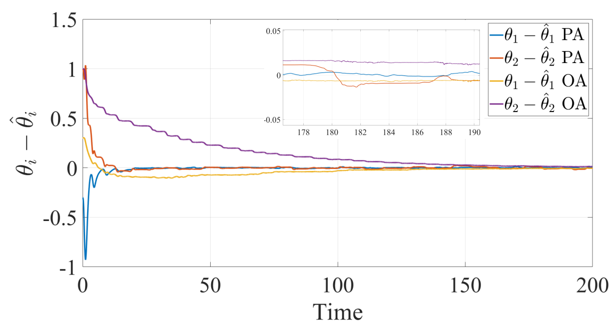

Figure 3: Evolution of the parameter-estimation error. Blue and orange lines depict the parameter-estimation error of the proposed approach (PA). Yellow and purple lines depict the parameter-estimation error of the approach in [35] (OA).

6. Application to a compartmental epidemiology model

Additionally, the proposed technique has been implemented in a compartmental model, which is one of the most used type of models in the epidemiology field. The idea is to segregate the population into homogeneous compartments, which represent the different states of the disease. The dynamics of these models depict the movement of individuals between disease states.

The most used compartmental model for depicting the dynamics of a disease is the susceptible-infected-recovered (SIR) model, which divides the population in three types [56]:

- Susceptible (s): Individuals that are not immune to the disease.
- Infected (i): Individuals that have contracted the disease. These individuals may transmit the disease to Susceptible ones.
- Recovered (r): Individuals that have moved from the infected group. Either, because they have recovered and are immune, or because of death.

However, in practice, it is very difficult to have a reliable measurement of the number of individuals in each compartment and this type of model does not include the influence of public interventions. Consequently, some authors [57] have proposed the inclusion of a fourth compartment generating a susceptible-infected-recovered-quarantined (SIRQ) model:

- Quarantined (q): Individuals that have been detected and either have been hospitalized or quarantined. This group contains infected individuals that have been diagnosed and susceptible individuals that have voluntarily quarantined itself.

Taking into account these compartments, the dynamics of a population of N individuals are depicted through the following ordinary differential equations [57]:

$$\begin{aligned}
\dot{s} &= -\beta s i - \delta_2 s \\
\dot{i} &= \beta s i - \gamma i - \delta_1 i \\
\dot{r} &= \gamma i \\
\dot{q} &= \delta_2 s + \delta_1 i
\end{aligned} \tag{42}$$

where β is the transmission rate and γ is the recovery rate. The factor δ_1 is the rate of infected that are being hospitalized and δ_2 is the rate of susceptible individuals being quarantined, which are time varying and assumed to be known. It is assumed that q is measurable and the rest of states are unmeasurable.

Remark 6.1. *The following relation holds for all t:*

$$s + i + r + q = N. \tag{43}$$

This fact will be exploited to design an observer only considering a state space model of s, i and q. Once s, i and q have been estimated, r can be deduced from (43).

The parameters β and γ are of particular interest as they provide information of the transmission of the disease and can help to design public interventions. For example, the factor $\dfrac{\beta}{\gamma}$ allows to compute the basic reproduction number of the disease, or the ratio $\dfrac{\gamma}{\delta_1}$ depicts the ratio of infected individuals which not being detected [57]. However, these parameters are usually unknown and the absence of data related to the states s, i and q, makes it difficult to estimates these parameters. Consequently, it is of interest to design an observer that can estimate online the states s, i and the unknown parameters β and γ, based on the measured state q.

For convenience, the concerned SIRQ model can be rewritten as in (35) by taking $\mathbf{x} = [s, i, q]^\intercal$, $\boldsymbol{\theta} = [\beta, \gamma]^\intercal$ and

$$\mathbf{A}(t) = \begin{bmatrix} -\delta_2 & 0 & 0 \\ 0 & -\delta_1 & 0 \\ \delta_2 & \delta_1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boldsymbol{\phi}(\mathbf{x}) = \begin{bmatrix} si & 0 \\ 0 & -i \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

From a observer design viewpoint, this problem is of particular interest for multiple reasons. First, the system presents singular inputs [55], e.g. the condition $\delta_1 = \delta_2$ drives the system unobservable, which prevents the transformation of the system to the standard observer canonical form [2]. Second, it is noticeable that the relative degree between the measured output, and the unknown parameters is 2, which is greater than one and lower than the system order, and motivates the implementation of the proposed technique.

