MORSE CELL DECOMPOSITION AND PARAMETRIZATION OF SURFACES FROM POINT CLOUDS

MARIA ALBERICH-CARRAMIÑANA, JAUME AMORÓS, FRANCO COLTRARO, CARME TORRAS, AND MIQUEL VERDAGUER

ABSTRACT. An algorithm for the reconstruction of a surface from a point sample is presented. It proceeds directly from the point-cloud to obtain a cellular decomposition of the surface derived from a Morse function. No intermediate triangulation or local implicit equations are used, saving on computation time and reconstruction-induced artifices. No a priori knowledge of surface topology, density or regularity of its point sample is required to run the algorithm. The results are a piecewise parametrization of the surface as a union of Morse cells, suitable for tasks such as noise-filtering or mesh-independent reparametrization, and a cell complex of small rank determining the surface topology. The algorithm can be applied to smooth surfaces with or without boundary, embedded in an ambient space of any dimension.

INTRODUCTION

Reconstruction of a surface in space from a sample of points on it is a question to which considerable attention has been devoted in the areas of Computational Geometry and Computer Graphics ([3]). The authors' goal was a fast algorithm for topology identification and parametrization of surfaces with boundary was required. The qualities were required for robotic handling of textiles, but are hoped to make the algorithm fit to study higher dimensional algebraic varieties.

Differential Topology has tackled the piecewise parametrization problem for manifolds through Morse functions. Applying this idea directly to the sample point cloud of a surface was suggested by [8],[9], who propose an algorithm for point clouds with a known, homogeneous density of sampling. Cazals et al ([2]) propose a Morse decomposition scheme from point clouds sampling manifolds of any dimension. The complexity of their algorithm preserves the interest in simpler schemes for low dimension.

The authors report in this work a complete Morse cell decomposition algorithm for surfaces of any topology, with or without boundary, which can be applied to sample point clouds without a priori knowledge of sampling density or regularity, or of surface topology. It can be applied to surfaces in any ambient dimension. We use the gradient flows of [8],[9] as the starting point, but then detect saddle points and their Morse cells by studying the level sections of these flows.

Date: 2022/3/2.

This project has received funding from the European Union's Horizon 2020 research and innovation program (grant agreement No. 741930) and the María de Maeztu Seal of Excellence to IRI (MDM-2016-0656).

The talk at the meeting EACA 2022 has been given by the third author.

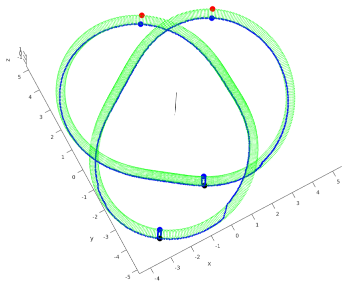

FIGURE 1. A sampled knotted torus: the black line is the direction of the height function; local maxima, resp. minima, are painted red, resp. black; saddle points are painted blue; their 1–cells are outlined in blue.

1. From Morse theory for manifolds...

Let M be a smooth compact manifold. A map $f : M \to \mathbb{R}$ is *Morse* if it is $\mathcal{C}^2$, has only finitely many critical points, and at all of these the Hessian $d^2f(P)$ is nondegenerate. Classical Morse theory (see [6]) shows that a generic Morse function f induces, through its gradient flow, two decompositions of the manifold M:
1. *As a CW complex* (see [7]): Each critical point of f, together with its unstable manifold for the vector field $-\nabla f$, forms a cell which is topologically a ball, whose boundary attaches to lower-dimensional cells. A global piecewise parametrization of M is achieved, and a Morse-Smale complex, with the critical points of f as a basis, computing the singular homology of M.
2. *As level sets*: M is foliated by the level sets $f^{-1}(c)$. For regular values c these level sets are submanifolds of M with codimension 1, with $f^{-1}(c_1) \cong f^{-1}(c_2)$ if no critical value of f lies between c_1 and c_2. The transformation of the level set when c crosses a critical value of f is a surgery ([6]).

The success of Morse theory comes from the fact that Morse functions, and the Morse-Smale transversality required for the above analysis, are generic among $\mathcal{C}^2$ maps from M to $\mathbb{R}$. For instance, the height function in a random direction in $\mathbb{R}^N$ has probability 1 of being a Morse-Smale function. Morse theory also extends to manifolds with boundary ([5]).

2. ...to point clouds

Let $X \subset \mathbb{R}^N$ be a point cloud sampling a compact surface S, possibly with boundary. Determine the neighbours of each point P in the sample, e.g. proceeding as outlined in Sect. 3. Choose a unit vector $v \in \mathbb{R}^N$ such that the height function $f(x) = x \cdot v$ has different values in all points of X, and define the gradient (or *upwards*), resp. -gradient (or *downwards*) flows of f by sending every point P in the cloud to its neighbour which maximizes the slope of growth of f, resp. makes f decrease with the most negative slope. Points where f cannot grow, resp. decrease, are local maxima, resp. minima of f in X.

Interpret the downwards flow of f on X as a graph with edges connecting each point in the cloud to its downwards neighbour. The intersections of this graph with the hyperplane $x \cdot v = c$ are point samples for the level set $f^{-1}(c)$ on the original surface S, with some noise added by the linear interpolation. This level set consists of finitely many simple curves, either closed or with edges in the boundary of S. The curves can be reconstructed, e.g. as explained in Sect. 3.

