A_∞ persistent homology estimates detailed topology from pointcloud datasets

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Contents

Abstract

Let X be a closed subspace of a metric space M. It is well known that, under mild hypotheses, one can estimate the Betti numbers of X from a finite set $P \subset M$ of points approximating X. In this paper, we show that one can also use P to estimate much more detailed topological properties of X. We achieve this by proving the stability of A_∞-persistent homology. In its most general case, this stability means that given a continuous function $f\colon Y \longrightarrow \mathbb{R}$ on a topological space Y, small perturbations in the function f imply at most small perturbations in the family of A_∞-barcodes. This work can be viewed as a proof of the stability of cup-product and generalised-Massey-products persistence.

The technical key of this paper consists of figuring out a setting which makes A_∞-persistence functorial.

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1. Introduction

Persistent homology (in the sense of [13, 24]) is a topological technique used to extract global structural information from datasets which may be high dimensional and contain noise.

About a decade ago, two results set the foundations of persistent homology as a robust mathematical theory. First, the *structural theorem* [13, §3] explained how the homology of a sequence of nested topological spaces can be split into simple pieces forming a barcode or a persistence diagram. Secondly, the *stability theorem* [17, Main Thm.] showed that small perturbations in the input sequence can produce at most small perturbations in the corresponding barcode.

These two milestones justified the use of barcodes as a meaningful characteristic which is robust to noise. They also provided the formalism to show that in order to estimate the homology groups of a closed subspace X of a metric space, in theory it is enough to have a sufficiently good finite sample P of X [17, Homology Inference Theorem]. To that end, one would only need to compute the barcode of the following sequence of nested spaces: for any given radius r, consider the union P_r of the balls of radius r centered at each point in P. Then, as r grows, so does the union P_r.

Persistent homology has been successfully applied to fields such as medicine [1, 6], sensor networks coverage [21] and molecular modelling [27, 38], among many others. However, persistent homology computes information only at the level of homology groups. Intuitively, this means that persistent homology cares about the number of connected components, tunnels, voids and higher-dimensional holes of objects, and this information is not always enough. For instance, work on signal processing [37] and image texture representation [12] shows that point clouds whose shapes are related to tori $\mathbb{T}$ and Klein bottles $\mathbb{K}$ arise naturally from data. With coefficients in the finite field of two elements $\mathbb{Z}_2 = \{0, 1\}$, homology groups do not distinguish $\mathbb{T}$ from $\mathbb{K}$, nor from a space as simple as a wedge of spheres $\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1$, but the fundamental group does, and so does the cohomology ring. It was then natural to enhance persistent homology with the discriminatory power of the fundamental group or the cohomology ring. Persistence approaches to the fundamental group can be found in [9, 15, 26], and the cup product is dealt with within the theory of A_∞ persistent homology, or A_∞-persistence [4, 5], for short. Beyond that, in order to use cohomology to detect that the Borromean rings are

non-trivially linked, the cup product is not enough, and ternary operations like Massey products are needed. Information at this ternary and n-ary level in general is included as well in the computations of A_∞ persistent homology. Indeed, A_∞-persistence provides a whole family of barcodes, and morally, the level-2 barcode describes cohomology classes which persistently cannot be expressed as a cup product, the level-3 barcode describes cohomology classes which persistently cannot be expressed as a triple Massey product and so on. Recently, the notion of A_∞-persistence has received much attention, with several authors investigating interesting alternative approaches to the one followed in this paper [28, 29].

Recent advances in generalizing the structure theorem [16, 18, 32] and in categorifying the stability theorem [10, 22] allow one to prove that a given version of persistence (such as A_∞ persistent homology) has a barcode decomposition and is stable, provided it is functorial. The issue is that A_∞ persistent homology is not functorial in general [3, Thm. 3.1]. Therefore, a big challenge consists of finding a non-trivial context in which we can guarantee the functoriality of A_∞-persistence. The main contribution of this work is the identification of one such context. Specifically, we introduce the category $\mathbf{Top}_n$ (Def. 4.1) and show that A_∞ persistent homology is functorial within this category (Thm. 4.3). Additionally, we illustrate that this is the largest category of its form (in a sense made explicit in Rmk. 4.4) for which such functoriality should be expected.

A crucial part of this paper is therefore devoted to proving the functoriality of A_∞ persistent homology (Thm. 4.3). From this, we then deduce that the barcodes from A_∞ persistent homology are robust to small perturbations of the input (Cor. 4.8 and Cor. 4.10), and that one can extrapolate A_∞ information of a metric space from a finite point-set approximation (Cor. 4.12).

This paper is organized as follows: In §2, we recall the basics of persistent homology and state the formal results we will use in §4 to study the stability of A_∞ persistent homology. In §3, we collect all definitions and properties we need to know about A_∞-structures in order to understand the theory of A_∞ persistent homology. All results in §4 are stated in terms of the category $\mathbf{Top}_n$ we define in Def. 4.1. The main theorem of the paper proves the functoriality of A_∞ persistent homology (Thm. 4.3). Rmk. 4.4 and Ex. 4.5 illustrates that $\mathbf{Top}_n$ is large enough in a particular sense. As a first corollary of Thm. 4.3, we provide a new structure theorem for A_∞ persistent homology for a case left aside to date (Cor. 4.6). To illustrate the higher discriminatory power of A_∞ persistent homology over classical persistence, Ex. 4.7 exhibits two persistent spaces X_*, Y_* with the same persistent homology barcodes but different A_∞ barcodes. This toy example also shows how cup product persistence is part of A_∞-persistence. We finish §4 with three important applications of the functoriality Thm. 4.3 - namely, we show that A_∞ persistent homology is stable with respect to perturbations in the

input function (Cor. 4.8) and perturbations in the input space (Cor. 4.10), and that we can recover A_∞ information of a metric space from a finite point sample (Cor. 4.12).

Note that when we focus on the second operation Δ_2 on an A_∞-coalgebra (such as in Ex. 4.7), or equivalently, on the cup product on an A_∞-algebra on cohomology, then all results in this paper hold without the need to restrict to the category $\mathbf{Top}_n \subseteq \mathbf{Top}$ and instead, we can work directly with the category of topological spaces **Top**. In particular, this paper proves the stability of the persistence of cup product with minimal restrictions.

Notation *Throughout the text, we will work over a fixed field $\mathbb{F}$. We will usually omit the field from the notation.* E.g., *we will denote by $H_*(X)$ the singular homology of X with coefficients in $\mathbb{F}$.*

We will present the results of this paper in terms of homology, but everything works as well for cohomology and for reduced (co)homology.

2. Persistence and functoriality

Let

$$K_0 \hookrightarrow K_1 \hookrightarrow \cdots \hookrightarrow K_N \tag{2.1}$$

be a finite sequence of nested topological spaces. In the context of persistence, sequences like this arise as sublevel sets of functions of the form $f\colon M \longrightarrow \mathbb{R}$, for some metric space M; for instance, by specifying

$$K_i \coloneqq f^{-1}(-\infty, i].$$

To give a more concrete example, given a closed subspace X of M, if one defines the distance function

$$d^X\colon M \to \mathbb{R}, \quad y \mapsto d(y, X),$$

then the sequence given by $K_i \coloneqq (d^X)^{-1}(-\infty, i]$ can be interpreted as a thickening of X.

Let us fix a particular homology degree of interest, $p \geq 0$ and assume that all these nested spaces have finite-dimensional homology groups, *i.e.*, $\dim_{\mathbb{F}} H_p(K_i) < \infty$, for all $0 \leq i \leq N$. We learned from [13] that we can decompose the p^{th} homology of the sequence (2.1) in simple pieces that can be represented in what is called a barcode or a persistence diagram. We now recall the formalism behind this in a higher level of generality which we will need later on.

Notation $\mathbb{R}$ *will denote the poset $(\mathbb{R}, \leq)$ of real numbers.* **Vect** *will denote the category of $\mathbb{F}$-vector spaces and linear maps, and* **Top** *will denote the category of topological spaces and continuous maps.*

We will use the notation $\mathcal{P}$ for any poset $(\mathcal{P}, \leq)$, i.e. any category whose objects are the elements of $\mathcal{P}$, and such that given two objects $x, y \in \mathcal{P}$, there is exactly one arrow $x \to y$ if $x \leq y$ and no arrow $x \to y$, otherwise.

DEFINITION 2.1. Let $\mathcal{C}$ be any category. A **generalized persistence module** (valued in $\mathcal{C}$) is a functor of the form $\mathbf{F} : \mathbb{R} \to \mathcal{C}$. When the category $\mathcal{C}$ is understood by the context, we call $\mathbf{F}$ a generalized persistence module or **GPM** for short. A morphisms between GPMs is a natural transformation between these functors. In this way, the collection of all GPMs forms a functor category $\mathcal{C}^{\mathbb{R}}$ which we call a GPM-category. The GPM-categories we focus on are $\mathbf{Top}^{\mathbb{R}}$ and $\mathbf{Vect}^{\mathbb{R}}$, whose objects are called **persistence spaces** and **persistence modules**, respectively.

A persistence module $\mathbf{V} \in \mathbf{Vect}^{\mathbb{R}}$ is **pointwise finite dimensional (p.f.d.)** if $\dim_{\mathbb{F}} \mathbf{V}(t) < \infty$ for every $t \in \mathbb{R}$.

DEFINITION 2.2. For a pair of continuous maps $f : X \longrightarrow \mathbb{R}, g : Y \longrightarrow \mathbb{R}$, let us define the distance

$$d_\infty(f, g) = \inf_{\Phi} ||f - g \circ \Phi||_\infty \tag{2.2}$$

where Φ ranges over all homeomorphisms of the form $\Phi : X \to Y$. We set $d_\infty(f, g) = \infty$ if X and Y are not homeomorphic.

The collection of all real-valued continuous functions forms a slice category $(\mathbf{Top} \downarrow \mathbb{R})$ which is equipped with the distance given in Def. 2.2. These functions are commonly used to construct persistence spaces via the sublevel-set filtration construction $\mathcal{S}$.

DEFINITION 2.3. The **sublevel-set filtration functor** $\mathcal{S}\colon (\mathbf{Top} \downarrow \mathbb{R}) \longrightarrow \mathbf{Top}^{\mathbb{R}}$ assigns to each continuous $f : X \to \mathbb{R}$ the persistence space $\mathcal{S}(f) : \mathbb{R} \to \mathbf{Top}$, $t \mapsto f^{-1}(-\infty, t]$.

Fix an integer $p \geq 0$. Consider the singular homology functor $H_p\colon \mathbf{Top} \to \mathbf{Vect}$, $X \mapsto H_p(X)$ that assigns to each space its p^{th} homology group with coefficients in the field $\mathbb{F}$.