The first step is to design an auxiliary signal, z, considering the insights presented in Section 3. It can be seen that the signal $z = \delta_2 s + \delta_1 i +$

$q = \mathbf{h}^\intercal(t)\mathbf{x}$, where $\mathbf{h}(t) = [\delta_2, \delta_1, 1]$, is relative degree 1 with respect to the unknown parameters, $\boldsymbol{\theta}$. Moreover, this signal can be reconstructed as $z = y + \dot{y}$, thus, it satisfies condition (12). Finally, as the pair $(\mathbf{A}(t), \mathbf{h}(t))$ is uniform completely observable, it is possible to implement the state observer (36)-(37).

Now, as the auxiliary signal has been designed taking into account the insights in Section 3, its value can be estimated through the high-gain observer proposed in the same section. Moreover, it is possible to solve equation (14) to design the parameter dynamics (13). Specifically, the parameter adaptation takes the form (13) with M defined in (40).

The validity of the proposed observer scheme has been tested in a numerical simulation. It will be simulated a case where the public intervention policies are modified over time which induces a decrease in the factor δ_1 and an increase of δ_2. It is considered that the compartments are normalized to the population total, i.e. $N = 1$. The parameters of the SIRQ system are summarized in Table 2.

Table 2: SIRQ true parameters, model parameters and observer parameters

Parameter	Value
True Parameters	
β	0.5
γ	0.01
δ_1	$0.0475 - 1.14 \cdot 10^{-4}t$
δ_2	$0.105 + 4.2 \cdot 10^{-4}t$
N	1
Model Parameters	
δ_1	$0.05 - 1.2 \cdot 10^{-4}t$
δ_2	$0.1 + 4 \cdot 10^{-4}t$
Observer Parameters	
σ	$2.2max\{\delta_1, \delta_2\}$
α_1	0.31
α_2	0.019
ε	0.712

To make the simulation more realistic, it is considered that the measurement of q is corrupted with high-frequency noise of mean zero and variance $9.65 \cdot 10^{-5}$. Moreover, it is assumed that the factors δ_1 and δ_1 are not exactly

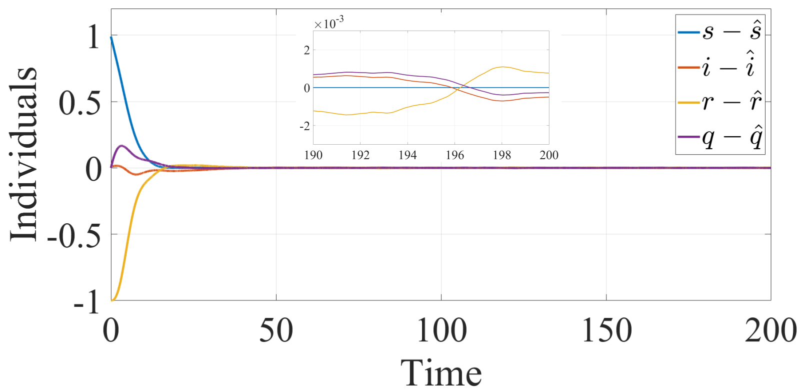

Figure 4: Evolution of the state-estimation error.

known and present a significant bias with respect to the real value. The model parameters used in the observer design are depicted in Table 2.

For these conditions, the factor σ^* is lower than $2|\lambda_{max}(\mathbf{A})| = 2max\{\delta_1, \delta_2\}$, thus, the adaptive observer parameter has been tuned as $\sigma = 2.2max\{\delta_1, \delta_2\}$, to ensure the satisfaction of the inequality (B.2). It is assumed that the derivative of the factors δ_i are unknown, and will be fixed at zero. This will introduce a bias in the estimation as commented in Remark 3.3, which will be reduced by a decreasing the parameter ε of the high-gain observer, as commented in Section 4. Finally, applying the insights presented in Section 2, the high-gain observer is designed to have adequate convergence rate and satisfy the condition (31), while presenting adequate noise performance. The observer parameters are summarized in Table 2.

