Nash equilibrium seeking in full-potential population games under capacity and migration constraints ⋆

Juan Martinez-Piazuelo [a], Nicanor Quijano [b], Carlos Ocampo-Martinez [a]

[a] *Automatic Control Department, Universitat Politècnica de Catalunya, Llorens i Artigas, 4-6, 08028, Barcelona, Spain*

[b] *Departamento de Ingeniería Eléctrica y Electrónica, Universidad de los Andes, Carrera 1 No. 18A-10, Bogotá, Colombia*

Abstract

This brief proposes a novel decision-making model for generalized Nash equilibrium seeking in the context of full-potential population games under capacity and migration constraints. The capacity constraints restrict the mass of players that are allowed to simultaneously play each strategy of the game, while the migration constraints introduce a networked interaction structure among the players and rule the strategic switches that players can make. In this brief, we consider both decoupled capacity constraints regarding individual strategies, as well as coupled capacity constraints regarding disjoint groups of strategies. As main technical contributions, we prove that the proposed decision-making protocol guarantees the forward time invariance of the feasible set, and we provide sufficient conditions on the connectivity level of the migration graph to guarantee the asymptotic stability of the set of generalized Nash equilibria of the underlying game when the game is a full-potential population game with concave potential function. Furthermore, we also provide an alternative discrete-time analysis of the proposed evolutionary game dynamics, which allows us to formulate a population-game-inspired distributed optimization algorithm that guarantees the hard satisfaction of the constraints over all iterations. Finally, the theoretical results are validated numerically on a constrained networked congestion game.

Key words: Evolutionary game theory; Generalized Nash equilibrium seeking; Distributed systems.

1 Introduction

Consider a large population of decision-making agents, represented by a continuum of mass $m \in \mathbb{R}_{>0}$, that are engaged in a game with a set of strategies $\mathcal{S} = \{1, 2, \ldots, n\}$, where $n \in \mathbb{Z}_{\geq 2}$. At any time, the mass of agents choosing strategy $i \in \mathcal{S}$ is given by $x_i \in \mathbb{R}_{\geq 0}$, and the strategic distribution of the population is described by the vector $\mathbf{x} = [x_1, x_2, \ldots, x_n]^\top \in \mathbb{R}^n_{\geq 0}$. Hence, the set of possible strategic distributions of the population is $\Delta = \left\{\mathbf{x} \in \mathbb{R}^n_{\geq 0} : \mathbf{1}^\top \mathbf{x} = m\right\}$, where $\mathbf{1}$ is the vector of ones with appropriate dimension. Moreover, each strategy $i \in \mathcal{S}$ has an associated payoff function $f_i : \mathbb{R}^n_{\geq 0} \to \mathbb{R}$, and thus the so-called payoff vector $\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x})]^\top \in \mathbb{R}^n$ provides the payoff for all strategies at $\mathbf{x} \in \Delta$. In summary, the players of the game are the population agents (modeled as a continuum of mass m), the set of available strategies is $\mathcal{S}$, and the payoffs perceived by the players are given by $\mathbf{f}(\cdot)$. Throughout, we refer to $\mathbf{f}(\cdot)$ as the population game to cope with the notation in the literature on population games (Sandholm 2010). Furthermore, we focus on the family of so-called full-potential population games.

Definition 1 (Sandholm 2010, Section 3.1.2) *The game $\mathbf{f} : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n$ is a full-potential game if there exists a continuously differentiable (potential) function $\varphi : \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ such that $\nabla\varphi(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$, i.e., $\partial\varphi(\cdot)/\partial x_i = f_i(\cdot)$, for all $i \in \mathcal{S}$.*

Under the considered framework, the population agents are regarded as non-cooperative players that seek to play the strategy leading to the highest payoff. To select which strategy to play, each player is equipped with

⋆ Juan Martinez-Piazuelo gratefully acknowledges the Universitat Politècnica de Catalunya and Banco Santander for the financial support of his predoctoral grant FPI-UPC. In addition, the authors would like to thank the project PID2020-115905RB-C21 (L-BEST) funded by MCIN/ AEI /10.13039/501100011033 for supporting this research. This paper was not presented at any IFAC meeting. Corresponding author Juan Martinez-Piazuelo.

Email addresses:
`juan.pablo.martinez.piazuelo@upc.edu` (Juan Martinez-Piazuelo), `nquijano@uniandes.edu.co` (Nicanor Quijano), `carlos.ocampo@upc.edu` (Carlos Ocampo-Martinez).

a stochastic alarm clock and a revision protocol. The clock of each player provides strategic revision opportunities according to a rate R exponential distribution. The revision protocol, on the other hand, is a map of the form $\rho_{ij} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that provides the conditional switch rate from strategy $i \in \mathcal{S}$ to strategy $j \in \mathcal{S}$, for all $i, j \in \mathcal{S}$ (for simplicity we let $\rho_{ij} \triangleq \rho_{ij}(\mathbf{x}, \mathbf{f}(\mathbf{x}))$). From the microscopic perspective, it is thus assumed that if a player playing $i \in \mathcal{S}$ receives a revision opportunity, then such a player switches to strategy $j \in \mathcal{S} \setminus \{i\}$ with probability ρ_{ij}/R, and remains at strategy $i \in \mathcal{S}$ with probability $1 - \sum_{j \in \mathcal{S} \setminus \{i\}} \rho_{ij}/R$. As in (Sandholm 2010, Section 4.1), it is assumed that R is large enough so that ρ_{ij}/R is a valid probability for all $i, j \in \mathcal{S}$ and all $\mathbf{x} \in \Delta$. Consequently, given that the number of players is large, and following the ideas in (Sandholm 2010, Section 4.2), the macroscopic dynamics that describe the evolution of the (mean) strategic distribution of the population are given by

$$\dot{x}_i = \sum_{j \in \mathcal{S}} x_j \rho_{ji} - \sum_{j \in \mathcal{S}} x_i \rho_{ij}, \quad \forall i \in \mathcal{S}. \tag{1}$$

Namely, $x_j \rho_{ji}$ is the mass of players switching from j to i, and $x_i \rho_{ij}$ is the mass of players switching from i to j. Therefore, given a game $\mathbf{f}(\cdot)$ and a revision protocol $\rho_{ij}(\cdot, \cdot)$, for all $i, j \in \mathcal{S}$, the temporal evolution of the strategic distribution $\mathbf{x}$ can be analyzed by means of (1). For instance, by analyzing the dynamics in (1) for a given game and revision protocol, one might determine whether the population reaches a Nash equilibrium (NE) of the game. That is, a self-enforceable strategic distribution where no player can increase her payoff by unilaterally deviating from her selected strategy.

Contributions of this brief: Based on the aforementioned model, in this brief we formulate a novel revision protocol that allows the players to asymptotically converge to a generalized Nash equilibrium (GNE) of the population game. Namely, a GNE is a self-enforceable strategic distribution of the population where: i) no player can increase her payoff by unilaterally deviating from her selected strategy; and ii) some coupled constraints over the players' decisions are satisfied. More precisely, in this brief we consider two types of constraints: capacity constraints and migration constraints. The capacity constraints restrict the mass of players that are allowed to simultaneously play each strategy of the game. Here, we consider both decoupled capacity constraints regarding individual strategies, as well as coupled capacity constraints regarding disjoint groups of strategies. Clearly, both classes of capacity constraints couple the decision-making process of the population players. On the other hand, the migration constraints are graphical interaction constraints that rule the strategic switches that players can make. Namely, the migration constraints determine whether a player playing strategy i can switch to strategy j, for all $i, j \in \mathcal{S}$. Note that the migration constraints do not couple the players decisions, but impose a network structure in the strategic interaction of the population. In summary, our main technical contributions are fourfold. First, we provide sufficient conditions on the migration graph to guarantee that the set of equilibria of the dynamics in (1) (under the proposed revision protocol) coincides with the set of generalized Nash equilibria of capacity-constrained population game. Second, we prove that the subset of Δ that satisfies the considered capacity constraints is positively invariant under the dynamics in (1) when the proposed revision protocol is considered. That is, if the initial strategic distribution of the population satisfies the capacity constraints, then the (mean) strategic distribution of the population satisfies the capacity constraints for all future times. Third, we provide sufficient conditions to guarantee the asymptotic stability of the set of equilibria of the dynamics in (1) under the proposed revision protocol and when the population game is a full-potential game (c.f., Definition 1). Finally, we analyze a discretized version of the dynamics in (1) given by

$$x_i[k+1] = x_i[k] + \epsilon \hat{x}_i[k], \quad \forall i \in \mathcal{S}, \tag{2}$$

where $\hat{x}_i[k] = \sum_{j \in \mathcal{S}} x_j[k]\rho_{ji}[k] - \sum_{j \in \mathcal{S}} x_i[k]\rho_{ij}[k]$; $\rho_{ij}[k] \triangleq \rho_{ij}(\mathbf{x}[k], \mathbf{f}(\mathbf{x}[k]))$; and $\epsilon \in \mathbb{R}_{>0}$ is the (fixed) discretization time. By analyzing the dynamics in (2), we provide theoretical upper bounds on the discretization time ϵ that ensure the preservation of the invariance and convergence properties of the dynamics in (1) in their discretized counterpart given by (2). Consequently, such a discrete-time analysis of (1) allows us to formulate a population-game-inspired distributed optimization algorithm (with fixed step size ϵ) that guarantees the hard satisfaction of the constraints over all iterations. Notice that the discrete-time analysis of the dynamics in (1) has also been studied in our preliminary work Martinez-Piazuelo, Diaz-Garcia, Quijano & Giraldo (2022), but only for the capacity-unconstrained case. Hence, in this brief we generalize such previous results to the context of GNE seeking. Moreover, in contrast with Martinez-Piazuelo, Diaz-Garcia, Quijano & Giraldo (2022), we provide the self-contained analyses for both the continuous-time and the discrete-time perspectives.