DEFINITION 2.4. The post composition functor $H_p \circ - : \mathbf{Top}^{\mathbb{R}} \to \mathbf{Vect}^{\mathbb{R}}$ assigns to each persistence space $X_*\colon \mathbb{R} \longrightarrow \mathbf{Top}$, $t \mapsto X_t$, the persistence module $H_pX_*\colon \mathbb{R} \longrightarrow \mathbf{Vect}$, $t \mapsto H_p(X_t)$. H_pX_* is called the **persistent p^{th} homology of X_*.** Analogously, the post composition functor $H_p\mathcal{S} \circ - : (\mathbf{Top} \downarrow \mathbb{R}) \to \mathbf{Vect}^{\mathbb{R}}$ assigns to each continuous function $f : X \to \mathbb{R}$, the persistence module $H_p\mathcal{S}(f)\colon \mathbb{R} \longrightarrow \mathbf{Vect}$, $t \mapsto H_p(f^{-1}(-\infty, t])$. $H_p\mathcal{S}(f)$ is called the **sublevel-set persistent p^{th} homology of $f : X \to \mathbb{R}$.**

There is a natural way of splitting a persistence module into elementary pieces called interval persistence modules:

DEFINITION 2.5. Given an interval $I \subseteq \mathbb{R} \cup \{+\infty\}$, the **interval persistence module** $C(I) \in \mathbf{Vect}^{\mathbb{R}}$ is given by

$$\mathbf{C}(I)(t) = \begin{cases} \mathbb{F} & \text{if } t \in I \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbf{C}(I)[s \leq t] = \begin{cases} Id_{\mathbb{F}} & \text{if } s,t \in I \\ 0 & \text{otherwise.} \end{cases}$$

THEOREM 2.6. ***(Structure theorem of persistence modules [18])*** *Every p.f.d. persistence module* $\mathbf{V} \in \mathbf{Vect}^{\mathbb{R}}$ *decomposes uniquely (up to isomorphism) into interval persistence modules* $C(I)$*,*

$$\mathbf{V} \cong \bigoplus_{I \in B(\mathbf{V})} \mathbf{C}(I),$$

where $B(\mathbf{V})$ *is a multiset (*i.e.*, a set of objects with multiplicities) of intervals of the form* $[a,b)$ *or* $(-\infty,b)$ *for some* $a \in \mathbb{R}$*,* $b \in \mathbb{R} \cup \{+\infty\}$*. This* $B(\mathbf{V})$ *is called the* ***barcode*** *of* $\mathbf{V}$*.*

In particular, the persistence module $\mathbf{V} \coloneqq H_p X_* \in \mathbf{Vect}^{\mathbb{R}}$ is uniquely determined by a barcode, as long as $\dim_{\mathbb{F}} H_p(X_t) < \infty$, for all $t \in \mathbb{R}$. In comparison, the original structure theorem [13, §3] required this persistence module to be indexed by the integers, $H_p X_* \in \mathbf{Vect}^{\mathbb{Z}}$. Additionally, [13] also required there to be finitely many points $r \in \mathbb{R}$ such that, for every sufficiently small $\epsilon > 0$, the linear map $H_p X_{r-\epsilon} \longrightarrow H_p X_{r+\epsilon}$ is not an isomorphism.

Each barcode has an associated persistence diagram, which is a multiset of points in the extended plane $(\mathbb{R} \cup \{-\infty\}) \times (\mathbb{R} \cup \{+\infty\})$.

DEFINITION 2.7. Given a barcode $B(\mathbf{V})$, its corresponding **persistence diagram** $Dgm(\mathbf{V})$ consists of the following points:

- For each interval $[a,b) \in B(\mathbf{V})$ with multiplicity k, the point of coordinates $(a,b) \in \mathbb{R} \times (\mathbb{R} \cup \{+\infty\})$ is included in $Dgm(\mathbf{V})$ with multiplicity k.
- For each interval $(-\infty,b) \in B(\mathbf{V})$ with multiplicity k, the point of coordinates $(-\infty,b) \in \{-\infty\} \times (\mathbb{R} \cup \{+\infty\})$ is included in $Dgm(\mathbf{V})$ with multiplicity k.
- For each $x \in \mathbb{R}$, the diagonal point $(x,x) \in \mathbb{R}^2$ is included in $Dgm(\mathbf{V})$ with infinite multiplicity.

Many stability theorems are presented in terms of a pseudo-distance between barcodes or persistence diagrams called the bottleneck distance. In order to give its definition, we use the following notation. For any $(p_1,p_2),(q_1,q_2) \in (\mathbb{R} \cup \{-\infty\}) \times (\mathbb{R} \cup \{+\infty\})$,

$$||(p_1,p_2) - (q_1,q_2)||_\infty \coloneqq \max\{|p_1 - q_1|, |p_2 - q_2|\},$$

where $|\cdot|$ denotes the absolute value with the convention that:

- If $p_2 = q_2 = +\infty$, then $|p_2 - q_2| \coloneqq 0$.
- If $p_1 = q_1 = -\infty$, then $|p_1 - q_1| \coloneqq 0$.
- If one, and only one, of p_2, q_2 is $+\infty$, then $|p_2 - q_2| \coloneqq +\infty$.
- If one, and only one, of p_1, q_1 is $-\infty$, then $|p_1 - q_1| \coloneqq +\infty$.

We think of each point of multiplicity k in a persistence diagram as k different points, and since all diagonal points are added with infinite multiplicity, there are always infinitely many bijections between any given pair of persistence diagrams.

Definition 2.8. The **bottleneck distance** d_B between two barcodes $B(\mathbf{V})$ and $B(\mathbf{W})$ (or between their associated persistence diagrams $Dgm(\mathbf{V})$ and $Dgm(\mathbf{W})$) is defined as

$$d_B(B(\mathbf{V}), B(\mathbf{W})) = d_B(Dgm(\mathbf{V}), Dgm(\mathbf{W})) \coloneqq \inf_{\gamma} \sup_{x} ||x - \gamma(x)||_\infty,$$

where x runs over all points in $Dgm(\mathbf{V})$ and γ runs over all bijections $\gamma\colon Dgm(\mathbf{V}) \longrightarrow Dgm(\mathbf{W})$.

See [32, §3.1] for a definition of the bottleneck distance in terms of matchings and see [31] to learn how to efficiently compute the bottleneck distance.

Example 2.9. Let $\mathbf{V}$ and $\mathbf{W}$ be the persistence modules given by the following barcodes: $B(\mathbf{V}) = \{[1, 6), [9, 10)\}, B(\mathbf{W}) = \{[2, 6.8)\}$. The bottleneck distance $d_B(B(\mathbf{V}), B(\mathbf{W}))$ can be computed as $\max\{x, y, z\}$, where x, y, z are those distances shown in Fig. 1. Hence, $d_B(B(\mathbf{V}), B(\mathbf{W})) = \max\{1, 0.8, 0.5\} = 1$.

In order to state the classical stability theorem for barcodes, we need one last concept.

Definition 2.10. Given a continuous function $f\colon X \longrightarrow \mathbb{R}$, its **homological critical values** of degree p are those points $t \in \mathbb{R}$ such that, for every sufficiently small $\epsilon > 0$, the linear map $H_pX_{t-\epsilon} \longrightarrow H_pX_{t+\epsilon}$ is not an isomorphism.

Theorem 2.11. ***(Stability of persistent homology [17, Main Thm.])*** *Let $p \geq 0$ be an integer, let X be a triangulable space and let $f, g\colon X \longrightarrow \mathbb{R}$ be continuous functions with finitely many homological critical values of degree p. If the persistence modules $H_p\mathcal{S}(f), H_p\mathcal{S}(g)\colon \mathbb{R} \longrightarrow$ **Vect** are p.f.d., then the bottleneck distance between the barcodes of f and g is bounded above by the supremum distance between the functions:*

$$d_B(B(H_p\mathcal{S}(f)), B(H_p\mathcal{S}(g))) \leq ||f - g||_\infty.$$

Thm. 2.11 guarantees that slight changes in the input function can only result in slight changes in the corresponding barcodes. Although this is considered the most classical stability result, it must be acknowledged that earlier work by the Italian team of M. d'Amico et al. obtained similar results for degree 0 homology [19].

We now recall the rest of the tools we need on the categorification of stability, which will help us prove that A_∞ persistent homology is stable (see §4). For that purpose, we will need to make some extra assumptions, but at the same time, we will also drop two of the assumptions

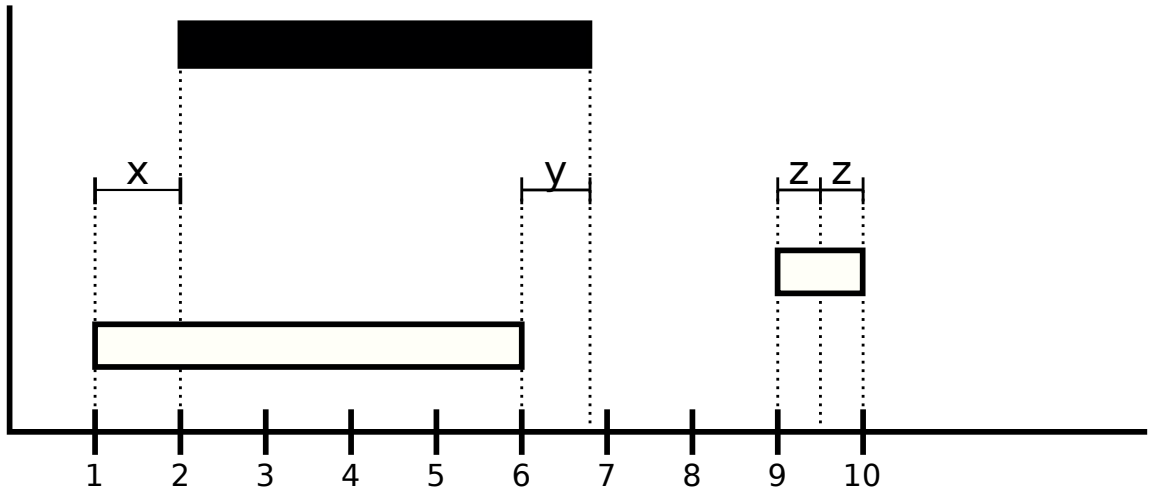
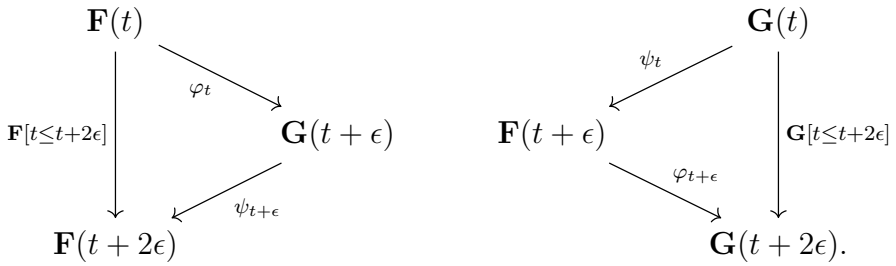