It is assumed that there is no prior information on the system. Therefore, the observer states are initialized at the origin.

The evolution of the state-estimation errors is depicted in Fig. 4. The unknown parameter-estimation and the true value is depicted in Fig. 5. It can be observed that even in the presence of significant sensor noise and uncertainty, the technique is capable of recovering the unknown states and parameters, which validates the robustness and performance of the adaptive observer scheme.

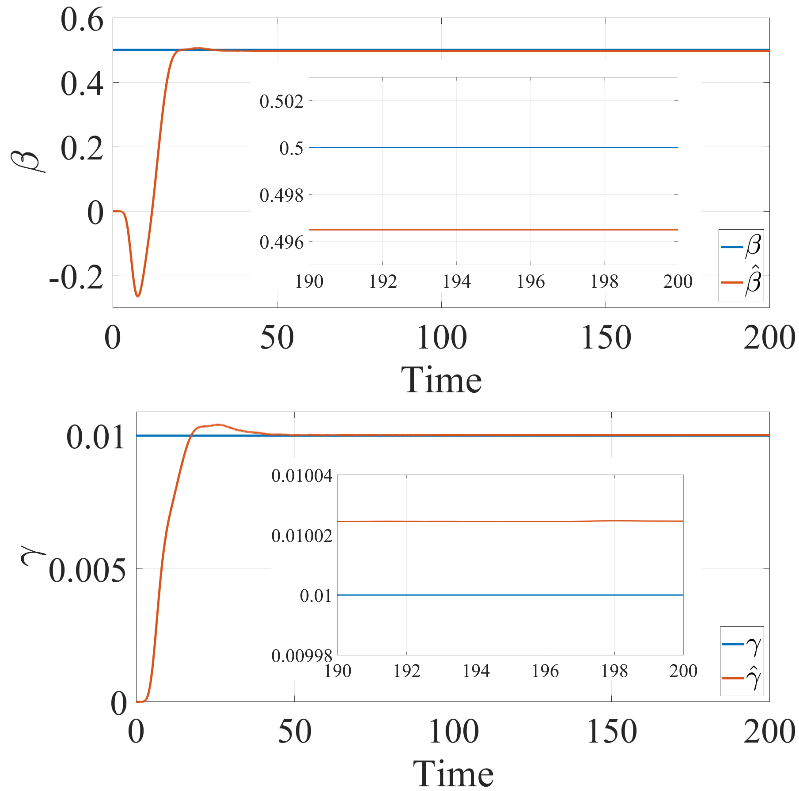

Figure 5: Evolution of the parameter-estimation and true value.

7. Conclusions and Discussion

This work has presented a methodology with provable convergence to relax the relative degree condition in adaptive observers even in the presence of sensor noise and uncertainty. The proposed approach is based on coupling the adaptive observer with a certain high-gain observer.

In order to analyse the convergence and accuracy of the approach, this work has obviated the common Barbalat's lemma argument and has proposed a new analysis based on a strict Lyapunov function. This analysis allows to prove the convergence of the high-gain observer and adaptive observer coupling and allows to study its performance in presence of sensor noise and unmodelled uncertainty. The proposed approach has been validated in a synthetic system and in a SIRQ epidemiology model.

It is expected that future works will implement the proposed strict Lyapunov methodology to study the performance of similar adaptive observers in presence of disturbances and noise, which is an analysis that has eluded the literature in adaptive observer design.

Furthermore, in future works, this approach will be implemented in other systems with higher relative degree and significant sensor noise and uncertainty, e.g. the estimation of the liquid water saturation and liquid water transport parameters in fuel cells [26, 27].