Related work: Recently, the problem of distributed NE/GNE seeking in classical non-cooperative multiplayer games has received significant attention. Such a problem refers to the task of designing decision-making algorithms that allow non-cooperative players to reach an NE/GNE of the game while interacting over a (non-complete) network. Regarding distributed NE seeking problems, a common approach relies on gradient play and consensus-based algorithms. Namely, gradient play is employed to update the players' decisions in the direction that minimizes their given cost functions, while consensus-based algorithms are applied to estimate the joint action profile of the players in partial-decision information scenarios ruled by the interaction network. Such an approach has been recently considered both from continuous-time (Ye & Hu 2021, Gadjov & Pavel 2019, De Persis & Grammatico 2019) and discrete-time perspectives (Tatarenko et al. 2021, Bianchi & Grammatico 2021). In the context of distributed GNE seeking, on the other hand, several approaches rely on primal-dual methods where dual variables are included to handle the coupled constraints over the players' decisions. In this context, consensus-based algorithms are also employed either to estimate the joint action profile and/or to enforce the players' agreement on the optimal dual variables. Recent approaches have considered such a framework both in full-decision information scenarios, where players directly observe the interfering action profile (Yi & Pavel 2019*a*,*b*, Chen et al. 2020), as well as in partial-decision information scenarios, where players indeed estimate the interfering actions through consensus-based methods (Pavel 2020, Yi & Pavel 2020, Belgioioso et al. 2021). Still in the context of distributed GNE seeking, some alternative approaches have considered penalty-based methods where the cost functions of the players are extended with penalty terms to handle the coupled constraints. As shown in Facchinei & Kanzow (2010), penalty-based approaches allow to recast the GNE problem as an NE seeking task that does not involve coupled dual variables. Some recent approaches in this direction are the ones in Sun & Hu (2021*a*,*b*) and Romano & Pavel (2020, 2021). As highlighted in Romano & Pavel (2021), a significant advantage of penalty-based methods over primal-dual approaches is that the former can be designed to enforce the constraints' satisfaction over the whole transient of the decision-making process and not only at the equilibrium of the game.

The problem of distributed NE/GNE seeking has also received some attention from the perspective of evolutionary game theory. For instance, the authors in Barreiro-Gomez et al. (2017) and Como et al. (2021) have introduced non-complete networked interaction structures within the aforementioned population games framework, and have provided sufficient conditions on the game and interaction network to guarantee the convergence to an NE for various classes of revision protocols. Such non-complete interaction schemes have been shown to be relevant for distributed optimization and control applications including distributed extremum seeking (Poveda & Quijano 2015) and distributed predictive control in resource allocation systems (Barreiro-Gomez et al. 2019), among others (Quijano et al. 2017). In the aforementioned approaches, the control objective is related to reaching an NE of the underlying game. Regarding GNE seeking problems in evolutionary games, on the other hand, different approaches have been recently proposed. Namely, Barreiro-Gomez et al. (2016) consider the migration-constrained population games of Barreiro-Gomez et al. (2017) in the context of density-dependent population games (i.e., population games where the total mass of players is not constant), and propose a primal-dual approach to include affine constraints over the strategic distribution of the population. The proposed approach is illustrated on distributed GNE seeking problems in water distribution systems. On a similar vein, Martinez-Piazuelo, Quijano & Ocampo-Martinez (2022) propose a primal-dual-based approach for GNE seeking in population games under affine equality constraints. The proposed method considers dynamic payoff mechanisms (Park et al. 2019), where the payoffs perceived by the players are determined by an auxiliary dynamical system in feedback interconnection with the population of players. The proposed approach is illustrated in the context of congestion games, yet only complete interaction structures are considered. Similar to the primal-dual classical game theoretical approaches, the aforementioned primal-dual GNE seeking methods for evolutionary games only guarantee the satisfaction of the constraints at the equilibrium of the game. To cope with such an issue, Barreiro-Gomez & Tembine (2018) propose a novel form of revision protocols to include capacity constraints that restrict the mass of players that can simultaneously choose the same strategy. The proposed approach guarantees the satisfaction of the capacity constraints over the whole transient of the decision-making process. Besides, the proposed revision protocols also incorporate migration constraints to accommodate for networked interaction structures over the players. Consequently, such an idea has been exploited in distributed GNE seeking applications under hard constraints regarding optimal frequency control (Barreiro-Gomez et al. 2018) and charging coordination of electric vehicles (Martinez-Piazuelo et al. 2020).

Motivated by Barreiro-Gomez & Tembine (2018), in this paper we propose a novel form of revision protocol that allows the hard satisfaction of strategic capacity constraints over the whole transient of the decision-making process, as well as the consideration of migration constraints that rule the interaction between the population players. In contrast with Barreiro-Gomez & Tembine (2018), however, our proposed revision protocol allows the consideration of both decoupled capacity constraints regarding individual strategies, as well as coupled capacity constraints regarding disjoint groups of strategies (the approach in Barreiro-Gomez & Tembine (2018) only allows for decoupled capacity constraints). We highlight that some particular coupled capacity constraints have also been considered in Martinez-Piazuelo et al. (2020) in the context of aggregative games. In contrast, the approach in this brief considers more general games, capacity constraints, and graphical interaction structures among the players. Finally, we highlight that in contrast with the aforementioned penalty-based methods studied in classical game theoretical perspectives, the approach proposed in this brief does not require the design of any penalty/barrier functions to ensure the satisfaction of the capacity constraints over the whole transient.

The remainder of this brief is organized as follows. First, we formulate our proposed revision protocol (Section 2). Second, we analyze the resulting population dynamics and their discretized counterpart (Section 3). Third, we present an illustrative example (Section 4). Finally, we provide some concluding remarks and future directions of research (Section 5). Besides, all the proofs of our theoretical developments are provided in Section 6.

2 Proposed revision protocol

In this section, we formally define the capacity and migration constraints considered in this paper, and we formulate our proposed revision protocol.

Capacity constraints: In this paper, we consider two types of capacity constraints: decoupled and coupled. The decoupled constraints restrict the maximum mass of players that can simultaneously play the same strategy. The coupled ones, on the other hand, restrict the maximum mass of players that can simultaneously play strategies that belong to the same group of strategies, here referred to as a policy. More formally, we consider that the set of strategies $\mathcal{S}$ is partitioned by a (given) set of $P \in \mathbb{Z}_{\geq 1}$ disjoint policies indexed by $\mathcal{P} = \{1, 2, \ldots, P\}$. Namely, the set $\mathcal{S}^p \subseteq \mathcal{S}$ denotes the set of strategies that belong to policy p, for all $p \in \mathcal{P}$, and $\mathcal{S} = \cup_{p\in\mathcal{P}}\mathcal{S}^p$ and $\mathcal{S}^p \cap \mathcal{S}^q = \emptyset$, for all $p, q \in \mathcal{P}$ with $p \neq q$, i.e., each strategy $i \in \mathcal{S}$ belongs to exactly one policy in $\mathcal{P}$. To ease the forthcoming discussions, we define the index-valued function $h : \mathcal{S} \to \mathcal{P}$, which provides the index of the policy for a given strategy. More precisely, for all $i \in \mathcal{S}$ and all $p \in \mathcal{P}$, $h(i) = p \Leftrightarrow i \in \mathcal{S}^p$.

To formally define the capacity constraints, let $d_i \in \mathbb{R}_{>0}$ be the maximum (decoupled) capacity of the strategy $i \in \mathcal{S}$, and let $c^{h(i)} \in \mathbb{R}_{>0}$ be the maximum (coupled) capacity of the policy $h(i) \in \mathcal{P}$. Additionally, let $\alpha_i(x_i) = d_i - x_i$ and $\beta_i(\mathbf{x}) = c^{h(i)} - \sum_{j\in\mathcal{S}^{h(i)}} x_j$, for all $i \in \mathcal{S}$, and let $\boldsymbol{\alpha}(\cdot) = [\alpha_1(\cdot), \alpha_2(\cdot), \ldots, \alpha_n(\cdot)]^\top \in \mathbb{R}^n$, and $\boldsymbol{\beta}(\cdot) = [\beta_1(\cdot), \beta_2(\cdot), \ldots, \beta_n(\cdot)]^\top \in \mathbb{R}^n$. Therefore, a strategic distribution (also referred to as population state) $\mathbf{x} \in \Delta$ is feasible if and only if $\mathbf{x} \in \mathcal{X}$, where

$$\mathcal{X} = \{\mathbf{x} \in \Delta \,:\, \boldsymbol{\alpha}(\mathbf{x}) \succeq \mathbf{0},\, \boldsymbol{\beta}(\mathbf{x}) \succeq \mathbf{0}\}\,. \tag{3}$$

Here, ($\succeq$) denotes the element-wise inequality, and $\mathbf{0}$ is the zero vector of appropriate dimension. Besides, we impose the following assumption on the capacity constraints.

Standing Assumption 1: The total population mass and the capacity constraints satisfy that $m < \sum_{i\in\mathcal{S}} d_i$, and that $m < \sum_{p\in\mathcal{P}} c^p$.

Under Standing Assumption 1, it follows that the feasible set $\mathcal{X}$ is nonempty. Moreover, note that because $\mathcal{X} \subseteq \Delta$, for every $d_i > m$ we could set $d_i = m$ without changing the feasible set $\mathcal{X}$. Similarly, for every $c^p > m$ we could set $c^p = m$ without changing $\mathcal{X}$. Hence, without loss of generality, under the considered framework it is always possible (if required) to upper bound the capacities by the total population mass m. This observation might be useful to obtain smaller α^* terms in the forthcoming Theorems 3 and 5.

To capture the information regarding the capacity constraints at a given population state $\mathbf{x}$, we further define the map $\phi_{ij} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, for all $i, j \in \mathcal{S}$, given by

$$\phi_{ij}(\mathbf{x}) = \begin{cases} [\alpha_j(x_j)]_0\,, & \text{if } h(i) = h(j), \\ \min\left\{[\alpha_j(x_j)]_0\,,\, [\beta_j(\mathbf{x})]_0\right\}, & \text{if } h(i) \neq h(j). \end{cases} \tag{4}$$

Here, $[\cdot]_0 \triangleq \max\{\cdot, 0\}$. In particular, observe that $\phi_{ij}(\mathbf{x}) > 0$ if and only if strategy $j \in \mathcal{S}$ and policy $h(j) \in \mathcal{P}$ have enough capacity to receive a player from strategy $i \in \mathcal{S}$ at the population state $\mathbf{x}$. Namely, $\phi_{ij}(\mathbf{x})$ provides the capacity of strategy $j \in \mathcal{S}$ to receive a player from strategy $i \in \mathcal{S}$ at the population state $\mathbf{x}$.