Figure 1. Illustration of the bottleneck distance. The two intervals in the barcode $B(\mathbf{V}) = \{[1,6),[9,10)\}$ are shown in white and the interval of the barcode $B(\mathbf{W}) = \{[2,6.8)\}$ is shown in black. Therefore, $d_B(B(\mathbf{V}),B(\mathbf{W})) = \max\{x,y,z\} = \max\{1,0.8,0.5\} = 1$. Indeed, the bottleneck distance $d_B(B(\mathbf{V}),B(\mathbf{W}))$ can be interpreted as the minimum amount one has to enlarge or shrink the ends of the intervals in $B(\mathbf{V})$ in order to obtain the intervals in $B(\mathbf{W})$. Just as we assumed every persistence diagram to contain each diagonal point with infinite multiplicity, here we can think of each barcode as having each interval of the form $[a,a) = \emptyset$, for $a \in \mathbb{R}$, with infinite multiplicity. This allows the trick of shrinking both ends of an interval by half of its length to make it disappear and the trick of enlarging both ends of an empty interval $[a,a)$ to form an interval $[a-\epsilon, a+\epsilon)$.

in Thm. 2.11 – namely, the triangulability of X, and the finiteness condition on the number of homological critical points.

We first loot at interleavings between functors, which provide us with a tool to compare persistence modules and to compare persistence spaces as well.

Definition 2.12. [14] Two GPMs $\mathbf{F},\mathbf{G} : \mathbb{R} \to \mathcal{C}$ are **ϵ-interleaved**, for $\epsilon \geq 0$, if there exists a pair of natural transformations $\varphi_t : \mathbf{F}(t) \to \mathbf{G}(t+\epsilon)$, $t \in \mathbb{R}$ and $\psi_t : \mathbf{G}(t) \to \mathbf{F}(t+\epsilon)$, $t \in \mathbb{R}$ such that the following diagrams commute



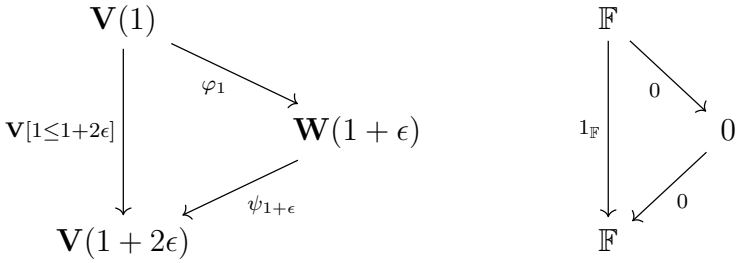
The **interleaving distance** between $\mathbf{F}$ and $\mathbf{G}$ is then defined as

$$d_I(\mathbf{F},\mathbf{G}) \coloneqq \inf\{\epsilon \geq 0 \mid \mathbf{F},\mathbf{G} \text{ are } \epsilon\text{-interleaved}\}.$$

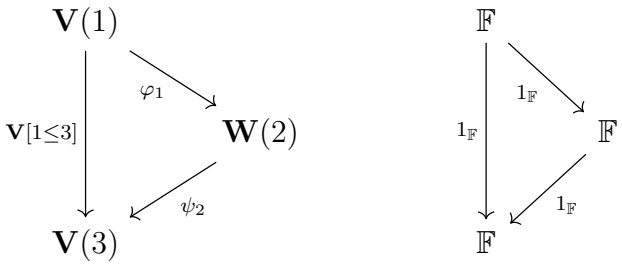
If $\mathbf{F}$ and $\mathbf{G}$ are not ϵ-interleaved for any ϵ, we set $d_I(\mathbf{F}, \mathbf{G}) = \infty$.

In a way, the interleaving distance $d_I(\mathbf{F}, \mathbf{G})$ measures how far the GPMs $\mathbf{F}$ and $\mathbf{G}$ are from being isomorphic. For instance, if two GPMs $\mathbf{F}$ and $\mathbf{G}$ are 0-interleaved, it means that they are isomorphic.

EXAMPLE 2.13. Let $\mathbf{V}$ and $\mathbf{W}$ be the persistence modules in Ex. 2.9 given by the following barcodes: $B(\mathbf{V}) = \{[1, 6), [9, 10)\}, B(\mathbf{W}) = \{[2, 6.8)\}$. The interleaving distance $d_I(\mathbf{V}, \mathbf{W})$ cannot be smaller than 1, since for every $\epsilon < 1$, the diagram on the left is not commutative (the values of this diagram are made explicit in the diagram next to it):



Notice, though, that for $\epsilon = 1$, the following diagram is commutative:



It is straightforward to check that, indeed, $d_I(\mathbf{V}, \mathbf{W}) = 1$.

THEOREM 2.14. *[10, Thm. 3.21] The interleaving distance d_I on a GPM-category $\mathcal{C}^{\mathbb{R}}$ is an extended pseudometric on the class of generalized persistence modules.*

It is no coincidence that the distances computed in Ex. 2.9 and Ex. 2.13 are the same. Indeed, there is a deep relation between p.f.d. persistence modules and their barcodes known as the isometry theorem.

THEOREM 2.15. ***(Isometry Theorem [14, 32])*** *The interleaving distance of a pair of p.f.d. persistence modules $\mathbf{V}, \mathbf{W} \in Vect^{\mathbb{R}}$ is equal to the bottleneck distance of their associated barcodes, i.e.*

$$d_I(\mathbf{V}, \mathbf{W}) = d_B(B(\mathbf{V}), B(\mathbf{W})).$$

F. Chazal et al. [14] proved that $d_I(\mathbf{V}, \mathbf{W}) \geq d_B(B(\mathbf{V}), B(\mathbf{W}))$, and more recently, M. Lesnick [32] proved the converse inequality.

As Thm. 2.11 illustrates, the study of the bottleneck distance d_B and its generalizations is of central importance to TDA. Thm. 2.15 allows us to prove useful results concerning d_B by understanding the interleaving distance.

We next state some results we will use on the interleaving distance. We start by recalling that some functors can be viewed as non-expansive maps:

THEOREM 2.16. *[10, Cor. of Thm. 3.16] Let $\mathbf{H} : \mathcal{C} \to \mathcal{D}$ be a functor between arbitrary categories $\mathcal{C}, \mathcal{D}$. Then the functor $\mathbf{H} \circ - : \mathcal{C}^{\mathbb{R}} \to \mathcal{D}^{\mathbb{R}}$, $\mathbf{F} \mapsto \mathbf{HF}$, defined by post-composing with $\mathbf{H}$, is 1-Lipschitz. That is, for any two GPMs $\mathbf{F}, \mathbf{G} : \mathbb{R} \to \mathcal{C}$, we have*

$$d_I(\mathbf{HF}, \mathbf{HG}) \leq d_I(\mathbf{F}, \mathbf{G}).$$

By Thm. 2.16, the persistent p^{th} homology $H_p \circ -$ forms a 1-Lipschitz map with respect to the interleaving distance. That is, for any pair of persistence spaces $\mathbb{X}_* : \mathbb{R} \to \mathbf{Top}$, $t \mapsto X_t$ and $Y_* : \mathbb{R} \to \mathbf{Top}$, $t \mapsto Y_t$ we have

$$d_I(H_p X_*, H_p Y_*) \leq d_I(X_*, Y_*).$$

THEOREM 2.17. *[14] **(The sublevel-set functor is non-expansive)** The sublevel-set filtration map $\mathcal{S} \colon (\mathbf{Top} \downarrow \mathbb{R}) \longrightarrow \mathbf{Top}^{\mathbb{R}}$ is 1-Lipschitz,* i.e., *if $f \colon X \longrightarrow \mathbb{R}$ and $g \colon Y \longrightarrow \mathbb{R}$ are continuous, then*

$$d_I(\mathcal{S}(f), \mathcal{S}(g)) \leq d_\infty(f, g).$$

3. Transferred A_∞-coalgebras on $H_*(X)$

One can extract topological information of a space from the structure of its homology. To do this beyond the Betti numbers, one can study the relationship between homology classes. For instance, a torus $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$ and a wedge of spheres $\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1$ have the same Betti numbers but their homology classes are related in very different ways. On the one hand, $\mathbb{T}$ is a surface of revolution whose generatrix curve is a circumference which creates non-trivial homology. This hints some relation between entities of dimension 1 (the curve) and 2 (the surface), which, in terms of cohomology, can be explained as follows: choosing an appropriate basis of the cohomology $H^*(\mathbb{T})$, the generator of $H^2(\mathbb{T})$ is the cup product of the two generators of $H^1(\mathbb{T})$. In contrast, the generatrix of a sphere $\mathbb{S}^2$ does not produce non-trivial homology in $\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1$. In terms of cohomology, the generator of $H^2(\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1)$ is nobody's cup product, and the cup product of the 2 generators of $H^1(\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1)$ is 0, regardless of the chosen basis of $H^*(\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1)$. A_∞-structures contain all the information provided by the cup product and much more. For instance, there are links which cannot be distinguished by the cup product alone, but which can be distinguished using A_∞-structures [34], [4, §3]. Later

in this section, we will mention more examples of spaces for which the A_∞-structure provide much more detailed topological information.

In this section, we list some basic notions and notation we will need to understand the meaning of A_∞ persistent homology and its stability. We will restrict ourselves to A_∞-coalgebra structures on graded vector spaces only, although they can be defined in more general contexts.

DEFINITION 3.1. An **A_∞-coalgebra structure** $\{\Delta_n\}_{n\geq 1}$ on a graded vector space C is a family of maps

$$\Delta_n \colon C \longrightarrow C^{\otimes n}$$

of degree $n-2$ such that, for all $n \geq 1$, the following *Stasheff identity* holds:

$$\mathrm{SI}(n)\colon \sum_{i=1}^{n}\sum_{j=0}^{n-i}(-1)^{i+j+ij}\left(1^{\otimes n-i-j}\otimes \Delta_i \otimes 1^{\otimes j}\right)\Delta_{n-i+1} = 0.$$

If $(C, \{\Delta_n\}_{n\geq 1})$ is an A_∞-coalgebra, the identity SI(1) states that Δ_1 is a differential on C and SI(3) states that the comultiplication Δ_2 is coassociative up to the chain homotopy Δ_3. Any differential graded coalgebra (C, ∂, Δ) can be viewed as an A_∞-coalgebra $(C, \{\Delta_n\}_{n\geq 1})$ by setting $\Delta_1 = \partial, \Delta_2 = \Delta$, and $\Delta_n = 0$ for all $n > 2$. An A_∞-coalgebra $(C, \{\Delta_n\}_{n\geq 1})$ is called **minimal** if $\Delta_1 = 0$.

