On Ellipse Intersections by Means of Distance Geometry

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Abstract. The problem of intersecting two ellipses arises as a frequent subproblem in computational kinematics and geometry. In this paper, an efficient solution method to this problem is presented using the concept of the power of a point with respect to an ellipse. The point-ellipse power appears in Distance Geometry as a generalization to the squared distance between two points. For establishing the intersection method, several algebraic forms of ellipses are reviewed and the interoperability of distinct definitions for the power of points and ellipses are outlined.

Keywords: distance geometry, ellipse constellations, squared distances, computational kinematics, power of a point with respect to an ellipse

1 Introduction

The problem to determine intersections between two ellipses can be identified as a basic geometric problem with relevant applications in the areas of computational kinematics and geometry. The ellipse intersection problem appears in robotics, for example, in the design analysis of regional 3R manipulators [22,21,19,25,20]. In this paper, we focus on the geometric problem itself and report, following the ideas of Wildberger [27] and Oomes [16], an efficient computation method based on the squared-distance representation of an ellipse with respect to its diagonals. The method relies on the concept of the power of a point and an ellipse and permits to distinguish several subcases of geometric constellations [1]. The power concept generalizes the Steiner power of a point with respect to a circle [3,26]. The paper works out that the point-ellipse power, and hence the intersection method, can be adopted to different algebraic forms. A particular application for a certain subcase of the present analysis to the kinematics of regional 3R manipulators is presented in a second, accompanying paper.
The paper has the following structure: Section 2 reviews the diagonal form of ellipses in light of other established forms. Section 3 compares distinct definitions for the point-ellipse power in terms of these forms. As a result of the comparison, Section 4 presents the intersection method. The paper is concluded with Section 5 and features an appendix which contains essential conversion formulae.

2 Ellipse Representations

This section outlines different algebraic representations for ellipses, organized in four paragraphs. Three squared-distance concepts are introduced in advance.

Quadrance of Directions and Points. In accordance with [27,3], the quadrance of a vector V, given in coordinates $\boldsymbol{v}$, is denoted by

$$Q_{\mathsf{V}} = \mathrm{Q}(\boldsymbol{v}) := \boldsymbol{v} * \boldsymbol{v} = \|\boldsymbol{v}\|^2 \;, \tag{1}$$

and is computed as the sum $\boldsymbol{v}*\boldsymbol{v} = \sum_i v_i \cdot v_i$. The quadrance between two points P_1 and P_2 is determined by the sum

$$Q_{12} = \mathrm{Q}(\boldsymbol{p}_1, \boldsymbol{p}_2) := (\boldsymbol{p}_2 - \boldsymbol{p}_1) * (\boldsymbol{p}_2 - \boldsymbol{p}_1) \;. \tag{2}$$

Quadrance of Points and Lines. For three dimensions $\boldsymbol{m} \in \mathbb{R}^3$, the axial square matrix $\boldsymbol{m}^{\circledcirc}$ is introduced and adopted for two dimensions $\boldsymbol{n} \in \mathbb{R}^2$ by

$$\boldsymbol{m}^{\circledcirc} := \begin{pmatrix} m_{\mathsf{y}}^2 + m_{\mathsf{z}}^2 & -m_{\mathsf{x}} m_{\mathsf{y}} & -m_{\mathsf{x}} m_{\mathsf{z}} \\ -m_{\mathsf{x}} m_{\mathsf{y}} & m_{\mathsf{x}}^2 + m_{\mathsf{z}}^2 & -m_{\mathsf{y}} m_{\mathsf{z}} \\ -m_{\mathsf{x}} m_{\mathsf{z}} & -m_{\mathsf{y}} m_{\mathsf{z}} & m_{\mathsf{x}}^2 + m_{\mathsf{y}}^2 \end{pmatrix} \qquad \boldsymbol{n}^{\circledcirc} := \begin{pmatrix} n_{\mathsf{y}}^2 & -n_{\mathsf{x}} n_{\mathsf{y}} \\ -n_{\mathsf{x}} n_{\mathsf{y}} & n_{\mathsf{x}}^2 \end{pmatrix} \;.$$

Thus, the quadrance of a point P and a line g is defined as

$$Q_{\mathsf{PG}} = \mathrm{Q}(\boldsymbol{p}, g) = \mathrm{Q}(\boldsymbol{p}, \boldsymbol{c}, \hat{\boldsymbol{n}}) := \mathrm{Q}(\hat{\boldsymbol{n}}^{\circledcirc} \cdot (\boldsymbol{p} - \boldsymbol{c})) \;, \tag{3}$$

where $\hat{\boldsymbol{n}} \in \mathbb{S}^1 \subset \mathbb{R}^2$ is the unit direction and $\boldsymbol{c} \in \mathbb{R}^2$ any point of the line g.