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Barbalat's lemma argument for parameter-estimation convergence

Consider the case without noise and uncertainty, i.e. $\mathbf{v} = \mathbf{w} = 0$. Taking into account the expression (6) and the parameter adaptation (7), it is shown that the Lyapunov function (5) has a semidefinite derivative, thus, it is possible to conclude that $\lim_{t\to\infty} \|\mathbf{x} - \hat{\mathbf{x}}\| = 0$. Then, as $\mathbf{f}, \boldsymbol{\phi}$ are Lipschitz and $\dot{\mathbf{e}}_x$ is uniformly continuous, it is possible to proof that $\lim_{t\to\infty} \dot{\mathbf{e}}_x = 0$ from the Barbalat's lemma. From this fact, it can be seen that the following also holds:

$$\lim_{t\to\infty} \|\mathbf{B}\boldsymbol{\phi}(\mathbf{x}, \mathbf{u})\mathbf{e}_\theta\| = 0. \tag{A.1}$$

From the fact that (5) is non-increasing and lower bounded by zero, it has a limit as $t \to \infty$. Hence, $\hat{\boldsymbol{\theta}}$ and $\mathbf{e}_\theta$ must converge to a constant. These facts do not prove that the parameter-estimation converges to the true value, as it still possible that $\lim_{t\to\infty} \|\mathbf{e}_\theta\| = c$, where c is some positive constant and $\lim_{t\to\infty} \|\mathbf{B}\boldsymbol{\phi}(\mathbf{x}, \mathbf{u})\| = 0$, and condition (A.1) would still be satisfied. However, if we assumed that the system is persistently exciting, this case is not possible and the parameter-estimation converges to the true value.

Definition Appendix A.1. *Denote $\mathbf{X}(\boldsymbol{x}_0, t)$ as the solution of (1), starting from the initial condition $\boldsymbol{x}_0$ at time 0. The curve $\boldsymbol{B}\boldsymbol{\phi}(\mathbf{X}(\boldsymbol{x}_0, t), \boldsymbol{u}(t))$ is persistently exciting if there exist some positive constants μ_1, μ_2 and T_0 such that $\forall t$:*

$$\mu_1 \boldsymbol{I} \geq \int_t^{t+T_0} \boldsymbol{\phi}(\tau)^\intercal \boldsymbol{B}^\intercal \boldsymbol{B}\boldsymbol{\phi}(\tau) d\tau \geq \mu_2 \boldsymbol{I}. \tag{A.2}$$

Appendix B. Proof of Theorem 4.5

The first step of the proof consists in proving the first two inequalities in (4), which requires showing that the time-varying matrix $\mathbf{P}(t)$ is upper and lower bounded. If the pair $(\mathbf{A}(t), \mathbf{h}(t))$ is uniform complete observable, then, the matrix $\mathbf{P}(t)$ is strictly positive [55]:

$$0 < \lambda_{min}(\mathbf{P}(t)), \quad \forall t. \tag{B.1}$$

Moreover, it also presents the following upper bound

$$\lambda_{max}(\mathbf{P}(t)) \leq \|\mathbf{P}(0)\| + \frac{H_{max}^2}{\sigma - 2|\lambda_{max}(\mathbf{A})|}, \quad \forall t. \tag{B.2}$$

Notice that, inequality (B.2) only holds if $\sigma > 2|\lambda_{max}(\mathbf{A})|$.

The second part of the proof consists in proving that the derivative of the Lyapunov function (3) satisfies the third inequality in (4) for some positive constants α_3, α_4 and α_5.

The state-estimation error dynamics between the adaptive observer equation (36) and the structure (35), $\mathbf{e}_x = \mathbf{x} - \hat{\mathbf{x}}$, are depicted by the following expression

$$\begin{aligned}\dot{\mathbf{e}}_x =&(\mathbf{A}(t) - \mathbf{P}(t)^{-1}\mathbf{h}(t)\mathbf{h}^\intercal(t))\mathbf{e}_x \\ &+ \mathbf{B}\boldsymbol{\phi}(\mathbf{x},t)\boldsymbol{\theta} - \mathbf{B}\boldsymbol{\phi}(\hat{\mathbf{x}},t)\hat{\boldsymbol{\theta}} + \mathbf{K}(t)\mathbf{v} + \mathbf{w}.\end{aligned} \tag{B.3}$$

where $\mathbf{v}$ is the sensor noise and $\mathbf{w}$ are the unmodelled process disturbances.