Migration constraints: Besides the aforementioned capacity constraints, we consider some graphical interaction constraints over the strategies of the game. Namely, let $\mathcal{G} = (\mathcal{S}, \mathcal{E}, \mathbf{W})$ be the migration graph of the game, where $\mathcal{S}$ is the set of nodes; $\mathcal{E} = \{(i,j) : i, j \in \mathcal{S}\}$ is the set of edges; and $\mathbf{W} \in \mathbb{R}^{n\times n}_{\geq 0}$ is the weighted adjacency matrix that describes the structure of the graph, i.e., $w_{ij} > 0$ if $(i,j) \in \mathcal{E}$ and $w_{ij} = 0$ otherwise. Moreover, let $\mathcal{N}_i = \{j \in \mathcal{S} : w_{ij} > 0\} \cup \{i\}$ be the set of strategies that the players playing $i \in \mathcal{S}$ can migrate to (and thus interact with). The interpretation of the migration constraints is as follows. A player playing $i \in \mathcal{S}$ is able to switch to $j \in \mathcal{S}$ only if $w_{ij} > 0$. In addition, a player playing $i \in \mathcal{S}$ has information regarding $j \in \mathcal{S}$ if and only if $w_{ij} > 0$. We further impose the following assumptions.

Standing Assumption 2: For all $i \in \mathcal{S}$, $f_i(\cdot)$ depends only on local information available over $\mathcal{N}_i$.

Standing Assumption 3: For all $i, j \in \mathcal{S}$, it holds that if $h(i) = h(j)$, then $w_{ij} > 0$. Additionally, $\mathbf{W} = \mathbf{W}^\top$.

Revision protocol: Based on the considered framework, in this paper we propose the revision protocol given by

$$\rho_{ij}\left(\mathbf{x}, \mathbf{f}\left(\mathbf{x}\right)\right) = w_{ij}\phi_{ij}\left(\mathbf{x}\right)\left[f_j\left(\mathbf{x}\right) - f_i\left(\mathbf{x}\right)\right]_0^\gamma\,, \quad \forall i, j \in \mathcal{S}, \tag{5}$$

where $[\cdot]_0^\gamma \triangleq \min\left\{[\cdot]_0\,, \gamma\right\}$; and $\gamma \in \mathbb{R}_{>0}$ is a (fixed) parameter of the decision-making mechanism and is assumed equal for all players. In particular, $\rho_{ij} > 0$ if and only if $w_{ij} > 0$, $\phi_{ij}(\mathbf{x}) > 0$, and $f_j(\mathbf{x}) > f_i(\mathbf{x})$. Hence, the proposed revision protocol considers the migration constraints imposed by $\mathcal{G}$, the capacity constraints characterized by $\phi_{ij}(\cdot)$, and the underlying game $\mathbf{f}(\cdot)$.

To end this section, we highlight two observations regarding the proposed revision protocol in (5). First, note that under Standing Assumptions 2 and 3, each player is able to evaluate her corresponding revision protocol in a distributed fashion without the need of centralized information schemes. In consequence, the discretized dynamics in (2) (under the revision protocol in (5)) can be implemented in a distributed fashion with the migration graph $\mathcal{G}$ playing the role of the communication graph between the nodes that update each $x_i[k]$. Second, notice that the saturation provided by γ implies that $\rho_{ij}(\mathbf{x}, \mathbf{f}(\mathbf{x})) \leq w_{ij}\phi_{ij}(\mathbf{x})\gamma$, for all $\mathbf{x} \in \Delta$. That is, the probability of a player switching from strategy i to strategy j is upper bounded by $w_{ij}\phi_{ij}(\mathbf{x})\gamma/R$, for all $i, j \in \mathcal{S}$, and such an upper bound is independent of the game $\mathbf{f}(\cdot)$. The fact that the maximum conditional switch rate between the strategies of the game is independent of the payoff vector $\mathbf{f}(\cdot)$ plays an important role in the invariance analysis of the discretized dynamics (c.f., Theorem 3 and Remark 1).

3 Analysis of the proposed dynamics

In this section, we analyze the dynamics in (1) (and (2)) for the case when the revision protocol is set according to (5). Notice that such dynamics are equivalent to

$$\dot{x}_i = \sum_{j\in\mathcal{S}} \theta_{ij}(\mathbf{x})\left(f_i(\mathbf{x}) - f_j(\mathbf{x})\right), \quad \forall i \in \mathcal{S}, \tag{6}$$

where, for all $i, j \in \mathcal{S}$,

$$\theta_{ij}(\mathbf{x}) = \begin{cases} x_i w_{ij}\phi_{ij}(\mathbf{x})\zeta_{ij}(\mathbf{x}), & \text{if } f_i(\mathbf{x}) < f_j(\mathbf{x}), \\ x_j w_{ji}\phi_{ji}(\mathbf{x})\zeta_{ji}(\mathbf{x}), & \text{if } f_i(\mathbf{x}) > f_j(\mathbf{x}), \\ 0, & \text{if } f_i(\mathbf{x}) = f_j(\mathbf{x}), \end{cases}$$

$$\zeta_{ij}(\mathbf{x}) = \begin{cases} 1, & \text{if } |f_i(\mathbf{x}) - f_j(\mathbf{x})| \leq \gamma, \\ \frac{\gamma}{|f_i(\mathbf{x}) - f_j(\mathbf{x})|}, & \text{if } |f_i(\mathbf{x}) - f_j(\mathbf{x})| > \gamma. \end{cases}$$

Clearly, $\theta_{ij}(\cdot) = \theta_{ji}(\cdot)$ and $\zeta_{ij}(\cdot) \leq 1$, for all $i, j \in \mathcal{S}$. Additionally, such dynamics can be expressed in matrix form as $\dot{\mathbf{x}} = \mathbf{L}(\mathbf{x})\mathbf{f}(\mathbf{x})$, where $\mathbf{L}(\mathbf{x}) \in \mathbb{R}^{n\times n}$ is a matrix whose elements are $\ell_{ii}(\mathbf{x}) = \sum_{j\in\mathcal{S}\setminus\{i\}} \theta_{ij}(\mathbf{x})$, and $\ell_{ij}(\mathbf{x}) = -\theta_{ij}(\mathbf{x})$, for all $i, j \in \mathcal{S}$ with $i \neq j$. Moreover, we provide the auxiliary Lemma 1 regarding $\mathbf{L}(\cdot)$.

Lemma 1 *For all $\mathbf{x} \in \mathcal{X}$, $\mathbf{L}(\mathbf{x})$ is positive semi-definite; $(\mathbf{f}(\mathbf{x}))^\top \mathbf{L}(\mathbf{x})\mathbf{f}(\mathbf{x}) = 0 \Leftrightarrow \theta_{ij}(\mathbf{x})(f_i(\mathbf{x}) - f_j(\mathbf{x})) = 0$, for all $i, j \in \mathcal{S}$; and $\mathbf{L}(\mathbf{x})\mathbf{f}(\mathbf{x}) = \mathbf{0} \Leftrightarrow (\mathbf{f}(\mathbf{x}))^\top \mathbf{L}(\mathbf{x})\mathbf{f}(\mathbf{x}) = 0$.*

3.1 Analysis of the set of equilibria

In this section, we provide sufficient conditions on the graph $\mathcal{G}$ to guarantee that the set of equilibria of the dynamics in (1) (and equivalently (2)) under the revision protocol in (5) coincides with the set of generalized Nash equilibria of the underlying capacity-constrained population game. Throughout, let $\mathbf{\Phi}_i(\cdot) = [\phi_{i1}(\cdot), \phi_{i2}(\cdot), \ldots, \phi_{in}(\cdot)]^\top \in \mathbb{R}^n$ and let $\mathcal{C}_i(\cdot) = \operatorname{supp}(\mathbf{\Phi}_i(\cdot))$, for all $i \in \mathcal{S}$ (here, $\operatorname{supp}(\mathbf{\Phi}_i(\mathbf{x}))$ denotes the support of vector $\mathbf{\Phi}_i(\mathbf{x})$). Namely, $\mathcal{C}_i(\mathbf{x})$ is the set of strategies that have available capacity to receive a player from strategy $i \in \mathcal{S}$ at the strategic distribution $\mathbf{x}$.

Definition 2 *Given a population game $\mathbf{f}(\cdot)$ and the feasible set $\mathcal{X}$ in (3), the set of generalized Nash equilibria of $\mathbf{f}(\cdot)$ is defined as*

$$\mathrm{GNE}(\mathbf{f}) = \left\{ \mathbf{x} \in \mathcal{X} : \begin{array}{c} x_i > 0 \Rightarrow f_i(\mathbf{x}) \geq f_j(\mathbf{x}), \\ \forall i \in \mathcal{S},\ j \in \mathcal{C}_i(\mathbf{x}) \end{array} \right\}.$$

Similarly, the set of graph-dependent generalized Nash equilibria of $\mathbf{f}(\cdot)$ is defined as

$$\mathrm{GNE}_{\mathcal{G}}(\mathbf{f}) = \left\{ \mathbf{x} \in \mathcal{X} : \begin{array}{c} x_i > 0 \Rightarrow f_i(\mathbf{x}) \geq f_j(\mathbf{x}), \\ \forall i \in \mathcal{S},\ j \in \mathcal{C}_i(\mathbf{x}) \cap \mathcal{N}_i \end{array} \right\}.$$

The interpretation of Definition 2 is as follows. At any $\mathbf{x}^* \in \mathrm{GNE}(\mathbf{f})$ no player can increase her payoff by unilaterally deviating from her selected strategy to any other strategy with available capacity. Similarly, at any $\mathbf{x}^* \in \mathrm{GNE}_{\mathcal{G}}(\mathbf{f})$ no player can increase her payoff by unilaterally deviating from her selected strategy to any other strategy in her neighborhood with available capacity. Clearly, $\mathrm{GNE}(\mathbf{f}) \subseteq \mathrm{GNE}_{\mathcal{G}}(\mathbf{f})$. Moreover, when the game $\mathbf{f}(\cdot)$ is continuous, it immediately follows that a GNE exists and, therefore, the set $\mathrm{GNE}(\mathbf{f})$ is nonempty.