Perform these level set intersections at n equispaced levels $c_i = c_0 + i \cdot h$ ranging from $c_0 = \min f(X)$ to $c_n = \max f(X)$. The number of level sections n must be selected, the idea is that all surface handles whose range in height is h or greater will be detected. Changes in the topology of the level set are now variations in the number of curves. Each surgery in the level set induced by its going over a saddle point of f can be tracked by two pairs of neighbours in the sample of a level set which end up in different connected components of the other level set after upwards or downwards flow.

FIGURE 2. Change in level set when crossing a critical value in a surface (left) and point cloud (right): note the change in neighbours among the 4 marked points after the flow.

Following the 4 points in these 2 pairs in the downwards flow, and their pairing according to closeness, the level at which the pairing changes marks the position of the saddle point. The unstable variety of this saddle point for the flow of $-\nabla f$ is approximated by taking the two pairs of neighbouring points at the level of f immediately below the saddle point, and averaging the downward orbits of each pair. These computations have a margin of error $O(d)$, where d is the local variation of height among neighbouring cloud points.

3. Implementation

The first step is the identification of a set of neighbours of each point in the cloud. Merging usual criteria, we take 2 points as neighbours when (i) their Voronoï cells in the decomposition of ambient space $\mathbb{R}^N$ induced by X are adjoining, and (ii) each point is among the k nearest neighbours of the other in the cloud, with $k \in [6, 12]$ as suggested by sphere packing on surfaces.

The boundary is identified in the point cloud by PCA analysis of tangent spaces at each point cloud using its neighbours (as in [1]). Neighbours of a boundary point cluster in a semispace.

Curve reconstruction is used for boundary parametrization, and later for level set reconstruction when we intersect the gradient flow graph with level hyperplanes. A variant of the NN-Crust algorithm of T. Dey ([3]) is used.

Once the saddle points of the height function and their (un)stable varieties have been found, the piecewise parametrization for the entire surface follows: 0-cells are the local minima, 1-cells have been parametrized at saddle point detection, and each 2-cell can be parametrized from the tree formed in it by the upwards flow to its unique maximum, e.g. with the shape-preserving algorithm of Floater ([4]).

Finally, the boundary relations given by the downwards flow on the cells give us the Morse-Smale complex and singular homology of the surface S.

Figure 1 shows our algorithm applied to a point sample from a torus, embedded in $\mathbb{R}^3$ along a (2,3)-toric knot. The algorithm correctly detects 2 local maxima, 2 local minima and 4 saddle points for the height function. Out of a point cloud of 30.000 points, a parametrization of the surface into 8 cells is found, and from this its topology.

4. Conclusions

The algorithm presented in this work successfully reconstructs a surface S by finding a Morse cellular decomposition from a cloud of sampled points. The advantages of this approach are:

- A global piecewise parametrization of the surface is found, with number of pieces O(1000) times smaller than that of sample points in typical examples.
- The topology of the surface can be deduced immediately from the cellular decomposition.
- The algorithm is robust: it always produces a surface, and it captures the topological features of the sampled surface with a size greater than the typical distance between sample points.
- In presence of noise in the sample points position, the parametrization allows the use of a range of filtering techniques while preserving the surface topology.
- The algorithm can be applied to surfaces in space $\mathbb{R}^N$ for any ambient dimension N.

The authors expect to extend this algorithm to the reconstruction of higher dimensional manifolds by iterating the hyperplane sections, reconstructing the manifold from lower dimensional slices. A further extension would be to the study of real algebraic varieties of any dimension, where point cloud samples can be obtained from their equations and refined where necessary. This is hoped to lead our method to detect a Whitney stratification, and the Morse cellular decomposition after [5] of the variety, by purely numeric methods.

References

[1] Cao, Li, Sun, Azadi, Zhang, *Efficient Curvature Estimation for Oriented Point Clouds.* arxiv.stat.ML.1905.10725

[2] Cazals, Roth, Robert, Christian, *Towards Morse Theory for Point Cloud Data*, Research Report RR-8331, INRIA. 2013, pp.37.

[3] T. Dey, *Curve and surface reconstruction. Algorithms with mathematical analysis.* CUP, 2006.

[4] M. Floater, *Parametrization and smooth approximation of surface triangulations*, Computer Aided Geometric Design **14** (1997) 231–250.

[5] Goresky, MacPherson, *Stratified Morse Theory.* Springer Verlag, 1988.

[6] M. Hirsch, *Differential Topology.* Springer Verlag, 1976.

[7] J. Munkres, *Elements of Algebraic Topology.* Addison-Wesley, 1984.

[8] Gao, Sarkar, Zhu, *Morse-Smale Decomposition, Cut Locus and Applications in Wireless Sensor Networks*, 2008.

[9] Zhu, Sarkar, Gao, *Topological Data Processing for Distributed Sensor Networks with Morse-Smale Decomposition*, IEEE INFOCOM 2009 (pp. 2911-2915).

Institut de Robòtica i Informàtica Industrial, CSIC-UPC. Barcelona, Spain
Email address: fcoltraro@iri.upc.edu