Consider the Lyapunov function candidate (3), then,

$$\begin{aligned}\dot{V}_x =& \mathbf{e}_x^\intercal(\dot{\mathbf{P}}(t) + \mathbf{A}(t)^\intercal\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}(t) - 2\mathbf{h}(t)\mathbf{h}^\intercal(t))\mathbf{e}_x \\ &+ 2(\mathbf{B}\boldsymbol{\phi}(\mathbf{x},t)\boldsymbol{\theta} - \mathbf{B}\boldsymbol{\phi}(\hat{\mathbf{x}},t)\hat{\boldsymbol{\theta}})^\intercal\mathbf{P}(t)\mathbf{e}_x \\ &+ \mathbf{e}_x^\intercal\mathbf{P}(t)\mathbf{K}(t)\mathbf{v} + \mathbf{e}_x^\intercal\mathbf{P}(t)\mathbf{w} \\ &\leq \mathbf{e}_x^\intercal(-\sigma\mathbf{P}(t))\mathbf{e}_x + 2\phi_{max}\boldsymbol{\theta}\|\mathbf{B}\|\|\mathbf{e}_x\|\|\mathbf{P}(t)\mathbf{e}_x\| \\ &+ \|\mathbf{e}_x\|\|\mathbf{P}(t)\mathbf{K}(t)\|\vartheta + \|\mathbf{e}_x\|\|\mathbf{P}(t)\|w_2 \\ &\leq (-\sigma\lambda_{min}(\mathbf{P}(t)) + 2\lambda_{max}(\mathbf{P}(t))\phi_{max}\boldsymbol{\theta}\|\mathbf{B}\|)\|\mathbf{e}_x\|^2 \\ &+ \|\mathbf{e}_x\|\lambda_{max}(\mathbf{P}(t))\|\mathbf{K}(t)\|\vartheta + \|\mathbf{e}_x\|\lambda_{max}(\mathbf{P}(t))w_2.\end{aligned} \tag{B.4}$$

From (B.4) it can be deduced that the Lyapunov function (3) satisfies (4) with:

$$\begin{aligned}\alpha_3 &= \sigma\lambda_{min}(\mathbf{P}(t)) - 2\lambda_{max}(\mathbf{P}(t))\phi_{max}\boldsymbol{\theta}\|\mathbf{B}\| \\ \alpha_4 &= \lambda_{max}(\mathbf{P}(t)) \\ \alpha_5 &= \lambda_{max}(\mathbf{P}(t))\|\mathbf{K}(t)\|\end{aligned}$$

Finally, it is necessary to show the conditions in which the factor α_3 is strictly positive. It can be seen that for a value σ that satisfies:

$$\sigma > \frac{2\lambda_{max}(\mathbf{P}(t))}{\lambda_{min}(\mathbf{P}(t))}\phi_{max}\boldsymbol{\theta}\|\mathbf{B}\| \triangleq \sigma^*,$$

the constant α_3 is strictly positive. As the matrix $\mathbf{P}(t)$ is upper and lower bounded, the factor σ^* is also upper bounded.

References

[1] S. K. Korovin, V. V. Fomichev, State observers for linear systems with uncertainty, Vol. 51, Walter de Gruyter, 2009.

[2] J.-P. Gauthier, H. Hammouri, S. Othman, et al., A simple observer for nonlinear systems applications to bioreactors, IEEE Trans. Autom. Control 37 (6) (1992) 875–880.

[3] D. Astolfi, L. Marconi, L. Praly, A. R. Teel, Low-power peaking-free high-gain observers, Automatica 98 (2018) 169 – 179. doi:https://doi.org/10.1016/j.automatica.2018.09.009.

[4] A. Zemouche, F. Zhang, F. Mazenc, R. Rajamani, High-gain nonlinear observer with lower tuning parameter, IEEE Trans. Autom. Control 64 (8) (2019) 3194–3209. doi:10.1109/TAC.2018.2882417.

[5] A. Kadrine, Z. Tir, O. P. Malik, M. A. Hamida, A. Reatti, A. Houari, Adaptive non-linear high gain observer based sensorless speed estimation of an induction motor, Journal of the Franklin Institute 357 (13) (2020) 8995–9024. doi:https://doi.org/10.1016/j.jfranklin.2020.06.013.

[6] A. Levant, Robust exact differentiation via sliding mode technique, Automatica 34 (3) (1998) 379 – 384.