Lemma 2 *Let $\mathbf{f}(\cdot)$ be continuous. Then, the set $\mathrm{GNE}(\mathbf{f})$ is nonempty and compact.*

Hence, from Lemma 2 it follows that a GNE always exists for any full-potential population game under the considered capacity constraints. We now provide formal connections between the set $\mathrm{GNE}(\mathbf{f})$ and the set of equilibria of the considered dynamics.

Lemma 3 *Consider the dynamics in (1) under the revision protocol in (5). A population state $\mathbf{x}^* \in \mathcal{X}$ is an equilibrium of these dynamics if and only if $\mathbf{x}^* \in \mathrm{GNE}_{\mathcal{G}}(\mathbf{f})$.*

Lemma 3 guarantees the coincidence of the set of equilibria of the dynamics and the set $\mathrm{GNE}_{\mathcal{G}}(\mathbf{f})$. Under some assumptions on the graph $\mathcal{G}$, it is possible to further prove that $\mathbf{x} \in \mathrm{GNE}_{\mathcal{G}}(\mathbf{f}) \Leftrightarrow \mathbf{x} \in \mathrm{GNE}(\mathbf{f})$.

Definition 3 *Given the feasible region $\mathcal{X}$ in (3), a strategy $s \in \mathcal{S}$ is said to be a support strategy at a state $\mathbf{x} \in \mathcal{X}$,* *if it holds that* $0 < x_s < d_s$ *and* $\beta_s(\mathbf{x}) > 0$. *Consequently,* $\tilde{\mathcal{S}}(\mathbf{x}) = \{s \in \mathcal{S} : 0 < x_s < d_s,\ \beta_s(\mathbf{x}) > 0\}$ *is the set of support strategies at the state* $\mathbf{x}$.

Assumption 1 *For every* $\mathbf{x}^* \in \mathrm{GNE}_{\mathcal{G}}(\mathbf{f})$ *it holds that: i) the set* $\tilde{\mathcal{S}}(\mathbf{x}^*)$ *is nonempty; ii) the subgraph of* $\mathcal{G}$ *that considers only the strategies in* $\tilde{\mathcal{S}}(\mathbf{x}^*)$ *is connected; and iii)* $\tilde{\mathcal{S}}(\mathbf{x}^*) \cap \mathcal{N}_i$ *is nonempty, for all* $i \in \mathcal{S}$.

Theorem 1 *Consider the dynamics in* (1) *under the revision protocol in* (5). *Besides, suppose that Assumption 1 holds. Then, a population state* $\mathbf{x}^* \in \mathcal{X}$ *is an equilibrium of the dynamics if and only if* $\mathbf{x}^* \in \mathrm{GNE}(\mathbf{f})$, *i.e.,* $\mathbf{L}(\mathbf{x}^*)\,\mathbf{f}(\mathbf{x}^*) = \mathbf{0} \Leftrightarrow \mathbf{x}^* \in \mathrm{GNE}(\mathbf{f})$.

We highlight that Assumption 1 is not a necessary condition for Theorem 1 to hold. Consider for instance the case where the set $\tilde{\mathcal{S}}(\mathbf{x}^*)$ is empty for some population state $\mathbf{x}^* \in \mathrm{GNE}_{\mathcal{G}}(\mathbf{f})$, but the migration graph $\mathcal{G}$ is complete. Under such a scenario it trivially holds that $\mathrm{GNE}(\mathbf{f}) = \mathrm{GNE}_{\mathcal{G}}(\mathbf{f})$, yet Assumption 1 does not hold. Moreover, observe that if $\mathrm{GNE}_{\mathcal{G}}(\mathbf{f}) \subseteq \mathrm{relint}(\mathcal{X})$, where $\mathrm{relint}(\mathcal{X})$ is the relative interior of $\mathcal{X}$, then Assumption 1 reduces to the standard connectivity of $\mathcal{G}$. In Section 4, we further illustrate a less restrictive scenario where Assumption 1 readily holds.

Besides the coincidence of $\mathrm{GNE}(\mathbf{f})$ and the set of equilibria of the considered dynamics, if $\mathbf{f}(\cdot)$ is a full-potential game (c.f., Definition 1), then it is possible to derive some additional results.

Lemma 4 *Let* $\mathbf{f}(\cdot)$ *be a full-potential game with concave potential function* $\varphi(\cdot)$. *Then,* $\mathbf{x}^* \in \mathrm{GNE}(\mathbf{f})$ *if and only if* $\mathbf{x}^* \in \arg\max_{\mathbf{x}\in\mathcal{X}} \varphi(\mathbf{x})$.

To end this section, we remark that the set of equilibria of the continuous-time dynamics in (1) is aligned with the set of equilibria of their discretized counter part in (2). Hence, the results of Lemma 3 and Theorem 1 are also valid for the discretized dynamics in (2) when the revision protocol in (5) is considered.

3.2 Invariance analysis

In this section, we show that the feasible set $\mathcal{X}$ is positively invariant under the dynamics in (1) when the proposed revision protocol in (5) is considered. Furthermore, we also provide sufficient conditions to extend such a result to the discretized dynamics in (2).

Theorem 2 *Consider the dynamics in* (1) *under the revision protocol in* (5), *and consider the feasible set* $\mathcal{X}$ *in* (3). *If* $\mathbf{x}(0) \in \mathcal{X}$, *then* $\mathbf{x}(t) \in \mathcal{X}$ *for all* $t \geq 0$.

Theorem 3 *Consider the dynamics in* (2) *under the revision protocol in* (5), *and consider the feasible set* $\mathcal{X}$ *in* (3). *Moreover, let* $\alpha^* = \max_{i\in\mathcal{S}} d_i$, $\eta^* = \max_{p\in\mathcal{P}} |\mathcal{S}^p|$, *and* $\delta^* = \max_{i\in\mathcal{S}} \sum_{j\in\mathcal{S}} w_{ij}$. *If* $0 < \epsilon \leq (\gamma\alpha^*\eta^*\delta^*)^{-1}$, *then it holds that* $\mathbf{x}[k] \in \mathcal{X} \Rightarrow \mathbf{x}[k+1] \in \mathcal{X}$, *for all* $k \geq 0$.

Remark 1 *Notice that the upper bound on* ϵ *in Theorem 3 is independent of the game* $\mathbf{f}(\cdot)$. *The parameters* α^*, η^*, *and* δ^* *are fully determined by the constraints of the problem, and the parameter* γ *is fixed by the revision protocol. Besides,* $\alpha^* \in \mathbb{R}_{>0}$ *because* $d_i \in \mathbb{R}_{>0}$, *for all* $i \in \mathcal{S}$; $\eta^* \in \mathbb{R}_{>0}$ *because* $\min_{p\in\mathcal{P}} |\mathcal{S}^p| \geq 1$; *and* $\delta^* \in \mathbb{R}_{>0}$ *because* $w_{ii} > 0$, *for all* $i \in \mathcal{S}$ *[c.f., Standing Assumption 3]. Hence,* $(\gamma\alpha^*\eta^*\delta^*)^{-1}$ *is positive and finite.*

3.3 Convergence analysis

In this section, we provide sufficient conditions to guarantee the asymptotic stability of the set $\mathrm{GNE}(\mathbf{f})$ under the dynamics in (1) when the revision protocol in (5) is considered and when the game is a full-potential population game with concave potential function (c.f., Definition 1). Besides, we also extend the analysis for the discretized dynamics in (2).

Assumption 2

- *i) The game* $\mathbf{f}(\cdot)$ *is a full-potential game with concave potential function* $\varphi(\cdot)$.
- *ii) Moreover,* $\varphi(\cdot)$ *is twice continuously differentiable and L-smooth under the Euclidean norm. Thus,* $\|\nabla\varphi(\mathbf{x}) - \nabla\varphi(\mathbf{y})\|_2 \leq L\|\mathbf{x} - \mathbf{y}\|_2$, *for some* $L \in \mathbb{R}_{>0}$ *and for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_{\geq 0}$.

Theorem 4 *Consider the dynamics in* (1) *under the revision protocol in* (5), *and consider the feasible set* $\mathcal{X}$ *in* (3). *Moreover, suppose that Assumptions 1 and 2i) hold, and let* $\mathbf{x}(0) \in \mathcal{X}$. *Then, the set* $\mathrm{GNE}(\mathbf{f})$ *is asymptotically stable under the considered dynamics.*

Theorem 5 *Consider the dynamics in* (2) *under the revision protocol in* (5), *and consider the feasible set* $\mathcal{X}$ *in* (3). *Moreover, suppose that Assumptions 1 and 2 hold, let* $\mathbf{x}[0] \in \mathcal{X}$, *and let* ϵ *satisfy the conditions of Theorem 3. If in addition* $\epsilon < (Lm\alpha^*\delta^*)^{-1}$, *then the set* $\mathrm{GNE}(\mathbf{f})$ *is asymptotically stable under the considered dynamics.*

Theorems 3 and 5 show that, under the proposed revision protocol in (5), the discretized dynamics in (2) serve as a discrete-time distributed optimization algorithm (with fixed step size ϵ) that allows the hard satisfaction of the capacity constraints for all forward times. Such a hard satisfaction of constraints is a significant advantage over other recent distributed optimization algorithms that only satisfy constraints in the asymptotic sense (Falsone et al. 2020, Liang et al. 2020, Yang et al. 2019). Moreover, similar to Remark 1, and using the facts that $L \in \mathbb{R}_{>0}$ and $m \in \mathbb{R}_{>0}$, it follows that the upper bound on ϵ in Theorem 5 is positive and finite. Furthermore, although the parameters α^*, η^*, and δ^* might be computed distributedly in some special cases (e.g., homogeneous capacity constraints under doubly stochastic migration graphs), the parameters L and m rely on global information that must be known in advance by the nodes that compute the discretized dynamics in (2). We leave it for future research to explore how to remove such an informational requirement.