DEFINITION 3.2. A **morphism of A_∞-coalgebras**

$$f\colon (C, \{\Delta_n\}_{n\geq 1}) \to (C', \{\Delta'_n\}_{n\geq 1})$$

is a family of linear maps

$$f_{(m)}\colon C \longrightarrow C'^{\otimes m}, \qquad m \geq 1,$$

of degree $m-1$, such that for each $i \geq 1$, the following identity holds:

$$\mathrm{MI}(i)\colon \sum_{\substack{p+q+k=i\\ q\geq 1, p,k\geq 0}} (1^{\otimes p}\otimes \Delta'_q \otimes 1^{\otimes k}) f_{(p+k+1)} = \sum_{\substack{k_1+\cdots+k_\ell=i\\ l,k_j\geq 1}} (f_{(k_1)}\otimes\cdots\otimes f_{(k_\ell)})\Delta_\ell.$$

We say that the morphism of A_∞-coalgebras f is:

- an **isomorphism** if $f_{(1)}$ is an isomorphism (of vector spaces),
- a **quasi-isomorphism** if $f_{(1)}$ induces an isomorphism (of vector spaces) in homology.

There are some trivial A_∞-coalgebra structures one can always endow a graded vector space C with, such as the one given by $\Delta_n = 0$ for all n. We hence only consider *transferred* A_∞-coalgebras.

DEFINITION 3.3. We will say that an A_∞-coalgebra $(H_*(X), \{\Delta_n\}_n)$ on the homology of a space X is a **transferred A_∞-coalgebra** (induced by X) if it is minimal and quasi-isomorphic to the A_∞-coalgebra

$$(C_*(X), \{\partial, \Delta, 0, 0, \ldots\}),$$

where $(C_*(X), \partial)$ denotes the singular chain complex of X and Δ denotes the Alexander-Whitney diagonal. We will drop the *'induced by X'* from the notation when no confusion is possible.

The dual of this notion consists of transferred A_∞-algebras $\{\mu_n\}_n$ on the cohomology of X, $H^*(X)$, where μ_2 coincides with the cup product. Hence, transferred A_∞-structures encode all the information in the homology groups of X and in its cohomology algebra as well, but there is more. For instance, T. Kadeishvili proved that under mild conditions on a topological space X, any transferred A_∞-algebra on its cohomology determines the cohomology of its loop space [30, Prop. 2], $H^*(\Omega X)$, whereas the cohomology ring of X alone does not; in [3, Thm. 1.3], one can find a way to build pairs of spaces with isomorphic homology groups and isomorphic cohomology algebras but non-isomorphic transferred A_∞-coalgebras; in [4, §3], [34], one can find examples of links which are told apart by transfered A_∞-structures, which cannot be told apart by using the cup product alone. This can be done thanks to the relation between Massey products and A_∞-structures [11].

An immediate consequence of Definition 3.3 is that all transferred A_∞-coalgebras on $H_*(X)$ induced by X are isomorphic. The following is a folklore result, of which one can find a proof in [4, Cor. 3.3].

THEOREM 3.4. ***(Homotopy invariance of* $\dim \operatorname{Ker} \Delta_{m|H_p(X)}$***)*** *Let $\{\Delta_n\}_n$ be a transferred A_∞-coalgebra structure on the homology of a space X, and let us set*

$$k(X) := \begin{cases} \min\{n \mid \Delta_n \neq 0\}, & \text{if } \{n \mid \Delta_n \neq 0\} \neq \emptyset \\ +\infty, & \text{otherwise.} \end{cases}$$

Then, the number $k(X) \in \mathbb{Z} \cup \{+\infty\}$ and the integers $\dim \operatorname{Ker} \Delta_{m|H_p(X)}$ (for integers $m \leq k(X)$ and $p \geq 0$) are independent of the choice of transferred A_∞-coalgebra structure on $H_*(X)$. Moreover, since homotopy equivalent spaces induce isomorphic transferred A_∞-coalgebras, $k(X)$ and every such $\dim \operatorname{Ker} \Delta_{m|H_p(X)}$ are invariants of the homotopy type of X.

We finally recall a classical results used for computing and working with transferred A_∞-structures which we will use in the proof of the functoriality of A_∞ persistent homology (Thm. 4.3).

THEOREM 3.5. *[30, 33] **(Homotopy Transfer Theorem)** Let the following be a diagram of chain complexes,*

$$\phi \circlearrowright (M,d) \underset{\iota}{\overset{\pi}{\rightleftarrows}} (N,d) \tag{3.1}$$

where (N,d) is a chain complex, (M,d) is a differential graded coalgebra with comultiplication Δ, and the degree 0 chain maps π and ι and the degree 1 chain homotopy ϕ make the following hold: $\pi\iota = \mathrm{id}_N$, $\pi\phi = \phi\iota = \phi^2 = 0$ and ϕ is a chain homotopy between id_M and $\iota\pi$, i.e., *$\phi d + d\phi = \iota\pi - \mathrm{id}_M$. Then, there is an explicit minimal A_∞-coalgebra structure $\{\Delta_n\}_n$ on N with $\Delta_2 = \pi^{\otimes 2}\Delta\iota$ and there are morphisms of A_∞-coalgebras*

$$M \underset{I}{\overset{\Pi}{\rightleftarrows}} N \ ,$$

such that $\Pi_{(1)} = \pi$ and $I_{(1)} = \iota$.

Building a diagram of the form (3.1) with $M \cong C_*(X)$ and $N \cong H_*(X)$ amounts to building a transferred A_∞-coalgebra structure on $H_*(X)$.

4. A_∞ persistent homology: topological estimation and stability

The barcode of $H_p(X_*)$ from classical persistent homology recovers information only at the level of homology groups. In contrast, the barcodes in A_∞ persistent homology (which we recall in this section) consist of partial information from A_∞-(co)algebras in (co)homology, therefore enhancing persistent homology with a greater discriminatory power. A_∞ persistent homology can be viewed as a way to relate the different bars in the classical persistence barcodes in the following sense: we can think of persistent cohomology as producing a barcode where each bar represents the lifespan of a cohomology class. With this information alone, we cannot know whether these classes can consistently be expressed as a cup product, or as a Massey product. However, the bars in the level-2 A_∞-persistent cohomology barcode correspond to cup indecomposables, the bars in level-3, to triple-Massey indecomposables, and so on.

It would therefore be desirable to have stability results for A_∞ persistent homology similar to those seen in §2 for classical persistence. The issue is that A_∞ persistent homology is not functorial in general. Indeed, [3, Thm. 3.1] illustrates that given transferred A_∞-coalgebra structures $\big(H_*(X), \{\Delta_n^X\}_n\big)$, $\big(H_*(Y), \{\Delta_n^Y\}_n\big)$ and a continuous map $f\colon X \longrightarrow Y$, the inclusion

$$H_p(f)(\mathrm{Ker}\,\Delta^X_{n\,|H_p(X)}) \subseteq \mathrm{Ker}\,\Delta^Y_{n\,|H_p(Y)} \tag{4.1}$$

does not need to hold, where $H_p(f)$ denotes the map induced in p^{th} homology by f. Here we tackle the challenge of finding a non-trivial context which guarantees the functoriality of A_∞-persistence. In this section, we introduce the category $\mathbf{Top}_n$ (Def. 4.1) and show that A_∞ persistent homology is functorial within this category (Thm. 4.3). Additionally, we illustrate

that this is the largest category of its form (in a sense made explicit in Rmk. 4.4) for which such functoriality should be expected. From Thm. 4.3, stability results for A_∞ persistent homology (Cor. 4.8, 4.10 and 4.12) will follow.

DEFINITION 4.1. Let $n \in \mathbb{Z} \cup \{+\infty\}$, $n > 1$, and let $\mathbf{Top}_n$ denote the category whose objects are topological spaces X such that $\Delta_m = 0$, for all $m < n$, where $\{\Delta_m\}_{m\geq 1}$ denotes any transferred A_∞-coalgebra structure on $H_*(X)$, and where the morphisms are continuous maps.

It follows from Def. 4.1 that for every integer $n > 1$, $\mathbf{Top}_\infty \subseteq \mathbf{Top}_{n+1} \subseteq \mathbf{Top}_n \subseteq \mathbf{Top}_2 = \mathbf{Top}$ are full subcategories. The higher $n \in \mathbb{Z} \cup \{+\infty\}$ is, the closer the objects in $\mathbf{Top}_n$ are to having the whole A_∞-coalgebra structure on their homology fully determined by their cohomology ring (a notion related to formality in the context of rational homotopy theory).

Given a topological space X, all the transferred A_∞-coalgebra structures $\{\Delta_m\}_m$ on $H_*(X)$ are isomorphic. The Axiom of Choice guarantees that we can fix a choice of transferred A_∞-coalgebra structure $\{\Delta_m\}_m$ on $H_*(X)$, for every $X \in \mathbf{Top}$. For a concrete example in a restricted scenario, let us consider filtered CW complexes, which are CW complexes $\langle c_0, \ldots, c_m\rangle$ whose cells are ordered $c_0, \ldots, c_m$ so that, for all $0 \leq i < m$, $\langle c_0, \ldots, c_i\rangle$ forms a subcomplex of $\langle c_0, \ldots, c_{i+1}\rangle$. H. Molina-Abril and P. Real [35, Alg. 1] used discrete vector fields to create a deterministic algorithm which computes a transferred A_∞-coalgebra structure on $H_*(X)$ for every filtered CW complex X. Once a method to build A_∞-coalgebra structures has been fixed, we can define $\kappa_{n,p}$:

DEFINITION 4.2. Let $\kappa_{n,p} : \mathbf{Top}_n \longrightarrow \mathbf{Vect}$ be the following assignment: for every object X in $\mathbf{Top}_n$, pick a particular transferred A_∞-coalgebra structure $\{\Delta_m\}_m$ on $H_*(X)$ and define

$$\kappa_{n,p}(X) \coloneqq \operatorname{Ker} \Delta_{n|H_p(X)} \subseteq H_p(X).$$