[7] J. Luna, R. Costa-Castelló, S. Strahl, Chattering free sliding mode observer estimation of liquid water fraction in proton exchange membrane fuel cells, Journal of the Franklin Institute 357 (18) (2020) 13816–13833. doi:https://doi.org/10.1016/j.jfranklin.2020.10.010.

[8] G. Phanomchoeng, R. Rajamani, D. Piyabongkarn, Nonlinear observer for bounded jacobian systems, with applications to automotive slip angle estimation, IEEE Trans. Autom. Control 56 (5) (2011) 1163–1170. doi:10.1109/TAC.2011.2108552.

[9] M. Arcak, P. Kokotović, Nonlinear observers: a circle criterion design and robustness analysis, Automatica 37 (12) (2001) 1923 – 1930. doi:https://doi.org/10.1016/S0005-1098(01)00160-1.

[10] R. Rajamani, W. Jeon, H. Movahedi, A. Zemouche, On the need for switched-gain observers for non-monotonic nonlinear systems, Automatica 114 (2020) 108814.

[11] A. J. Krener, A. Isidori, Linearization by output injection and nonlinear observers, Syst. Control. Lett. 3 (1) (1983) 47 – 52. doi:https://doi.org/10.1016/0167-6911(83)90037-3.

[12] N. Kazantzis, C. Kravaris, Nonlinear observer design using lyapunov's auxiliary theorem, Syst. Control. Lett. 34 (5) (1998) 241 – 247. doi:https://doi.org/10.1016/S0167-6911(98)00017-6.

[13] V. Andrieu, L. Praly, On the existence of a kazantzis–kravaris/luenberger observer, SIAM Journal on Control and Optimization 45 (2) (2006) 432–456. doi:10.1137/040617066.

[14] P. Bernard, V. Andrieu, Luenberger observers for nonautonomous nonlinear systems, IEEE Trans. Autom. Control 64 (1) (2019) 270–281. doi:10.1109/TAC.2018.2872202.

[15] D. Karagiannis, D. Carnevale, A. Astolfi, Invariant manifold based reduced-order observer design for nonlinear systems, IEEE Trans. Autom. Control 53 (11) (2008) 2602–2614. doi:10.1109/TAC.2008.2007045.

[16] D. Karagiannis, M. Sassano, A. Astolfi, Dynamic scaling and observer design with application to adaptive control, Automatica 45 (12) (2009) 2883 – 2889.

[17] A. Martinelli, Nonlinear unknown input observability: Extension of the observability rank condition, IEEE Trans. Autom. Control 64 (1) (2019) 222–237. doi:10.1109/TAC.2018.2798806.

[18] G. C. Goodwin, M. M. Seron, Fundamental performance properties of a general class of observers for linear systems having predictable disturbances, Automatica 113 (2020) 108717. doi:https://doi.org/10.1016/j.automatica.2019.108717.

[19] Z. Jiang, A. Teel, L. Praly, Small-gain theorem for iss systems and applications, Math. Control. Signal Sys. 7 (1994) 95–120. doi:10.1007/BF01211469.

[20] G. C. Goodwin, A critique of observers used in the context of feedback control, in: Z. Chen, A. Mendes, Y. Yan, S. Chen (Eds.), Intelligent Robotics and Applications, Springer International Publishing, Cham, 2018, pp. 1–24.

[21] U. Montanaro, R. Costa-Castelló, J. M. Olm, C. L. Barros, Experimental validation of a continuous-time mcsi algorithm with bounded adaptive

gains, Journal of the Franklin Institute 356 (12) (2019) 5881 – 5897. doi:https://doi.org/10.1016/j.jfranklin.2019.03.012.

[22] P. A. Ioannou, J. Sun, Robust adaptive control, Englewood Cliffs, NJ: Prentice Hall, 1995.

[23] G. Bastin, M. R. Gevers, Stable adaptive observers for nonlinear time-varying systems, IEEE Trans. Autom. Control 33 (7) (1988) 650–658. doi:10.1109/9.1273.

[24] Young Man Cho, R. Rajamani, A systematic approach to adaptive observer synthesis for nonlinear systems, IEEE Trans. Autom. Control 42 (4) (1997) 534–537. doi:10.1109/9.566664.