4 An illustrative example

Population games have been recently applied to model various control problems in the context of dynamic resource allocation (Quijano et al. 2017, Barreiro-Gomez & Tembine 2018, Martinez-Piazuelo et al. 2020). In this section, we illustrate our developed theory on a (constrained) congestion game, which is a game-theoretical abstraction useful to model several of the aforementioned engineering problems and have been also considered in recent researches in the field of population games (Park et al. 2019, Martinez-Piazuelo, Quijano & Ocampo-Martinez 2022).

Consider a population of players, represented by a continuum of mass $m = 1$, that seek to go from A to B using 7 possible roads as shown in Fig. 1a. Namely, $\mathcal{S} = \{1, 2, 3, 4, 5\} \triangleq \{\{r_1, r_2\}, \{r_1, r_3\}, \{r_4\}, \{r_5, r_7\}, \{r_6, r_7\}\}$. Furthermore, let the strategies be partitioned into 3 policies as $\mathcal{S}^1 = \{1, 2\}$, $\mathcal{S}^2 = \{3\}$, $\mathcal{S}^3 = \{4, 5\}$. The corresponding capacity constraints are $x_1 \leq 0.1$, $x_2 \leq 0.4$, $x_3 \leq 1.1$, $x_4 \leq 0.2$, $x_5 \leq 0.3$, $x_1 + x_2 \leq 0.3$, and $x_4 + x_5 \leq 0.5$, and the migration constraints are depicted in Fig. 1b. Observe that since $m < 1.1$ and $m > 0.3+0.5$, it follows that for every $\mathbf{x} \in \mathcal{X}$, $0 < x_3 < 1.1$. Thus, Strategy 3 (r_4) plays the role of a support strategy (c.f., Definition 3) and Assumption 1 holds. Besides, from the considered parameters it follows that $\alpha^* = 1.1$, $\eta^* = 2$, and $\delta^* = 1$. Now, similar to Park et al. (2019) (yet without loss of generality), let the payoffs be determined by $f_1(\mathbf{x}) = -2x_1 - x_2$, $f_2(\mathbf{x}) = -x_1 - 2x_2$, $f_3(\mathbf{x}) = -2\nu(t)x_3$, $f_4(\mathbf{x}) = -2x_4 - x_5$, and $f_5(\mathbf{x}) = -x_4 - 2x_5$, where $\nu(t) \in \{1, 2\}$ is an exogenous time-varying signal that decreases/increases the congestion cost of r_4. It is straightforward to verify that the game $\mathbf{f}(\cdot)$ satisfies Assumption 2 with a concave quadratic potential function with $L = 4$. Finally, for our numerical simulation we set $\gamma = 1$, and so $\epsilon = 0.22$ satisfies the conditions of Theorems 3 and 5. As illustration, Fig. 2 depicts the temporal evolution of the (discretized) dynamics in (2) under the proposed revision protocol of (5). Note that both the asymptotic stability of $\mathrm{GNE}(\mathbf{f})$ and the invariance of $\mathcal{X}$ are verified.

5 Concluding remarks

In this brief, we have proposed and analyzed a novel decision-making mechanism for generalized Nash equilibrium seeking in full-potential population games under capacity and migration constraints. Furthermore, we have also developed a discrete-time analysis of the proposed dynamics and derived a discrete-time population-game-inspired distributed optimization algorithm that allows the satisfaction of the capacity constraints for all times. Future research should explore the extension of the developed theory to more general families of games, e.g., stable/monotone games, as well as the characterization of the convergence rate of the resulting dynamics.

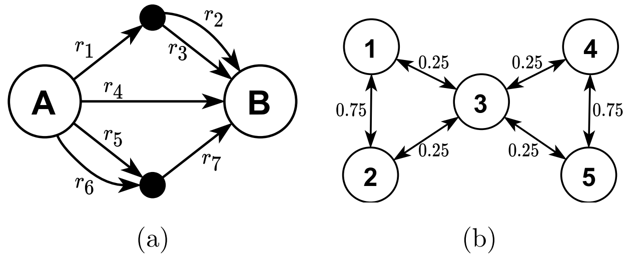

Fig. 1. Considered congestion game. (a) Considered topology for the congestion game. (b) Migration graph $\mathcal{G}$.

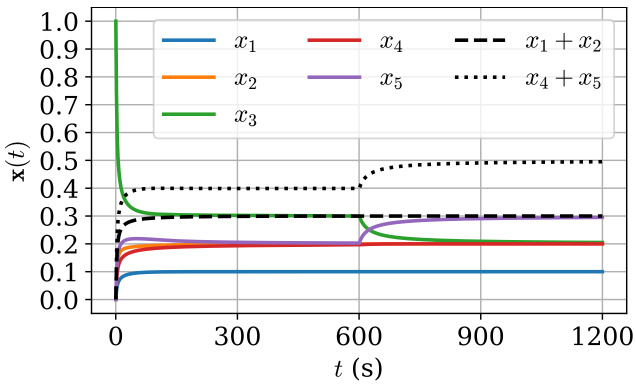

Fig. 2. Temporal evolution of the population state $\mathbf{x}$, with $\mathbf{x}(0) = [0, 0, 1, 0, 0]^\top \in \mathcal{X}$. For $t \leq 600$, $\nu(t) = 1$ and $\mathbf{x}$ converges to $\mathbf{x}^* = [0.1, 0.2, 0.3, 0.2, 0.2]^\top \in \mathrm{GNE}(\mathbf{f})$. For $t > 600$, $\nu(t) = 2$ and $\mathbf{x}$ converges to $\mathbf{x}^* = [0.1, 0.2, 0.2, 0.2, 0.3]^\top \in \mathrm{GNE}(\mathbf{f})$.

6 Proofs

6.1 Proof of Lemma 1

First, observe that for every $\mathbf{x} \in \mathcal{X}$, $\mathbf{L}(\mathbf{x})$ is a symmetric diagonally dominant real matrix with non-negative diagonal elements. Hence, $\mathbf{L}(\mathbf{x})$ is positive semi-definite for all $\mathbf{x} \in \mathcal{X}$. Second, note that $(\mathbf{f}(\mathbf{x}))^\top \mathbf{L}(\mathbf{x})\mathbf{f}(\mathbf{x}) = \sum_{(i,j)\in\mathcal{E}} \theta_{ij}(\mathbf{x}) (f_i(\mathbf{x}) - f_j(\mathbf{x}))^2$ (this follows from the quadratic form of Laplacian matrices). Since $\mathbf{x} \in \mathcal{X} \Rightarrow \theta_{ij}(\mathbf{x}) \geq 0$, for all $i, j \in \mathcal{S}$, it holds that $(\mathbf{f}(\mathbf{x}))^\top \mathbf{L}(\mathbf{x})\mathbf{f}(\mathbf{x}) = 0$ if and only if $\theta_{ij}(\mathbf{x}) (f_i(\mathbf{x}) - f_j(\mathbf{x})) = 0$, for all $i, j \in \mathcal{S}$. Finally, it is obvious that $\mathbf{L}(\mathbf{x})\mathbf{f}(\mathbf{x}) = \mathbf{0} \Rightarrow (\mathbf{f}(\mathbf{x}))^\top \mathbf{L}(\mathbf{x})\mathbf{f}(\mathbf{x}) = 0$. Moreover, from (6) it is straightforward to check that if $\theta_{ij}(\mathbf{x}) (f_i(\mathbf{x}) - f_j(\mathbf{x})) = 0$, for all $i, j \in \mathcal{S}$, then $\dot{x}_i = 0$, for all $i \in \mathcal{S}$, and thus $\mathbf{L}(\mathbf{x})\mathbf{f}(\mathbf{x}) = \mathbf{0}$. ∎

6.2 Proof of Lemma 2

Notice that GNE($\mathbf{f}$) in Definition 2 is equivalent to GNE($\mathbf{f}$) $= \left\{\mathbf{x} \in \mathcal{X} : \mathbf{x} \in \arg\max_{\mathbf{y}\in\mathcal{X}} \mathbf{y}^\top \mathbf{f}(\mathbf{x})\right\}$, which implies that $\mathbf{x}^* \in \mathrm{GNE}(\mathbf{f}) \Leftrightarrow (\mathbf{x}^*)^\top \mathbf{f}(\mathbf{x}^*) \geq \mathbf{y}^\top \mathbf{f}(\mathbf{x}^*)$, for all $\mathbf{y} \in \mathcal{X}$. Consequently, the considered set GNE($\mathbf{f}$) coincides with the set of solutions of the variational inequality VI$(\mathcal{X}, -\mathbf{f})$ given by: find $\mathbf{x} \in \mathcal{X}$ such that $(\mathbf{y}-\mathbf{x})^\top(-\mathbf{f}(\mathbf{x})) \geq 0$, for all $\mathbf{y} \in \mathcal{X}$. Since $\mathbf{f}(\cdot)$ is continuous and $\mathcal{X}$ is nonempty, convex, and compact, it follows from (Facchinei & Pang 2003, Corollary 2.2.5) that GNE($\mathbf{f}$) is nonempty and compact. ■

6.3 Proof of Lemma 3

Note that $\mathbf{x}^* \in \mathcal{X}$ is an equilibrium of the dynamics if and only if $\mathbf{L}(\mathbf{x}^*)\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$. Therefore, using Lemma 1, it follows that $\mathbf{x}^*$ is an equilibrium of the dynamics if and only if $\theta_{ij}(\mathbf{x}^*)(f_i(\mathbf{x}^*) - f_j(\mathbf{x}^*)) = 0$, for all $i, j \in \mathcal{S}$. Hence, we must prove that such a condition holds if and only if $\mathbf{x}^* \in \mathrm{GNE}_\mathcal{G}(\mathbf{f})$.