For every morphism $f : X \longrightarrow Y$ in $\mathbf{Top}_n$, define

$$\kappa_{n,p}(f) : \kappa_{n,p}(X) \longrightarrow \kappa_{n,p}(Y)$$

as the map

$$H_p(f) : H_p(X) \longrightarrow H_p(Y)$$

induced by f in degree-p homology, restricted to $\kappa_{n,p}(X)$. Being pedantic with the notation, this would be

$$\kappa_{n,p}(f) = H_p(f)_{|\operatorname{Ker} \Delta_{n|H_p(X)}}.$$

THEOREM 4.3. **(Functoriality of $\kappa_{n,p}$)** *For each pair of integers $n \geq 2$, $p \geq 0$, the category $\mathbf{Top}_n$ is well defined, the assignment*

$$\kappa_{n,p} : \mathbf{Top}_n \longrightarrow \mathbf{Vect}$$

in Def. 4.2 defines a functor and the integer $\dim \kappa_{n,p}(X)$ *does not depend on the choice of* A_∞*-coalgebra made in Def. 4.2.*

PROOF. By definition, the objects in $\mathbf{Top}_n$ are the topological spaces X such that

$$\min\{m \,|\, \Delta_m \neq 0\} \geq n,$$

where $\{\Delta_m\}_m$ denotes any transferred A_∞-coalgebra structure on $H_*(X)$. Hence, Thm. 3.4 guarantees that $\mathbf{Top}_n$ is well defined for all $n \in \mathbb{Z} \cup \{+\infty\}$ such that $n > 1$. In particular, the property of X being in $\mathbf{Top}_n$ does not depend on the choice of A_∞-coalgebra on its homology. Rather, it only depends on the homotopy type of X. Thm. 3.4 also guarantees that if $X \in \mathbf{Top}_n$ for some integer $n > 1$, then the integer $\dim \kappa_{n,p}(X) = \dim \operatorname{Ker} \Delta_{n|H_p(X)}$ does not depend on the choice of A_∞-coalgebra and it is indeed a homotopy invariant of X, for all $p \geq 0$.

Shifting the focus now to $\kappa_{n,p}$, the Axiom of Choice makes the assignment $X \mapsto \kappa_{n,p}(X)$ well defined. To prove that the assignment

$$f \mapsto (\kappa_{n,p}(f)\colon \kappa_{n,p}(X) \to \kappa_{n,p}(Y))$$

is well defined too, is equivalent to showing that

$$\operatorname{Im} \kappa_{n,p}(f) \subseteq \kappa_{n,p}(Y)$$

holds. For this, it suffices to show that the inclusion in (4.1), $H_p(f)(\operatorname{Ker} \Delta^X_{n\,|H_p(X)}) \subseteq \operatorname{Ker} \Delta^Y_{n\,|H_p(Y)}$, holds if $f : X \longrightarrow Y$ denotes a morphism in $\mathbf{Top}_n$ and $\{\Delta^X_m\}_m$ and $\{\Delta^Y_m\}_m$ denote any transferred A_∞-coalgebra structures on $H_*(X)$ and $H_*(Y)$, respectively. To show that (4.1) holds, notice that any continuous function $f : X \longrightarrow Y$ induces maps of differential graded coalgebras at the singular chain level $C_*(f) : C_*(X) \longrightarrow C_*(Y)$. Applying Thm. 3.5 once with $M \cong C_*(X)$ and once with $M \cong C_*(Y)$, we can infer that $C_*(f)$, in turn, induces a morphism of A_∞-coalgebras

$$\{F_{(m)}\}_m : (H_*(X), \{\Delta^X_n\}_n) \longrightarrow (H_*(Y), \{\Delta^Y_n\}_n)$$

where $F_{(1)} = H_*(f)$.

The key point here is that if $X, Y \in \mathbf{Top}_n$, then the identity $\mathrm{MI}(n)$ in Def. 3.2 becomes $\Delta^Y_n H_*(f) = H_*(f)^{\otimes n} \Delta^X_n$, and thus, the inclusion in (4.1) does hold.

Now that we have checked that $\kappa_{n,p}(f)\colon \kappa_{n,p}(X) \to \kappa_{n,p}(Y)$ is well defined, notice that $\kappa_{n,p}(f)$ is the restriction of the map $H_p(f)$, and therefore, the functoriality of $\kappa_{n,p}$ follows from that of the homology functor $H_p : \mathbf{Top} \longrightarrow \mathbf{Vect}$. □

REMARK 4.4. In general, one cannot expect to find a larger category in the family $\{\mathbf{Top}_m\}_{m \in \mathbb{Z} \cup \{\infty\}, m>1}$ for which

$$\kappa_{n,p} : \mathbf{Top}_m \longrightarrow \mathbf{Vect}$$

would be functorial, as explained next.

Consider we try to redefine $\kappa_{n,p}$ as in Def. 4.2 but on a general Top_m,

$$\kappa_{n,p} : \mathbf{Top}_m \longrightarrow \mathbf{Vect}.$$

Thm. 4.3 shows that

(a) $\kappa_{n,p} : \mathbf{Top}_m \longrightarrow \mathbf{Vect}$ defines a functor, and
(b) $\dim \kappa_{n,p}(X)$ does not depend on the choice of A_∞-coalgebra, for any $X \in \mathbf{Top}_m$,

for any values $m \geq n \geq 2$, but it does not guarantee these two properties to hold if $2 \leq m < n$. Actually, we should not expect to have such properties to hold for $m < n$ in general, as counterexamples such as Ex. 4.5 show.

In this example, we use adaptations of the definitions of $\mathbf{Top}_m$ and $\kappa_{n,p}$ so that all the A_∞-coalgebras considered in their definitions are transferred A_∞-coalgebras on reduced rational homology. With these reduced versions of $\mathbf{Top}_m$ and $\kappa_{n,p}$, we will now recall an example for which $\dim \kappa_{3,7}(X)$ does depend on the choice of A_∞-coalgebra if $X \in \mathbf{Top}_2 - \mathbf{Top}_3$, failing to satisfy property (b), and we will extend this example to exhibit a case in which

$$\kappa_{3,7} : \mathbf{Top}_2 \longrightarrow \mathbf{Vect}$$

does not define a functor.

Example 4.5. Let us denote a wedge of a complex projective plane and a 7-sphere as X, and let us omit the rational coefficients from the notation to simply keep $\widetilde{H}_*(X) := \widetilde{H}_*(\mathbb{C}P^2 \vee \mathbb{S}^7; \mathbb{Q})$. Ex. 1 in [4] presents two transferred A_∞-coalgebra structures $\{\Delta_n^V\}_n$ and $\{\Delta_n^W\}_n$ on $\widetilde{H}_*(X)$. From the computations in [4, Ex. 1], it is clear that $X \in \mathbf{Top}_2 - \mathbf{Top}_3$. An intuitive reason why this happens is that the cup product of the complex projective plane is non-trivial. The A_∞-structure $\{\Delta_n^V\}_n$ satisfies

$$\dim \operatorname{Ker} \Delta_3^V{}_{|\widetilde{H}_7(X)} = \dim \widetilde{H}_7(X) = 1,$$

whereas $\{\Delta_n^W\}_n$ satisfies

$$\dim \operatorname{Ker} \Delta_3^W{}_{|\widetilde{H}_7(X)} = 0.$$

This directly shows an example for which $\dim \kappa_{3,7}(X)$ does depend on the choice of A_∞-coalgebra if $X \in \mathbf{Top}_2 - \mathbf{Top}_3$, failing to satisfy property (b).

Let us now attach a cell to X by setting $Y = X \vee \mathbb{S}^1$. Notice that the homology of X and Y only differ in degree 1, where we have $\widetilde{H}_1(X) = 0$ and $\widetilde{H}_1(Y) \cong \mathbb{Q}$. With the techniques from [40, III.3.(6)], it is easy to see that we can extend the A_∞-coalgebra $\left(\widetilde{H}_*(X), \{\Delta_n^W\}_n\right)$

to a transferred A_∞-coalgebra $\left(\widetilde{H}_*(Y), \{\Delta_n\}_n\right)$, by simply setting $\Delta_{n|\widetilde{H}_p(Y)} \equiv \Delta^W_{n\ |\widetilde{H}_p(X)}$ for $p \neq 1$ and $\Delta_{n|\widetilde{H}_1(Y)} \equiv 0$. In particular,

$$\dim \operatorname{Ker} \Delta_{3|\widetilde{H}_7(Y)} = \dim \operatorname{Ker} \Delta^W_{3\ |\widetilde{H}_7(X)} = 0.$$

This leads us to the following situation. Let us denote by

$$\iota := X \longrightarrow Y$$

the inclusion map of X within Y. If α denotes the generator of $\widetilde{H}_7(X) \cong \mathbb{Q}$, then its image by the map induced in homology, $\widetilde{H}_7(\iota)(\alpha)$, generates $\widetilde{H}_7(Y) \cong \mathbb{Q}$. In particular, $0 \neq \widetilde{H}_7(\iota)(\alpha) \notin \operatorname{Ker} \Delta_{3|\widetilde{H}_7(Y)}$, and therefore,

$$\widetilde{H}_7(\iota)\left(\operatorname{Ker} \Delta_{3|\widetilde{H}_7(X)}\right) \not\subset \operatorname{Ker} \Delta_{3|\widetilde{H}_7(Y)}.$$

This means we cannot define $\kappa_{3,7}$ on functions of $Top_2 - Top_3$. In particular,

$$\kappa_{3,7} : \mathbf{Top}_2 \longrightarrow \mathbf{Vect}$$

does not define a functor.

From this moment on, we assume we have fixed a choice of a functor $\kappa_{n,p} : \mathbf{Top}_n \longrightarrow \mathbf{Vect}$ as in Def. 4.2. We will next see how the existence of barcodes in A_∞ persistent homology and their stability follow from the functoriality of $\kappa_{n,p}$.