[25] I. Y. Tyukin, E. Steur, H. Nijmeijer, C. van Leeuwen, Adaptive observers and parameter estimation for a class of systems nonlinear in the parameters, Automatica 49 (8) (2013) 2409–2423. doi:https://doi.org/10.1016/j.automatica.2013.05.008.

[26] A. Cecilia, M. Serra, R. Costa-castelló, PEMFC state and parameter estimation through a high-gain based adaptive observer, in: 21st IFAC World Congress, 2020.

[27] A. Cecilia, M. Serra, R. Costa-Castelló, Nonlinear adaptive observation of the liquid water saturation in polymer electrolyte membrane fuel cells, Journal of Power Sources 492 (2021) 229641. doi:https://doi.org/10.1016/j.jpowsour.2021.229641.

[28] A. Cecilia, S. Sahoo, T. Dragičević, R. Costa-Castelló, F. Blaabjerg, Detection and mitigation of false data in cooperative dc microgrids with unknown constant power loads, IEEE Transactions on Power Electronics 36 (8) (2021) 9565–9577. doi:10.1109/TPEL.2021.3053845.

[29] A. Cecilia, S. Sahoo, T. Dragicevic, R. Costa-Castello, F. Blaabjerg, On addressing the security and stability issues due to false data injection attacks in dc microgrids an adaptive observer approach, IEEE Transactions on Power Electronics (2021, to appear). doi:10.1109/TPEL.2021.3114990.

[30] R. Marino, P. Tomei, Global adaptive observers for nonlinear systems via filtered transformations, IEEE Trans. Autom. Control 37 (8) (1992) 1239–1245. doi:10.1109/9.151117.

[31] G. Besançon, Remarks on nonlinear adaptive observer design, Syst. Control. Lett. 41 (4) (2000) 271 – 280. doi:https://doi.org/10.1016/S0167-6911(00)00065-7.

[32] Q. Zhang, Adaptive observer for multiple-input-multiple-output (mimo) linear time-varying systems, IEEE Transactions on Automatic Control 47 (3) (2002) 525–529. doi:10.1109/9.989154.

[33] D. Efimov, C. Edwards, A. Zolghadri, Enhancement of adaptive observer robustness applying sliding mode techniques, Automatica 72 (2016) 53–56. doi:https://doi.org/10.1016/j.automatica.2016.05.029.

[34] M. Farza, M. M'Saad, T. Maatoug, M. Kamoun, Adaptive observers for nonlinearly parameterized class of nonlinear systems, Automatica 45 (10) (2009) 2292–2299. doi:https://doi.org/10.1016/j.automatica.2009.06.008.

[35] G. Besanöon, Q. Zhang, H. Hammouri, High-gain observer based state and parameter estimation in nonlinear systems, IFAC Proceedings Volumes 37 (13) (2004) 327–332, 6th IFAC Symposium on Nonlinear Control Systems 2004 (NOLCOS 2004), Stuttgart, Germany, 1-3 September, 2004. doi:https://doi.org/10.1016/S1474-6670(17)31244-2.

[36] R. Ortega, E. Panteley, Adaptation is unnecessary in l1-adaptive control: What makes an adaptive controller "adaptive"?, IEEE Control Systems Magazine 36 (1) (2016) 47–52. doi:10.1109/MCS.2015.2495023.

[37] O. Stamnes, O. M. Aamo, G. Kaasa, Adaptive redesign of nonlinear observers, IEEE Trans. Autom. Control 56 (5) (2011) 1152–1157. doi:10.1109/TAC.2011.2107090.

[38] M. Bin, L. Marconi, Model identification and adaptive state observation for a class of nonlinear systems, IEEE Trans. Autom. Control (2020, to appear). doi:10.1109/TAC.2020.3041238.

[39] G. Besançon, A. Ţiclea, On adaptive observers for systems with state and parameter nonlinearities, IFAC-PapersOnLine

50 (1) (2017) 15416–15421, 20th IFAC World Congress. doi:https://doi.org/10.1016/j.ifacol.2017.08.1870.

[40] C. Afri, V. Andrieu, L. Bako, P. Dufour, State and parameter estimation: A nonlinear luenberger observer approach, IEEE Transactions on Automatic Control 62 (2) (2016) 973–980.