(Sufficiency) Let $\mathbf{x}^* \in \mathrm{GNE}_\mathcal{G}(\mathbf{f})$. If $x_i^* > 0$, then $f_i(\mathbf{x}^*) \geq f_j(\mathbf{x}^*)$, for all $j \in \mathcal{C}_i(\mathbf{x}^*) \cap \mathcal{N}_i$. Moreover, if $f_i(\mathbf{x}^*) > f_j(\mathbf{x}^*)$, then $x_j^* = 0$ and $\theta_{ij}(\mathbf{x}^*) = \theta_{ji}(\mathbf{x}^*) = 0$. Thus, $\theta_{ij}(\mathbf{x}^*)(f_i(\mathbf{x}^*) - f_j(\mathbf{x}^*)) = 0$, $\forall i, j \in \mathcal{S}$.

(Necessity) Let $\theta_{ij}(\mathbf{x}^*)(f_i(\mathbf{x}^*) - f_j(\mathbf{x}^*)) = 0$, for all $i, j \in \mathcal{S}$, but suppose that $\mathbf{x}^* \in \mathcal{X} \setminus \mathrm{GNE}_\mathcal{G}(\mathbf{f})$. Hence, there is some $x_i^* > 0$ such that $f_i(\mathbf{x}^*) < f_j(\mathbf{x}^*)$ for some $j \in \mathcal{C}_i(\mathbf{x}^*) \cap \mathcal{N}_i$. Thus, $\theta_{ij}(\mathbf{x}^*) = w_{ij} x_i^* \phi_{ij}(\mathbf{x}^*) \zeta_{ij}(\mathbf{x}^*) > 0$, leading to a contradiction. ■

6.4 Proof of Theorem 1

(Sufficiency) Recall from Lemma 3 that a state $\mathbf{x}^* \in \mathcal{X}$ is an equilibrium of the dynamics if and only if $\mathbf{x}^* \in \mathrm{GNE}_\mathcal{G}(\mathbf{f})$. Since $\mathrm{GNE}(\mathbf{f}) \subseteq \mathrm{GNE}_\mathcal{G}(\mathbf{f})$, every $\mathbf{x}^* \in \mathrm{GNE}(\mathbf{f})$ is also an equilibrium of the dynamics.

(Necessity) Recall Lemma 3. To prove that $\mathbf{x}^* \in \mathcal{X}$ is an equilibrium of the dynamics only if $\mathbf{x}^* \in \mathrm{GNE}(\mathbf{f})$, we must show that, under the given assumptions, it holds that $\mathbf{x}^* \in \mathrm{GNE}_\mathcal{G}(\mathbf{f}) \Rightarrow \mathbf{x}^* \in \mathrm{GNE}(\mathbf{f})$. Let $\mathbf{x}^* \in \mathrm{GNE}_\mathcal{G}(\mathbf{f})$ but suppose that $\mathbf{x}^* \notin \mathrm{GNE}(\mathbf{f})$. By definition, there is an $i \in \mathcal{S}$ and a $j \in \mathcal{C}_i(\mathbf{x}^*) \setminus \mathcal{N}_i$ such that $x_i^* > 0$ and $f_i(\mathbf{x}^*) < f_j(\mathbf{x}^*)$. Moreover, from the Standing Assumption 3 we can conclude that $h(i) \neq h(j)$. Now, from Assumption 1 and the fact that $\mathbf{x}^* \in \mathrm{GNE}_\mathcal{G}(\mathbf{f})$, it follows that there is some connected path of support strategies between i and j. Furthermore, by definition, all the support strategies have the same payoffs. Thus, let $s, z \in \tilde{\mathcal{S}}(\mathbf{x}^*)$ so that $f_s(\mathbf{x}^*) = f_z(\mathbf{x}^*)$. In addition, let $s \in \mathcal{N}_i$ and $z \in \mathcal{N}_j$. Since $h(i) \neq h(j)$ and $j \in \mathcal{C}_i(\mathbf{x}^*)$, it follows that $j \in \mathcal{C}_z(\mathbf{x}^*)$. Moreover, since $\mathbf{x}^* \in \mathrm{GNE}_\mathcal{G}(\mathbf{f})$, it must hold that $f_j(\mathbf{x}^*) = f_z(\mathbf{x}^*) = f_s(\mathbf{x}^*) \leq f_i(\mathbf{x}^*)$. Clearly, this is a contradiction with $f_j(\mathbf{x}^*) > f_i(\mathbf{x}^*)$, and, in consequence, $\mathbf{x}^* \in \mathrm{GNE}(\mathbf{f})$. ■

6.5 Proof of Lemma 4

Recall that GNE($\mathbf{f}$) in Definition 2 is equivalent to GNE($\mathbf{f}$) $= \left\{\mathbf{x} \in \mathcal{X} : \mathbf{x} \in \arg\max_{\mathbf{y}\in\mathcal{X}} \mathbf{y}^\top \mathbf{f}(\mathbf{x})\right\}$. Since $\varphi(\cdot)$ is concave and differentiable and $\varphi(\cdot) = \nabla \mathbf{f}(\cdot)$, it follows that $\mathbf{x}^* \in \arg\max_{\mathbf{x}\in\mathcal{X}} \varphi(\mathbf{x})$ if and only if $(\mathbf{x}-\mathbf{x}^*)^\top \mathbf{f}(\mathbf{x}^*) \leq 0$, for all $\mathbf{x} \in \mathcal{X}$, which is equivalent to $(\mathbf{x}^*)^\top \mathbf{f}(\mathbf{x}^*) \geq \mathbf{x}^\top \mathbf{f}(\mathbf{x}^*)$, for all $\mathbf{x} \in \mathcal{X}$. Clearly, it follows that $(\mathbf{x}^*)^\top \mathbf{f}(\mathbf{x}^*) \geq \mathbf{x}^\top \mathbf{f}(\mathbf{x}^*)$, for all $\mathbf{x} \in \mathcal{X}$, if and only if $\mathbf{x}^* \in \mathrm{GNE}(\mathbf{f})$. ■

6.6 Proof of Theorem 2

First, note that Δ is positively invariant under the considered dynamics. To see this, observe that $\sum_{i\in\mathcal{S}} \dot{x}_i = \sum_{i\in\mathcal{S}}\sum_{j\in\mathcal{S}} x_j \rho_{ji} - \sum_{i\in\mathcal{S}}\sum_{j\in\mathcal{S}} x_i \rho_{ij} = 0$. Hence, $\sum_{i\in\mathcal{S}} x_i(0) = m \Rightarrow \sum_{i\in\mathcal{S}} x_i(t) = m$, for all $t \geq 0$. Additionally, notice that $x_i = 0$ implies that $\dot{x}_i \geq 0$. Thus, $\mathbf{x}(0) \in \mathbb{R}^n_{\geq 0} \Rightarrow \mathbf{x}(t) \in \mathbb{R}^n_{\geq 0}$, for all $t \geq 0$. Second, note that if $\alpha_i(\mathbf{x}_i) = 0$, then $\rho_{ji} = 0$ for all $j \in \mathcal{S}$, and so $\dot{x}_i \leq 0$. Hence, $\boldsymbol{\alpha}(\mathbf{x}(0)) \succeq \mathbf{0} \Rightarrow \boldsymbol{\alpha}(\mathbf{x}(t)) \succeq \mathbf{0}$, for all $t \geq 0$. Finally, observe that if $\beta_i(\mathbf{x}) = 0$, then $\rho_{ji} = 0$ for all $j \in \mathcal{S} \setminus \mathcal{S}^{h(i)}$, and so $\sum_{s\in\mathcal{S}^{h(i)}} \dot{x}_s \leq 0$. Thus, $\boldsymbol{\beta}(\mathbf{x}(0)) \succeq \mathbf{0} \Rightarrow \boldsymbol{\beta}(\mathbf{x}(t)) \succeq \mathbf{0}$, for all $t \geq 0$. ■

6.7 Proof of Theorem 3

We divide the proof in four parts. First, we prove that $\mathbf{x}[k] \in \mathcal{X} \Rightarrow \mathbf{1}^\top \mathbf{x}[k+1] = m$. Second, we prove that $\mathbf{x}[k] \in \mathcal{X} \Rightarrow \mathbf{x}[k+1] \succeq \mathbf{0}$. Third, we prove that $\mathbf{x}[k] \in \mathcal{X} \Rightarrow \boldsymbol{\alpha}(\mathbf{x}[k+1]) \succeq \mathbf{0}$. Finally, we prove that $\mathbf{x}[k] \in \mathcal{X} \Rightarrow \boldsymbol{\beta}(\mathbf{x}[k+1]) \succeq \mathbf{0}$. Together these facts lead to the desired result [c.f., (3)].

$\left(\mathbf{x}[k] \in \mathcal{X} \Rightarrow \mathbf{1}^\top \mathbf{x}[k+1] = m\right)$ Let $\mathbf{x}[k] \in \mathcal{X}$. As in the proof of Theorem 2, note that $\sum_{i\in\mathcal{S}} \hat{x}_i[k] = 0$, for all k. Hence, $\mathbf{1}^\top \mathbf{x}[k] = m \Rightarrow \mathbf{1}^\top \mathbf{x}[k+1] = m$.

$(\mathbf{x}[k] \in \mathcal{X} \Rightarrow \mathbf{x}[k+1] \succeq \mathbf{0})$ Let $\mathbf{x}[k] \in \mathcal{X}$. From (2),

$$x_i[k+1] \geq x_i[k] - \epsilon \sum_{j\in\mathcal{S}} x_i[k]\rho_{ij}[k] = \left(1 - \epsilon \sum_{j\in\mathcal{S}} \rho_{ij}[k]\right) x_i[k].$$

Therefore, to guarantee the non-negativity of $x_i[k+1]$ we must show that $1 \geq \epsilon \sum_{j\in\mathcal{S}} \rho_{ij}[k]$, for all $i \in \mathcal{S}$. Here, since $\phi_{ij}(\mathbf{x}[k]) \leq \alpha^*$ and $[f_j(\mathbf{x}[k]) - f_i(\mathbf{x}[k])]_0^\gamma \leq \gamma$, for all $i, j \in \mathcal{S}$, it holds that

$$\begin{aligned}\epsilon \sum_{j\in\mathcal{S}} \rho_{ij}[k] &\leq \epsilon\gamma\alpha^* \sum_{j\in\mathcal{S}} w_{ij} \quad \text{[using (5)]}\\ &\leq \epsilon\gamma\alpha^*\delta^* \quad [\text{since } \delta^* = \max_{i\in\mathcal{S}} \sum_{j\in\mathcal{S}} w_{ij}]\\ &\leq 1 \quad [\text{since } \epsilon \leq (\gamma\alpha^*\eta^*\delta^*)^{-1} \text{ and } \eta^* \geq 1].\end{aligned}$$

Thus, $\mathbf{x}[k] \in \mathcal{X} \Rightarrow \mathbf{x}[k+1] \succeq \mathbf{0}$.