Corollary 4.6. ***(Structure theorem of A_∞ persistent homology in $\mathbf{Top}_n$)*** *Fix integers $p \geq 0$ and $n \geq 1$. Let $X_* \colon \mathbb{R} \longrightarrow \mathbf{Top}_n$ be a persistence space such that $\dim_{\mathbb{F}} H_p(X_t) < \infty$ for all $t \in \mathbb{R}$. Then the A_∞ persistent homology module $\kappa_{n,p}X_*$ decomposes uniquely (up to isomorphism) into interval persistence modules $C(I)$,*

$$\kappa_{n,p}X_* \cong \bigoplus_{I \in B(\kappa_{n,p}X_*)} \mathbf{C}(I),$$

*where $B\left(\kappa_{n,p}X_*\right)$ is a multiset of intervals of the form $[a, b)$ for some $a \in \mathbb{R}$, $b \in (a, +\infty] \subseteq \mathbb{R} \cup \{+\infty\}$. We call $B\left(\kappa_{n,p}X_*\right)$ the **$\mathbf{\Delta}_{n,p}$-barcode** of X_*.*

Proof. Since $\kappa_{n,p}(X_t)$ is a vector subspace of $H_p(X_t)$, we have

$$\dim_{\mathbb{F}} \kappa_{n,p}(X_t) \leq \dim_{\mathbb{F}} H_p(X_t) < \infty$$

for all $t \in \mathbb{R}$, by assumption. Hence, $\kappa_{n,p}X_*$ forms a p.f.d. persistence module and Thm. 2.6 guarantees that we can uniquely decompose it as a direct sum of interval persistence modules. □

The barcode decomposition result in Cor. 4.6 deals with persistence spaces of the form $X_*\colon \mathbb{R} \longrightarrow \mathbf{Top}_n$. An analogous theorem was proved in [4] for persistence spaces $X_*\colon \mathbb{Z} \longrightarrow \mathcal{C}$ indexed by the integers and valued in a category $\mathcal{C}$ potentially larger than $\mathbf{Top}_n$ ($\mathbf{Top}_n \subset \mathcal{C} \subset \mathbf{Top}$). When the considered persistence spaces have the form $X_*\colon \mathbb{Z} \longrightarrow \mathbf{Top}$, the inclusion in (4.1) fails. This stops the analogue of $\kappa_{n,p}(X_*)$ from forming a persistence module, and one needs to resort to zigzag modules. The structure and interpretation of the corresponding A_∞ persistence zigzag modules was studied in detail in [3].

To state and prove the rest of the results in this paper, we will fix the following notation and assumptions:

Assumptions *All the sublevel sets of a real-valued functions $f\colon X \longrightarrow \mathbb{R}$, have finite-dimensional p^{th} homology,* i.e., $\dim_{\mathbb{F}} H_p\left(f^{-1}[-\infty, t)\right) < \infty$.

Notation

- *d_B denotes the bottleneck distance (Def. 2.8).*
- *d_∞ denotes the l^∞ distance in (2.2).*
- *$||\cdot||_\infty$ denotes the supremum norm.*
- *$B_{n,p}(f) \coloneqq B\left(\kappa_{n,p}\mathcal{S}(f)\right)$ denotes the $\Delta_{n,p}$-barcode of the sublevel-set filtration of a continuous real-valued function $f\colon X \longrightarrow \mathbb{R}$.*
- *$d^X : M \longrightarrow \mathbb{R}$ denotes the real-valued function defined by $d^X(y) = d(y, X)$, for all $y \in M$, given any closed subspace X of a metric space (M, d).*
- *$X^{+\delta}$ denotes the (open) δ-**thickening** of X, $\left(d^X\right)^{-1}(-\infty, \delta)$, for $\delta > 0$.*
- *$X^{+\delta]}$ denotes the (closed) δ-**thickening** of X, $\left(d^X\right)^{-1}(-\infty, \delta]$, for $\delta \geq 0$.*
- *$B_{n,p}(X)$ denotes the barcode $B_{n,p}(d^X)$.*

As we will mention after Cor. 4.10, $B_{n,p}(X)$ can be interpreted as the $\Delta_{n,p}$-barcode of the Čech-complex filtration of X.

Let us start with a toy example in which two persistent spaces result in the same barcode within classical persistent homology but result in barcodes at an infinite bottleneck distance away from each other within A_∞ persistent homology. This example illustrates the superior discriminatory power of A_∞ persistent homology and helps understanding what the barcodes in A_∞-persistence encode.

EXAMPLE 4.7. Let us work with reduced homology. We start by defining the category $\widetilde{\mathbf{Top}}$ as the analogue of $\mathbf{Top}_n$ (Def. 4.1) in this setting: for $n \in \mathbb{Z} \cup \{+\infty\}$, $n > 1$, let $\widetilde{\mathbf{Top}}_n$ denote the category whose objects are topological spaces X such that $\Delta_m = 0$, for all $m < n$, where $\{\Delta_m\}_{m \geq 1}$ denotes any transferred A_∞-coalgebra structure on the reduced homology $\widetilde{H}_*(X; \mathbb{F})$, and where the morphisms are continuous maps.

Let X_*, Y_* be two persistence spaces consisting of thickeniing filtrations of two point clouds P and Q, in the sense that

$$X_t \coloneqq P^{+t]} \quad \text{and} \quad Y_t \coloneqq Q^{+t]}, \quad \text{for all } t \in \mathbb{R}.$$

Assume that P and Q are sampled from a torus $\mathbb{T}$ and a wedge of spheres $\mathbb{S}^2 \vee \mathbb{S}^1 \vee \mathbb{S}^1$, respectively, in such a way that the non-trivial degree 2 homology of the persistence spaces appears at time t_0 and vanishes at time t_1 in both cases, *i.e.,*

$$H_2(X_t) = H_2(Y_t) = 0, \quad \text{for all } t \in (-\infty, t_0) \cup [t_1, +\infty)$$

and so that the homotopy types of the steps in-between are known:

$$X_t \simeq \mathbb{T} \quad \text{and} \quad Y_t \simeq \mathbb{S}^2 \vee \mathbb{S}^1 \vee \mathbb{S}^1, \quad \text{for all } t \in [t_0, t_1).$$

where all connecting maps are homotopic to the identity. Denote by $B_p(X_*)$ and $B_p(Y_*)$ the barcodes describing the evolution of $\widetilde{H}_p(X_t)$ and $\widetilde{H}_p(Y_t)$, respectively. Recall that

$$\widetilde{H}_p(\mathbb{T}) \cong \widetilde{H}_p(\mathbb{S}^2 \vee \mathbb{S}^1 \vee \mathbb{S}^1) \cong \begin{cases} \mathbb{F}^2, & p = 1 \\ \mathbb{F}, & p = 2 \\ 0, & p \neq 1, 2. \end{cases}$$

In particular, both $B_2(X_*)$ and $B_2(Y_*)$ consist of a single interval $[t_0, t_1)$. Hence,

$$d_B\left(B_2(X_*), B_2(Y_*)\right) = 0,$$

and classical persistence in degree 2 does not tell the two point clouds apart.

Now let us also denote by α^* and β^* the two generators of $\widetilde{H}^1(\mathbb{T})$ and let γ^* be the generator of $\widetilde{H}^2(\mathbb{T})$. It is well known that we can choose these cohomology generators so that the cup product relates them via the equality $\alpha^* \smile \beta^* = \gamma^*$. Dually, if we denote by α and β the two generators of $\widetilde{H}_1(\mathbb{T})$ and let γ be the generator of $\widetilde{H}_2(\mathbb{T})$, then

$$\Delta\gamma = \alpha \otimes \beta.$$

Hence, any transferred A_∞-coalgebra $\left(\widetilde{H}_*(\mathbb{T}), \{\Delta_n\}_n\right)$ will have

$$\Delta_2(\gamma) \neq 0, \tag{4.2}$$

and thus

$$X_t \in \widetilde{Top}_2 \setminus \widetilde{Top}_3,$$

for all $t \in [t_0, t_1)$. Therefore, as long as we only compare $\Delta_{n,p}$-barcodes of X_* for $n \leq 2$, we can guarantee that stability results such as Cor. 4.8 hold. Denote by $B_{2,2}(X_*)$ and $B_{2,2}(Y_*)$ the $\Delta_{2,2}$-barcode describing the evolution of $\operatorname{Ker} \Delta_{2|\widetilde{H}_2(X_t)}$ and $\operatorname{Ker} \Delta_{2|\widetilde{H}_2(Y_t)}$, respectively. Eq. (4.2) shows that

$$\operatorname{Ker} \Delta_{2|\widetilde{H}_2(X_t)} = 0 \subsetneq \widetilde{H}_2(X_t),$$

for all $t \in [t_0, t_1)$. Hence, $B_{2,2}(X_*)$ is empty - it consists of no intervals.

On the other hand, using the same reasoning on $\mathbb{S}^2 \vee \mathbb{S}^1 \vee \mathbb{S}^1$, any transferred A_∞-coalgebra $\left(\widetilde{H}_*(\mathbb{S}^2 \vee \mathbb{S}^1 \vee \mathbb{S}^1), \{\Delta_n\}_n\right)$ will have $\Delta_2 = 0$, which is,

$$\operatorname{Ker} \Delta_{2_{|\widetilde{H}_2(Y_t)}} = \widetilde{H}_2(Y_t),$$

for all $t \in [t_0, t_1)$. Hence, $B_{2,2}(Y_*) = B_2(Y_*)$ still consists of a single interval $[t_0, t_1)$ and we conclude that

$$d_B\left(B_{2,2}(X_*), B_{2,2}(Y_*)\right) = \frac{t_1 - t_0}{2}.$$

The more the degree 2 homology persist in the filtrations (*i.e.,* the greater the difference $t_1 - t_0$ is), the greater the bottleneck distance between the A_∞-persistence barcodes $B_{2,2}(X_*)$ and $B_{2,2}(Y_*)$ is too.

We now prove the first result on the stability of A_∞ persistent homology by providing a generalization of Thm. 2.11.