[41] A. Pyrkin, A. Bobtsov, R. Ortega, A. Vedyakov, S. Aranovskiy, Adaptive state observers using dynamic regressor extension and mixing, Systems & Control Letters 133 (2019) 104519. doi:https://doi.org/10.1016/j.sysconle.2019.104519.

[42] Y. Xing, J. Na, R. Costa-Castelló, G. Gao, Adaptive parameter estimation-based observer design for nonlinear systems, in: 2020 59th IEEE Conference on Decision and Control (CDC), 2020, pp. 4170–4175. doi:10.1109/CDC42340.2020.9304367.

[43] Y. H. Kim, F. L. Lewis, C. T. Abdallah, A dynamic recurrent neural-network-based adaptive observer for a class of nonlinear systems, Automatica 33 (8) (1997) 1539–1543. doi:https://doi.org/10.1016/S0005-1098(97)00065-4.

[44] A. Abbaspour, P. Aboutalebi, K. K. Yen, A. Sargolzaei, Neural adaptive observer-based sensor and actuator fault detection in nonlinear systems: Application in uav, ISA Transactions 67 (2017) 317–329. doi:https://doi.org/10.1016/j.isatra.2016.11.005.

[45] J. A. Farrell, M. M. Polycarpou, Adaptive approximation based control: unifying neural, fuzzy and traditional adaptive approximation approaches, Vol. 48, John Wiley & Sons, 2006.

[46] Y. Mu, H. Zhang, H. Ren, Y. Cai, Fuzzy adaptive observer-based fault and disturbance reconstructions for t-s fuzzy systems, IEEE Transactions on Circuits and Systems II: Express Briefs 68 (7) (2021) 2453–2457. doi:10.1109/TCSII.2021.3049243.

[47] G. Pillonetto, F. Dinuzzo, T. Chen, G. De Nicolao, L. Ljung, Kernel methods in system identification, machine learning and function estimation: A survey, Automatica 50 (3) (2014) 657–682. doi:https://doi.org/10.1016/j.automatica.2014.01.001.

[48] H. Hammouri, J. de Leon Morales, Observer synthesis for state-affine systems, in: 29th IEEE Conference on Decision and Control, 1990, pp. 784–785 vol.2. doi:10.1109/CDC.1990.203695.

[49] J. Gauthier, G. Bornard, Observability for any u(t) of a class of nonlinear systems, IEEE Trans. Autom. Control 26 (4) (1981) 922–926. doi:10.1109/TAC.1981.1102743.

[50] H. K. Khalil, L. Praly, High-gain observers in nonlinear feedback control, Int. J. Robust Nonlinear Control. 24 (6) (2014) 993–1015. doi:https://doi.org/10.1002/rnc.3051.

[51] A. Levant, Higher-order sliding modes, differentiation and output-feedback control, Int. J. Control 76 (9-10) (2003) 924–941. doi:10.1080/0020717031000099029.

[52] F. Mazenc, Strict lyapunov functions for time-varying systems, Automatica 39 (2) (2003) 349 – 353. doi:https://doi.org/10.1016/S0005-1098(02)00233-9.

[53] A. Loría, E. Panteley, M. Maghenem, Strict lyapunov functions for model reference adaptive control: Application to lagrangian systems, IEEE Trans. Autom. Control 64 (7) (2019) 3040–3045. doi:10.1109/TAC.2018.2874723.

[54] J. G. Rueda-Escobedo, J. A. Moreno, Strong lyapunov functions for two classical problems in adaptive control, Automatica 124 (2021) 109250. doi:https://doi.org/10.1016/j.automatica.2020.109250.

[55] G. Besançon, G. Bornard, H. Hammouri, Observer synthesis for a class of nonlinear control systems, European Journal of Control 2 (3) (1996) 176 – 192. doi:https://doi.org/10.1016/S0947-3580(96)70043-2.

[56] K. Roosa, G. Chowell, Assessing parameter identifiability in compartmental dynamic models using a computational approach: application to infectious disease transmission models, Theoretical Biology and Medical Modelling 16 (1) (2019) 1–15.

[57] W. Zheng, Total variation regularization for compartmental epidemic models with time-varying dynamics, arXiv preprint arXiv:2004.00412 (2020).