($\mathbf{x}[k] \in \mathcal{X} \Rightarrow \boldsymbol{\alpha}(\mathbf{x}[k+1]) \succeq \mathbf{0}$) Let $\mathbf{x}[k] \in \mathcal{X}$. From (2) and the definition of $\alpha_i(\cdot)$ it follows that

$$\begin{aligned}\alpha_i(x_i[k+1]) &= \alpha_i(x_i[k]) - \epsilon\hat{x}_i[k]\\ &\geq \alpha_i(x_i[k]) - \epsilon\sum_{j\in\mathcal{S}} x_j[k]\rho_{ji}[k]\\ &\geq \alpha_i(x_i[k]) - \epsilon\alpha_i(x_i[k])\gamma\sum_{j\in\mathcal{S}} w_{ji}x_j[k]\\ &= \left(1 - \epsilon\gamma\sum_{j\in\mathcal{S}} w_{ji}x_j[k]\right)\alpha_i(x_i[k])\\ &\geq (1-\epsilon\gamma\alpha^*\delta^*)\alpha_i(x_i[k]),\end{aligned}$$

where we have used the facts that $\phi_{ji}(\mathbf{x}[k]) \leq \alpha_i(x_i[k])$ and that $\mathbf{x}[k] \in \mathcal{X} \Rightarrow x_j[k] \leq \alpha^*$, for all $i, j \in \mathcal{S}$. Hence, with $\epsilon \leq (\gamma\alpha^*\eta^*\delta^*)^{-1}$, it follows that $1 \geq \epsilon\gamma\alpha^*\delta^*$ (because $\eta^* \geq 1$), and, consequently, $\alpha_i(x_i[k]) \geq 0 \Rightarrow \alpha_i(x_i[k+1]) \geq 0$, for all $i \in \mathcal{S}$.

($\mathbf{x}[k] \in \mathcal{X} \Rightarrow \boldsymbol{\beta}(\mathbf{x}[k+1]) \succeq \mathbf{0}$) Let $\mathbf{x}[k] \in \mathcal{X}$. From (2), notice that the change of mass regarding policy $p \in \mathcal{P}$ is

$$\begin{aligned}\sum_{i\in\mathcal{S}^p} \hat{x}_i[k] &= \sum_{i\in\mathcal{S}^p}\sum_{j\in\mathcal{S}} x_j[k]\rho_{ji}[k] - \sum_{i\in\mathcal{S}^p}\sum_{j\in\mathcal{S}} x_i[k]\rho_{ij}[k]\\ &= \sum_{i\in\mathcal{S}^p}\sum_{j\notin\mathcal{S}^p} x_j[k]\rho_{ji}[k] - \sum_{i\in\mathcal{S}^p}\sum_{j\notin\mathcal{S}^p} x_i[k]\rho_{ij}[k]\\ &\leq \sum_{i\in\mathcal{S}^p}\sum_{j\notin\mathcal{S}^p} x_j[k]\rho_{ji}[k].\end{aligned}$$

Here, $j \notin \mathcal{S}^p$ is to be understood as $j \in \mathcal{S} \setminus \mathcal{S}^p$. Hence, from the definition of $\beta_i(\cdot)$ it follows that

$$\begin{aligned}\beta_i(\mathbf{x}[k+1]) &= \beta_i(\mathbf{x}[k]) - \epsilon\sum_{z\in\mathcal{S}^{h(i)}} \hat{x}_z[k]\\ &\geq \beta_i(\mathbf{x}[k]) - \epsilon\sum_{z\in\mathcal{S}^{h(i)}}\sum_{j\notin\mathcal{S}^{h(i)}} x_j[k]\rho_{jz}[k].\end{aligned}$$

Here, observe that $\rho_{jz}[k] \leq w_{jz}\phi_{jz}(\mathbf{x}[k])\gamma$, and that $\phi_{jz}(\mathbf{x}[k]) \leq \beta z(\mathbf{x}[k]) = \beta_i(\mathbf{x}[k])$ because $h(j) \neq h(z)$ and $h(z) = h(i)$. Thus, $\rho_{jz}[k] \leq w_{jz}\beta_i(\mathbf{x}[k])\gamma$ and

$$\begin{aligned}\beta_i(\mathbf{x}[k+1]) &\geq \left(1 - \epsilon\gamma\sum_{z\in\mathcal{S}^{h(i)}}\sum_{j\notin\mathcal{S}^{h(i)}} w_{jz}x_j[k]\right)\beta_i(\mathbf{x}[k])\\ &\geq (1-\epsilon\gamma\alpha^*\eta^*\delta^*)\,\beta_i(\mathbf{x}[k]]),\end{aligned}$$

where we have used the facts that $\mathbf{x}[k] \in \mathcal{X} \Rightarrow x_j[k] \leq \alpha^*$; that $\sum_{j\notin\mathcal{S}^{h(i)}} w_{jz} \leq \delta^*$; and that $\sum_{z\in\mathcal{S}^{h(i)}} 1 \leq \eta^*$. Consequently, if $\epsilon \leq (\gamma\alpha^*\eta^*\delta^*)^{-1}$, then $\beta_i(\mathbf{x}[k]) \geq 0 \Rightarrow \beta_i(\mathbf{x}[k+1]) \geq 0$, for all $i \in \mathcal{S}$. ■

6.8 Proof of Theorem 4

First, from Theorem 2 it holds that $\mathbf{x}(t) \in \mathcal{X}$ for all $t \geq 0$. Second, recall Lemma 4 and consider the Lyapunov function candidate $V(\mathbf{x}) = \varphi(\mathbf{x}^*) - \varphi(\mathbf{x})$, where $\mathbf{x}^* \in \text{GNE}(\mathbf{f})$. Clearly, $\nabla V(\mathbf{x}) = -\mathbf{f}(\mathbf{x})$, and, in consequence, $(\nabla V(\mathbf{x}))^\top \dot{\mathbf{x}} = -(\mathbf{f}(\mathbf{x}))^\top \mathbf{L}(\mathbf{x})\mathbf{f}(\mathbf{x})$. Hence, using Lemma 1 and Theorem 1 we conclude that $(\nabla V(\mathbf{x}))^\top \dot{\mathbf{x}} \leq 0$ for all $\mathbf{x} \in \mathcal{X}$, and that $(\nabla V(\mathbf{x}))^\top \dot{\mathbf{x}} = 0$ if and only if $\mathbf{x} \in \text{GNE}(\mathbf{f})$. Thus, $\text{GNE}(\mathbf{f})$ is asymptotically stable under the considered dynamics. ■

6.9 Proof of Theorem 5

First, observe that from Theorem 3 it holds that $\mathbf{x}[k] \in \mathcal{X}$, for all $k \geq 0$. Second, recall Lemma 4 and consider the Lyapunov function $V(\mathbf{x}) = \varphi(\mathbf{x}^*) - \varphi(\mathbf{x})$, where $\mathbf{x}^* \in \text{GNE}(\mathbf{f})$. To prove the asymptotic stability of $\text{GNE}(\mathbf{f})$, we must show that $V(\mathbf{x}[k+1]) - V(\mathbf{x}[k]) < 0$, for all $\mathbf{x}[k] \in \mathcal{X} \setminus \text{GNE}(\mathbf{f})$. Thus, let $\hat{V}[k] \triangleq V(\mathbf{x}[k+1]) - V(\mathbf{x}[k])$. Clearly, $\hat{V}[k] = \varphi(\mathbf{x}) - \varphi(\mathbf{x} + \epsilon\mathbf{L}\mathbf{f})$, where we have set $\mathbf{x} \triangleq \mathbf{x}[k]$, $\mathbf{L} \triangleq \mathbf{L}(\mathbf{x}[k])$, and $\mathbf{f} \triangleq \mathbf{f}(\mathbf{x}[k])$. Since $\varphi(\cdot)$ is twice continuously differentiable, using (Nocedal & Wright 2006, Theorem 2.1) and $\mathbf{L} = \mathbf{L}^\top$, it follows that $\varphi(\mathbf{x}+\epsilon\mathbf{L}\mathbf{f}) = \varphi(\mathbf{x}) + \epsilon\mathbf{f}^\top\mathbf{L}\mathbf{f} + \frac{\epsilon^2}{2}\mathbf{f}^\top\mathbf{L}\mathbf{H}(q)\mathbf{L}\mathbf{f}$, where $\mathbf{H}_q \triangleq \nabla^2\varphi(\mathbf{x} + q\epsilon\mathbf{L}\mathbf{f})$ for some $q \in (0,1)$. Thus, $\hat{V}[k] = -\epsilon\left(\mathbf{f}^\top\mathbf{L}\mathbf{f} - (\epsilon/2)\mathbf{f}^\top\mathbf{L}\hat{\mathbf{H}}(q)\mathbf{L}\mathbf{f}\right)$, with $\hat{\mathbf{H}}_q = -\mathbf{H}_q$. Since $\epsilon > 0$, to guarantee $\hat{V}[k] < 0$ it must hold that $\mathbf{f}^\top\mathbf{L}\mathbf{f} > (\epsilon/2)\mathbf{f}^\top\mathbf{L}\hat{\mathbf{H}}_q\mathbf{L}\mathbf{f}$. Applying the eigen-decomposition on $\mathbf{L}$, that is $\mathbf{L} = \mathbf{P}\boldsymbol{\Lambda}\mathbf{P}^\top$, it follows that