COROLLARY 4.8. ***(Stability of A_∞ persistent homology for functions)*** *Let $n > 0$ be an integer and let $f\colon X \longrightarrow \mathbb{R}$ and $g\colon Y \longrightarrow \mathbb{R}$ be two continuous maps. Assume that all sublevel-sets of f and g are in $\mathbf{Top}_n$. Then, for all $p \geq 0$, the bottleneck distance between the $\Delta_{n,p}$-barcodes of f and g is bounded above by the l^∞ distance between the functions:*

$$d_B\left(B_{n,p}(f), B_{n,p}(g)\right) \leq d_\infty(f, g).$$

In particular, if $X = Y$, then

$$d_B\left(B_{n,p}(f), B_{n,p}(g)\right) \leq ||f - g||_\infty.$$

PROOF. The assumption of each $f^{-1}[-\infty, t)$ being in $\mathbf{Top}_n$ turns the sublevel-set filtration of f into a persistence space of the form $\mathcal{S}(f)\colon \mathbb{R} \longrightarrow \mathbf{Top}_n$. Since $\kappa_{n,p}$ is a functor (Thm. 4.3) and $\dim_{\mathbb{F}} H_p\left(f^{-1}[-\infty, t)\right) < \infty$ for all t by assumption, the composition $\kappa_{n,p}\mathcal{S}(f)$ is a p.f.d. persistence module. Thm. 2.6 guarantees the existence and uniqueness of the $\Delta_{n,p}$-barcode $B_{n,p}(f) \coloneqq B\left(\kappa_{n,p}\mathcal{S}(f)\right)$. The same holds for g, and Thm. 2.15 then shows that

$$d_B\left(B\left(\kappa_{n,p}\mathcal{S}(f)\right), B\left(\kappa_{n,p}\mathcal{S}(g)\right)\right) \leq d_I\left(\kappa_{n,p}\mathcal{S}(f), \kappa_{n,p}\mathcal{S}(g)\right),$$

where d_I denotes the interleaving distance. We then use Thm. 2.16 to exploit the functoriality of $\kappa_{n,p}$ and conclude that

$$d_I\left(\kappa_{n,p}\mathcal{S}(f), \kappa_{n,p}\mathcal{S}(g)\right) \leq d_I\left(\mathcal{S}(f), \mathcal{S}(g)\right).$$

Thm. 2.17 asserts that the sublevet-set functor is 1-Lipschitz. Hence:

$$d_I\left(\mathcal{S}(f), \mathcal{S}(g)\right) \leq d_\infty(f, g).$$

These 3 inequalities together show that

$$d_B\left(B_{n,p}(f), B_{n,p}(g)\right) \leq d_\infty(f,g).$$

In particular, if $X = Y$, then, since the identity $\Phi\colon X \longrightarrow Y$ is a homeomorphism,

$$d_\infty(f,g) \leq ||f-g||_\infty,$$

and the second claim in Cor. 4.8 holds as well. □

DEFINITION 4.9. The **Hausdorff distance** between two non-empty subsets X, Y of a metric space (M,d) is defined as

$$d_H(X,Y) = \inf\{\delta \geq 0\,;\ X \subseteq Y^{+\delta]} \text{ and } Y \subseteq X^{+\delta]}\}.$$

By definition of the Hausdorff distance d_H, we have that $||d^X - d^Y||_\infty = d_H(X,Y)$. This leads to the following straightforward corollary of Cor. 4.8:

COROLLARY 4.10. ***(Stability of A_∞ persistent homology for metric spaces)*** *Let $n \geq 1$ be an integer. Let X and Y be closed subspaces of a metric space M such that the thickenings $X^{+\tau]}$ and $Y^{+\tau]}$ are in $\mathbf{Top}_n$ for all $\tau \geq 0$. Then, for all $p \geq 0$, the bottleneck distance between the $\Delta_{n,p}$-barcodes $B_{n,p}(X)$ and $B_{n,p}(Y)$ is bounded above by the Hausdorff distance between the spaces X and Y:*

$$d_B(B_{n,p}(X), B_{n,p}(Y)) \leq d_H(X,Y).$$

The condition $X^{+\tau]}, Y^{+\tau]} \in \mathbf{Top}_n$ for all $\tau \geq 0$ is equivalent to assuming all sublevel sets of d^X and d^Y to be in $\mathbf{Top}_n$. Hence, Cor. 4.10 follows directly from applying Cor. 4.8 to $f = d^X, g = d^Y$.

For computational purposes, one tends to work with simplicial or cubical complexes. Let X be a finite subset of a metric space (M,d). For any $\epsilon > 0$, the Čech complex $\check{\mathcal{C}}_\epsilon(X)$ is a simplicial complex defined as the nerve of the set of open balls of radius ϵ centered at all points in X[1]. The Nerve Lemma [8, Corollary 3 in Section 9] shows that $\check{\mathcal{C}}_\epsilon(X)$ is homotopy equivalent to the sublevel set $X^{+\epsilon} = \left(d^X\right)^{-1}(-\infty, \epsilon)$, provided that all intersections of balls are either empty or contractible. Hence, Cor. 4.10 can be interpreted as a result on the stability of the A_∞ persistent homology of the Čech-complex filtration.

We finally come to extrapolating topological properties of a closed subspace X of a metric space M from a finite set P of points which may have been inaccurately sampled from X. For smooth manifolds X in low dimensions, some methods [**2**, **23**, **36**] can reconstruct X from P so that one can estimate the Betti numbers of X. V. Robins [**39**], V. de Silva and G. Carlsson [**20**] started estimating these Betti numbers via persistent homology instead.

[1]Note that $\check{\mathcal{C}}_\epsilon(X)$ is sometimes defined using the radius $\epsilon/2$.

One of the advantages of this approach is that it can be used for smooth and non-smooth spaces X of all dimensions, and it avoids having to choose an optimal thickening amount which may sometimes be impossible to find. Later on, stability would play a crucial role in this homology estimation task [7, 17, 25]. F. Chazal and A. Lieutier [15] were the first to estimate topological information of X not captured by the homology groups. Namely, they approximated the fundamental group $\pi_1(X)$. More on fundamental group of point clouds can be found in [9]. The current work goes further in this direction by showing that we can also estimate A_∞ information of X, as we will see in Cor. 4.12.

We now define an analogous concept to that of homological critical value (Def. 2.10) for A_∞-structures.

Definition 4.11. Given a continuous map $f\colon X \longrightarrow \mathbb{R}$ whose sublevel sets are in $\mathbf{Top}_n$ and a functor $\kappa_{n,p}$ defined as in Def. 4.2, we say that a real number $a \in \mathbb{R}$ is a $\boldsymbol{\Delta_{n,p}}$ **critical value** of f if the map

$$\kappa_{n,p}\mathcal{S}(f)(a-\epsilon) \longrightarrow \kappa_{n,p}\mathcal{S}(f)(a+\epsilon)$$

induced in homology by the inclusion

$$f^{-1}(-\infty, a-\epsilon] \hookrightarrow f^{-1}(-\infty, a+\epsilon]$$

is not an isomorphism for all sufficiently small $\epsilon > 0$. We then define the $\boldsymbol{\Delta_{n,p}}$ **feature size** of X, denoted by $\Delta_{n,p}fs(X)$, as the infimum over all positive $\Delta_{n,p}$ critical values of the distance function d^X.

Just as the homological feature size used in [17], the $\Delta_{n,p}$ feature size depends not only on the topology of X, but also on its geometry.

Given n, p and a functor $\kappa_{n,p}$ defined as in Def. 4.2, the following result computes A_∞-information of a δ-thickening of X via the A_∞ persistent homology of P, where we can think of P as a finite approximation to X.

Corollary 4.12. ***($\boldsymbol{A_\infty}$ inference)*** *Let X, P be closed subspaces of a metric space M such that the thickenings $X^{+\tau]}$ and $P^{+\tau]}$ are in $\mathbf{Top}_n$ for all $\tau \geq 0$. Let $\epsilon > 0$ be a real number with $d_H(X,P) < \epsilon < \frac{\Delta_{n,p}fs(X)}{4}$. For all sufficiently small $\delta > 0$, the topological invariant $\dim \kappa_{n,p}(X^{+\delta})$ coincides with the number of intervals in the $\Delta_{n,p}$-barcode $B_{n,p}(P)$ which contain the interval $[\epsilon, 3\epsilon]$.*

The lower bound on ϵ appearing in Cor. 4.12 describes how accurately P approximates X. The better P approximates X, the smaller $d_H(X,P)$ is. The upper bound on ϵ appearing in Cor. 4.12, $\frac{\Delta_{n,p}fs(X)}{4}$, depends on the metrics of X and M.

The proof of Cor. 4.12 is basically that of [17, Homology Inference Theorem], changing the persistent homology functor by the A_∞ persistent homology functor $\kappa_{n,p}$. We include the proof anyway for the sake of completeness.

PROOF. Let $f, g\colon M \longrightarrow \mathbb{R}$ be continuous maps whose sublevel sets are all in $\mathbf{Top}_n$. Let us define

$$F_i \coloneqq \kappa_{n,p}\left(f^{-1}(-\infty, i]\right) = \kappa_{n,p}\mathcal{S}\left(f\right)(i) \quad \text{and} \quad G_i \coloneqq \kappa_{n,p}\left(g^{-1}(-\infty, i]\right) = \kappa_{n,p}\mathcal{S}\left(g\right)(i)$$

for all $i \in \mathbb{R}$. Since $\kappa_{n,p}$ is functorial (Thm. 4.3), we can consider the map

$$\kappa_{n,p}\left(f^{-1}(-\infty, i] \hookrightarrow f^{-1}(-\infty, j]\right)$$

for any real numbers $i < j$, which we will denote by $f_i^j\colon F_i \longrightarrow F_j$. Define $g_i^j\colon G_i \longrightarrow G_j$ analogously. Finally, let us set

$$F_i^j \coloneqq \operatorname{Im} f_i^j \quad \text{and} \quad G_i^j \coloneqq \operatorname{Im} g_i^j.$$

This way, $\dim F_i^j$ and $\dim G_i^j$ are precisely the number of intervals which contain $[i, j)$ in the A_∞-barcodes $B_{n,p}(f)$ and $B_{n,p}(g)$, respectively. If $f = d^X$ or $f = d^P$, we will denote F_i^j by X_i^j or P_i^j, respectively.

If $||f - g||_\infty < \epsilon$, then $f^{-1}(-\infty, i] \subseteq g^{-1}(-\infty, i + \epsilon]$ for all $i \in \mathbb{R}$. By the functoriality of $\kappa_{n,p}$ (Thm. 4.3), this inclusion induces a map $\varphi_i\colon F_i \longrightarrow G_{i+\epsilon}$. Exchanging f and g, we also have a map $\psi_i\colon G_i \longrightarrow F_{i+\epsilon}$. We can fit these maps into the following diagram:

$$\begin{array}{ccc} F_{i-\epsilon} & \xrightarrow{f_{i-\epsilon}^{j+\epsilon}} & F_{j+\epsilon} \\ {\scriptstyle \varphi_{i-\epsilon}}\downarrow & & \uparrow{\scriptstyle \psi_j} \\ G_i & \xrightarrow[g_i^j]{} & G_j \end{array}$$

Since all maps in the diagram are induced by inclusions, the diagram commutes, for any reals $i < j$. Some diagram chasing shows that

$$F_{i-\epsilon}^{j+\epsilon} \subseteq \psi_j\left(G_i^j\right),$$

and hence

$$\dim F_{i-\epsilon}^{j+\epsilon} \leq \dim G_i^j. \tag{4.3}$$

Now set $f = d^X, g = d^P, i = \epsilon + \delta$ and $j = 3\epsilon + \delta$. Since $||d^X - d^P||_\infty = d_H(X, P)$ and we assumed the $d_H(X, P) < \epsilon$ and that all sublevel sets of d^X and d^P are in $\mathbf{Top}_n$, we can use (4.3) and rewrite it as

$$\dim X_\delta^{4\epsilon+\delta} \underset{23}{\leq} \dim P_{\epsilon+\delta}^{3\epsilon+\delta}. \tag{4.4}$$

Analogously, letting $f = d^P, g = d^X$ and $i = j = 2\epsilon + \delta$, (4.3) yields

$$\dim P^{3\epsilon+\delta}_{\epsilon+\delta} \leq \dim X^{2\epsilon+\delta}_{2\epsilon+\delta}. \tag{4.5}$$

Choosing δ small enough so that $4\epsilon + \delta < \Delta_{n,p} fs(X)$, there are no $\Delta_{n,p}$ critical values of d^X in $[\delta, 4\epsilon + \delta]$. In particular, $\dim X^{4\epsilon+\delta}_{\delta} = \dim X^{2\epsilon+\delta}_{2\epsilon+\delta}$, and the inequalities in (4.4) and (4.5) becomes equalities. Again, since there are no $\Delta_{n,p}$ critical values of d^X in $[\delta, 4\epsilon + \delta]$, $\dim X^{4\epsilon+\delta}_{\delta} = \dim X^{\delta}_{\delta} = \dim \kappa_{n,p}(X^{+\delta})$.