$$\begin{aligned}\mathbf{f}^\top\mathbf{L}\hat{\mathbf{H}}_q\mathbf{L}\mathbf{f} &= \mathbf{f}^\top\mathbf{P}\boldsymbol{\Lambda}^{1/2}\left(\boldsymbol{\Lambda}^{1/2}\mathbf{P}^\top\hat{\mathbf{H}}_q\mathbf{P}\boldsymbol{\Lambda}^{1/2}\right)\boldsymbol{\Lambda}^{1/2}\mathbf{P}^\top\mathbf{f}\\ &\leq \left\|\boldsymbol{\Lambda}^{1/2}\right\|_2 \left\|\mathbf{P}^\top\hat{\mathbf{H}}_q\mathbf{P}\right\|_2 \left\|\boldsymbol{\Lambda}^{1/2}\right\|_2 \mathbf{f}^\top\mathbf{L}\mathbf{f}\\ &= r\left(\hat{\mathbf{H}}_q\right) r(\mathbf{L})\,\mathbf{f}^\top\mathbf{L}\mathbf{f},\end{aligned}$$

where $r(\mathbf{Z})$ is the spectral radius of $\mathbf{Z}$, and $r\left(\hat{\mathbf{H}}_q\right) = r\left(\mathbf{P}^\top\hat{\mathbf{H}}_q\mathbf{P}\right)$ because $\mathbf{P}^\top\hat{\mathbf{H}}_q\mathbf{P}$ is similar to $\hat{\mathbf{H}}_q$. Moreover, from Assumption 2ii) it follows that $r\left(\hat{\mathbf{H}}_q\right) \leq L$, for all $\mathbf{x} \in \mathcal{X}$ and all $q \in (0,1)$. In contrast, from the Gershgorin Circle Theorem, $r(\mathbf{L}) \leq 2\bar{\ell}$, where $\bar{\ell} = \max_{\mathbf{x}\in\mathcal{X}, i\in\mathcal{S}} \ell_{ii}(\mathbf{x})$, i.e., $\bar{\ell}$ is an upper bound on the maximum diagonal element of $\mathbf{L}$ over $\mathcal{X}$. In consequence, $\hat{V}[k] \leq -\epsilon\left(\mathbf{f}^\top\mathbf{L}\mathbf{f} - \epsilon L\bar{\ell}\mathbf{f}^\top\mathbf{L}\mathbf{f}\right) \leq -\epsilon\left(1-\epsilon Lm\alpha^*\delta^*\right)\mathbf{f}^\top\mathbf{L}\mathbf{f}$, where we have used the definitions of $\ell_{ii}(\cdot)$ and $\theta_{ij}(\cdot)$ to assert that $\bar{\ell} \leq m\alpha^*\delta^*$. Since $\mathbf{L}$ is positive semi-definite [c.f., Lemma 1], it holds that $0 < \epsilon < (Lm\alpha^*\delta^*)^{-1}$ implies that $\hat{V}[k] \leq 0$, for all $k \geq 0$, and that $\hat{V}[k] = 0 \Leftrightarrow \mathbf{f}^\top\mathbf{L}\mathbf{f} = 0$. Therefore, using Lemma 1 and Theorem 1, we further conclude that $\hat{V}[k] = 0 \Leftrightarrow \mathbf{x}[k] \in \mathrm{GNE}(\mathbf{f})$. Thus, the set $\mathrm{GNE}(\mathbf{f})$ is asymptotically stable under the considered dynamics. ■

References

Barreiro-Gomez, J., Dörfler, F. & Tembine, H. (2018), Distributed robust population games with applications to optimal frequency control in power systems, *in* 'Proceedings of the Annual American Control Conference (ACC)', pp. 5762–5767.

Barreiro-Gomez, J., Obando, G., Ocampo-Martinez, C. & Quijano, N. (2019), 'Evolutionary-games approach for distributed predictive control involving resource allocation', *IET Control Theory & Applications* **13**(6), 772–782.

Barreiro-Gomez, J., Obando, G. & Quijano, N. (2017), 'Distributed population dynamics: Optimization and control applications', *IEEE Transactions on Systems, Man, and Cybernetics: Systems* **47**(2), 304–314.

Barreiro-Gomez, J., Quijano, N. & Ocampo-Martinez, C. (2016), 'Constrained distributed optimization: A population dynamics approach', *Automatica* **69**, 101–116.

Barreiro-Gomez, J. & Tembine, H. (2018), 'Constrained evolutionary games by using a mixture of imitation dynamics', *Automatica* **97**, 254 – 262.

Belgioioso, G., Nedić, A. & Grammatico, S. (2021), 'Distributed generalized Nash equilibrium seeking in aggregative games on time-varying networks', *IEEE Transactions on Automatic Control* **66**(5), 2061–2075.

Bianchi, M. & Grammatico, S. (2021), 'Fully distributed Nash equilibrium seeking over time-varying communication networks with linear convergence rate', *IEEE Control Systems Letters* **5**(2), 499–504.

Chen, G., Ming, Y., Hong, Y. & Yi, P. (2020), 'Distributed algorithm for ϵ-generalized Nash equilibria with uncertain coupled constraints', *Automatica* **123**, 109313.

Como, G., Fagnani, F. & Zino, L. (2021), 'Imitation dynamics in population games on community networks', *IEEE Transactions on Control of Network Systems* **8**(1), 65–76.

De Persis, C. & Grammatico, S. (2019), 'Distributed averaging integral Nash equilibrium seeking on networks', *Automatica* **110**, 108548.

Facchinei, F. & Kanzow, C. (2010), 'Penalty methods for the solution of generalized Nash equilibrium problems', *SIAM Journal on Optimization* **20**(5), 2228–2253.

Facchinei, F. & Pang, J.-S. (2003), *Finite-dimensional variational inequalities and complementarity problems*, Vol. 1, Springer.

Falsone, A., Notarnicola, I., Notarstefano, G. & Prandini, M. (2020), 'Tracking-admm for distributed constraint-coupled optimization', *Automatica* **117**, 108962.

Gadjov, D. & Pavel, L. (2019), 'A passivity-based approach to Nash equilibrium seeking over networks', *IEEE Transactions on Automatic Control* **64**(3), 1077–1092.

Liang, S., Wang, L. Y. & Yin, G. (2020), 'Distributed smooth convex optimization with coupled constraints', *IEEE Transactions on Automatic Control* **65**(1), 347–353.

Martinez-Piazuelo, J., Diaz-Garcia, G., Quijano, N. & Giraldo, L. F. (2022), 'Discrete-time distributed population dynamics for optimization and control', *IEEE Transactions on Systems, Man, and Cybernetics: Systems* **(in press)**.

Martinez-Piazuelo, J., Quijano, N. & Ocampo-Martinez, C. (2020), Decentralized charging coordination of electric vehicles using multi-population games, *in* 'Proceedings of the 59th IEEE Conference on Decision and Control (CDC)', pp. 506–511.

Martinez-Piazuelo, J., Quijano, N. & Ocampo-Martinez, C. (2022), 'A payoff dynamics model for equality-constrained population games', *IEEE Control Systems Letters* **6**, 530–535.

Nocedal, J. & Wright, S. (2006), *Numerical optimization*, Springer Science & Business Media.

Park, S., Martins, N. C. & Shamma, J. S. (2019), From population games to payoff dynamics models: A passivity-based approach, *in* 'Proceedings of the 58th IEEE Conference on Decision and Control (CDC)', pp. 6584–6601.

Pavel, L. (2020), 'Distributed GNE seeking under partial-decision information over networks via a doubly-augmented operator splitting approach', *IEEE Transactions on Automatic Control* **65**(4), 1584–1597.

Poveda, J. I. & Quijano, N. (2015), 'Shahshahani gradient-like extremum seeking', *Automatica* **58**, 51–59.

Quijano, N., Ocampo-Martinez, C., Barreiro-Gomez, J., Obando, G., Pantoja, A. & Mojica-Nava, E. (2017), 'The role of population games and evolutionary dynamics in distributed control systems: The advantages of evolutionary game theory', *IEEE Control Systems Magazine* **37**(1), 70–97.

Romano, A. R. & Pavel, L. (2020), GNE seeking in games with passive dynamic agents via inexact-penalty methods, *in* 'Proceeding of the 59th IEEE Conference on Decision and Control (CDC)', pp. 500–505.

Romano, A. R. & Pavel, L. (2021), 'An inexact-penalty method for GNE seeking in games with dynamic agents', *arXiv preprint arXiv:2104.11609* .

Sandholm, W. H. (2010), *Population games and evolutionary dynamics*, MIT Press.

Sun, C. & Hu, G. (2021*a*), 'Continuous-time penalty methods for Nash equilibrium seeking of a nonsmooth generalized noncooperative game', *IEEE Transactions on Automatic Control* **66**(10), 4895–4902.

Sun, C. & Hu, G. (2021*b*), 'Distributed generalized Nash equilibrium seeking for monotone generalized noncooperative games by a regularized penalized dynamical system', *IEEE Transactions on Cybernetics* **51**(11), 5532–5545.

Tatarenko, T., Shi, W. & Nedić, A. (2021), 'Geometric convergence of gradient play algorithms for distributed Nash equilibrium seeking', *IEEE Transactions on Automatic Control* **66**(11), 5342–5353.

Yang, T., Yi, X., Wu, J., Yuan, Y., Wu, D., Meng, Z., Hong, Y., Wang, H., Lin, Z. & Johansson, K. H. (2019), 'A survey of distributed optimization', *Annual Reviews in Control* **47**, 278–305.

Ye, M. & Hu, G. (2021), 'Adaptive approaches for fully distributed Nash equilibrium seeking in networked games', *Automatica* **129**, 109661.

Yi, P. & Pavel, L. (2019*a*), 'Distributed generalized Nash equilibria computation of monotone games via double-layer preconditioned proximal-point algorithms', *IEEE Transactions on Control of Network Systems* **6**(1), 299–311.

Yi, P. & Pavel, L. (2019*b*), 'An operator splitting approach for distributed generalized Nash equilibria computation', *Automatica* **102**, 111–121.

Yi, P. & Pavel, L. (2020), 'Asynchronous distributed algorithms for seeking generalized Nash equilibria under full and partial-decision information', *IEEE Transactions on Cybernetics* **50**(6), 2514–2526.