Since $\epsilon > 0$, $\dim P^{3\epsilon+\delta}_{\epsilon+\delta} = \dim P^{3\epsilon}_{\epsilon}$ for every $\delta > 0$ small enough. Therefore, for sufficiently small $\delta > 0$,

$$\dim \kappa_{n,p}(X^{+\delta}) = \dim P^{3\epsilon+\delta}_{\epsilon+\delta} = \dim P^{3\epsilon}_{\epsilon},$$

concluding the proof. □

Actually, the proof of Cor. 4.12 clearly shows that we do not need to assume that all sublevel sets of d^X and d^P are in $\mathbf{Top}_n$. Rather, it is enough to assume that $X^{+\tau]} \in \mathbf{Top}_n$ for all $\tau \leq 4\epsilon + \delta$ and that $P^{+\tau]} \in \mathbf{Top}_n$ for all $\tau \leq 3\epsilon + \delta$.

In most real world situations and for small enough values of $\delta > 0$, X is a retract of $X^{+\delta}$ and therefore the homotopy types of X and $X^{+\delta}$ coincide. Thus, we can see Cor. 4.12 as estimating topological properties of X from a (possibly finite) closed subset $P \subseteq M$ approximating X.

Note that when we focus on the second operation Δ_2 on an A_∞-coalgebra $(H_*(X), \{\Delta_n\}_n)$ (such as in Ex. 4.7), or equivalently, on the cup product in cohomology (which is the second operation $\mu_2 = \smile$ on an A_∞-algebra $(H^*(X), \{\mu_n\}_n)$), then all results in this paper hold without the need to restrict to a category $\mathbf{Top}_n \subseteq \mathbf{Top}$ for $n > 2$ and instead, we can work directly with the category of topological spaces $\mathbf{Top}$. In particular, this paper proves the stability of the persistence of cup product with minimal restrictions.

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References

[1] A. Adcock, G. Carlsson, and D. Rubin. Classification of hepatic lesions using the matching metric. *Comput. Vis. Image Und.*, 121:36–42, 2014.

[2] N. Amenta and N. Bern. Surface reconstruction by Voronoi filtering. *Discrete Comput. Geom.*, 22:481–504, 1999.

[3] F. Belchí. *A_∞-persistence*. PhD thesis, Universidad de Málaga, 2015.

[4] F. Belchí. Optimising the topological information of the A_∞-persistence groups. *Discrete & Computational Geometry*, 62(1):29–54, 2019.

[5] F. Belchí and A. Murillo. A_∞-persistence. *Appl. Algebra Engrg. Comm. Comput.*, 26(1-2):121–139, 2015.

[6] F. Belchí, M. Pirashvili, J. Conway, M. Bennett, R. Djukanovic, and J. Brodzki. Lung Topology Characteristics in patients with Chronic Obstructive Pulmonary Disease. *Scientific Reports*, 8(1), Mar 2018.

[7] A.J. Blumberg, I. Gal, M. A. Mandell, and M. Pancia. Robust statistics, hypothesis testing, and confidence intervals for persistent homology on metric measure spaces. *Foundations of Computational Mathematics*, 14(4):745–789, 2014.

[8] K. Borsuk. On the imbedding of systems of compacta in simplicial complexes. *Fund. Math.*, 35:217–234, 1948.

[9] P. Brendel, P. Dłotko, G. Ellis, M. Juda, and M. Mrozek. Computing fundamental groups from point clouds. *Appl. Algebra Engrg. Comm. Comput.*, 26(1-2):27–48, 2015.

[10] P. Bubenik, V. de Silva, and J. Scott. Metrics for generalized persistence modules. *Foundations of Computational Mathematics*, 15(6):1501–1531, 2015.

[11] U. Buijs, J. M. Moreno-Fernández, and A. Murillo. A_∞ structures and Massey products. *Mediterranean Journal of Mathematics*, 17(1):31, 2020.

[12] G. Carlsson, T. Ishkhanov, V. de Silva, and A. Zomorodian. On the local behavior of spaces of natural images. *Intl. J. Computer Vision*, 76(1):1–12, 2008.

[13] G. Carlsson and A. Zomorodian. Computing persistent homology. *Discrete Comput. Geom.*, 33(2):249–274, 2005.

[14] F. Chazal, D. Cohen-Steiner, M. Glisse, L. J. Guibas, and S. Y. Oudot. Proximity of persistence modules and their diagrams. In *Proceedings of the 25th annual symposium on Computational geometry*, SCG '09, pages 237–246, New York, NY, USA, 2009. ACM.

[15] F. Chazal and A. Lieutier. Weak feature size and persistent homology: computing homology of solids in $\mathbb{R}^n$ from noisy point samples. In *Proc. 21st Ann. Sympos. Comput. Geom.*, pages 255–262, 2005.

[16] Frédéric Chazal, Vin De Silva, Marc Glisse, and Steve Oudot. *The structure and stability of persistence modules*. Springer, 2016.

[17] D. Cohen-Steiner, H. Edelsbrunner, and J. Harer. Stability of persistence diagrams. *Discrete Comput. Geom.*, 37:103–120, 2007.

[18] W. Crawley-Boevey. Decomposition of pointwise finite-dimensional persistence modules. *Journal of Algebra and Its Applications*, 14(05):1550066, 2015.

[19] M. D'Amico, P. Frosini, and C. Landi. Optimal matching between reduced size functions. Technical Report 35, DISMI, Università di Modena e Reggio Emilia, 2003.

[20] V. de Silva and G. Carlsson. Topological estimation using witness complexes. In *SPBG04 Symposium on Point-Based Graphics*, pages 157–166, 2004.

[21] V. de Silva and R. Ghrist. Coverage in sensor networks via persistent homology. *Alg. & Geom. Top.*, 7:339–358, 2007.

[22] V. de Silva, Elizabeth M., and A. Stefanou. Theory of interleavings on categories with a flow. *Theory and Applications of Categories*, 33(21):583–607, 2018.

[23] T. K. Dey and S. Goswami. Provable surface reconstruction from noisy samples. In *Proc. 20th Ann. Sympos. Comput. Geom.*, pages 330–339, 2004.

[24] H. Edelsbrunner, D. Letscher, and A. Zomorodian. Topological persistence and simplification. *Discrete Comput. Geom.*, 28:511–533, 2002.

[25] B. T. Fasy, F. Lecci, A. Rinaldo, L. Wasserman, S. Balakrishnan, and A. Singh. Confidence sets for persistence diagrams. *Ann. Statist.*, 42(6):2301–2339, 2014.

[26] P. Frosini and M. Mulazzani. Size homotopy groups for computation of natural size distances. *Bulletin of the Belgian Mathematical Society-Simon Stevin*, 6(3):455–464, 1999.

[27] M. Gameiro, Y. Hiraoka, S. Izumi, M. Kramar, K. Mischaikow, and V. Nanda. A topological measurement of protein compressibility. *Japan Journal of Industrial and Applied Mathematics*, 32(1):1–17, 2015.

[28] G. Ginot and J. Leray. Multiplicative persistent distances. arXiv:1905.12307, 2019.

[29] E. Herscovich. A higher homotopic extension of persistent (co)homology. *Journal of Homotopy and Related Structures*, 13(3):599–633, Sep 2018.

[30] T. Kadeishvili. On the homology theory of fibrations. *Russian Mathematical Surveys*, 35(3):231–238, 1980. Previously published in Russian in *Uspekhi Mat. Nauk* 35(3):183–188, 1980.

[31] M. Kerber, D. Morozov, and A. Nigmetov. Geometry helps to compare persistence diagrams. In *2016 Proceedings of the Eighteenth Workshop on Algorithm Engineering and Experiments (ALENEX)*, pages 103–112. Society for Industrial and Applied Mathematics, 06 2016.

[32] M. Lesnick. The theory of the interleaving distance on multidimensional persistence modules. *Foundations of Computational Mathematics*, 15(3):613–650, 2015.

[33] J.-L. Loday and B. Vallette. *Algebraic operads*. Number 346 in Grundlehren der mathematischen Wissenschaften. Springer, 2012.

[34] W. S. Massey. Higher order linking numbers. *Journal of Knot Theory and Its Ramifications*, 7:393–414, 1998. Originally published in Conference on Algebraic Topology, ed. Victor Gugenheim; a collection of papers presented at the University of Illinois at Chicago Circle, June 17–June 28, 1968.

[35] H. Molina-Abril and P. Real. Cell AT-models for digital volumes. In *Graph-Based Representations in Pattern Recogn. (7th IAPR-TC-15 International Workshop, GbRPR 2009)*, volume 5534 of *Lecture Notes in Computer Science*, pages 314–323. Springer, 2009.

[36] P. Niyogi, S. Smale, and S. Weinberger. Finding the homology of submanifolds with high confidence from random samples. *Discrete Comput. Geom.*, 39:419–441, 2008.

[37] J. A. Perea. Persistent homology of toroidal sliding window embeddings. In *2016 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pages 6435–6439, March 2016.

[38] M. Pirashvili, L. Steinberg, F. Belchi Guillamon, M. Niranjan, J. G. Frey, and J. Brodzki. Improved understanding of aqueous solubility modeling through topological data analysis. *Journal of Cheminformatics*, 10(1):54, Nov 2018.

[39] V. Robins. Towards computing homology from finite approximations. In *Proceedings of the 14th Summer Conference on General Topology and its Applications (Brookville, NY, 1999)*, volume 24, pages 503–532 (2001), 1999.

[40] D. Tanré. *Homotopie rationnelle: modèles de Chen, Quillen, Sullivan*, volume 1025 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983.