Classic Forms. The classic Apollonian definition [11] of an ellipse $\mathcal{E} \subset \mathbb{R}^2$ relies on the focus, a point $\boldsymbol{f} \in \mathbb{R}^2$, the directrix, a line $g \subset \mathbb{R}^2$, and the eccentricity, a scalar $\varepsilon \in \mathbb{R}$, such that it can be summarized $\mathcal{E} \cong (\, \boldsymbol{f}, g, \varepsilon\,)$. The eccentricity ε is characterized [17] by the equations $\varepsilon = \frac{f}{a} = \sqrt{1 - \frac{b^2}{a^2}} < 1$ with focal-distance $f = \sqrt{\mathrm{Q}(\boldsymbol{c}, \boldsymbol{f})}$ and directrix-distance $h = \frac{a}{\varepsilon} = \sqrt{\mathrm{Q}(\boldsymbol{c}, g)}$ for the center $\boldsymbol{c}$, and with $a^2 = \mathrm{Q}(\boldsymbol{a})$ and $b^2 = \mathrm{Q}(\boldsymbol{b})$ for the major and the minor axes vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ (next paragraph). With these quantities, a quadrance-based definition of $\mathcal{E}$ is obtained [27,11] with

$$\mathcal{E} = \left\{ \boldsymbol{p} \in \mathbb{R}^2 \;:\; \frac{\mathrm{Q}(\boldsymbol{p}, \boldsymbol{f})}{\mathrm{Q}(\boldsymbol{p}, g)} = \varepsilon^2 \right\} \tag{4}$$

using the quadrances of (2) and (3). The equations $h \cdot f = a^2$ and $p \cdot a = b^2$ hold, with the parameter $p = a \cdot (1 - \varepsilon^2)$. Another classic definition[3] for an ellipse is achieved via the 'gardener's construction' [17,11] in the distance-based formula

$$\mathcal{E} = \left\{ \, \boldsymbol{p} \in \mathbb{R}^2 \;:\; \|\boldsymbol{p} - \boldsymbol{f}_+\| + \|\boldsymbol{p} - \boldsymbol{f}_-\| = 2 \cdot a \, \right\} . \tag{5}$$

The number $a \geq f$ is the major radius and it holds that $f = \frac{1}{2} \cdot \|\boldsymbol{f}_+ - \boldsymbol{f}_-\| = \|\boldsymbol{c} - \boldsymbol{f}\| = \sqrt{a^2 - b^2}$. An example ellipse is indicated in Figure 1 including its focal points $\boldsymbol{f}_+$ and $\boldsymbol{f}_-$ and its corresponding directrices.

[3] *"An ellipse is defined as the locus of a point P, the sum of whose distances from two given points, F_+ and F_-, is constant."* [17]

CAB Forms. Another specification of an ellipse $\mathcal{E}$ relies on a vector triplet of a center vector $\boldsymbol{c} \in \mathbb{R}^2$, a major axis vector $\boldsymbol{a} \in \mathbb{R}^2$, and a minor[4] axis vector $\boldsymbol{b} \in \mathbb{R}^2$, compactly denoted by $\mathcal{E} \cong (\, \boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b}\,)$. By means of the 'affine version' of Euler's identity $\boldsymbol{z}(\phi) = \boldsymbol{c} + \cos(\phi) \cdot \boldsymbol{a} + \sin(\phi) \cdot \boldsymbol{b}$ in terms of an angle $\phi \in \mathbb{R}$, the vectors $\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b}$, and the Weierstraß substitution, $\cos(\phi) = \frac{1-\lambda^2}{1+\lambda^2}$ and $\sin(\phi) = \frac{2\lambda}{1+\lambda^2}$, the ellipse is parametrized rationally in $\lambda \in \mathbb{R}$ as follows:

$$\mathcal{E} = \left\{ \boldsymbol{p} \in \mathbb{R}^2 \ : \ \boldsymbol{p} = \boldsymbol{z}(\lambda) = \boldsymbol{c} + \frac{1-\lambda^2}{1+\lambda^2} \cdot \boldsymbol{a} + \frac{2\lambda}{1+\lambda^2} \cdot \boldsymbol{b} \right\} . \tag{6}$$

An implicit form for $\mathcal{E}$ is given via $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b})$ by the affine-bilinear form [8]

$$\mathcal{E} = \left\{ \boldsymbol{p} \in \mathbb{R}^2 \ : \ (\boldsymbol{p} - \boldsymbol{c})^\top \cdot \boldsymbol{K} \cdot (\boldsymbol{p} - \boldsymbol{c}) = 1 \right\} \tag{7}$$

and the symmetric matrix $\boldsymbol{K} = \frac{\boldsymbol{a} \cdot \boldsymbol{a}^\top}{\mathrm{Q}^2(\boldsymbol{a})} + \frac{\boldsymbol{b} \cdot \boldsymbol{b}^\top}{\mathrm{Q}^2(\boldsymbol{b})} \in \mathbb{R}^{2\times 2}$. The example in Figure 1 includes the center $\boldsymbol{c}$ and the axis vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Homogeneous Form. For a homogeneous symmetric matrix $\boldsymbol{H} \in \mathbb{R}^{3\times 3}$ with components and determinants [14] denoted as

$$\boldsymbol{H} := \begin{pmatrix} a_{\mathsf{xx}} & a_{\mathsf{xy}} & b_{\mathsf{xo}} \\ a_{\mathsf{xy}} & a_{\mathsf{yy}} & b_{\mathsf{yo}} \\ b_{\mathsf{xo}} & b_{\mathsf{yo}} & c_{\mathsf{oo}} \end{pmatrix} \qquad \varDelta := \det(\boldsymbol{H}) \qquad \delta := -\det \begin{pmatrix} a_{\mathsf{xx}} & a_{\mathsf{xy}} \\ a_{\mathsf{xy}} & a_{\mathsf{yy}} \end{pmatrix} , \tag{8}$$

an ellipse $\mathcal{E}$ is given, for $\delta < 0$, as all points $\boldsymbol{p} = (x, y)^\top \in \mathbb{R}^2$ that attain 'zero potential' with respect to $F(x, y) = 0$, defined as the biquadratic polynomial

$$\mathcal{E} = \left\{ \boldsymbol{p} \in \mathbb{R}^2 \ : \ F(x, y) := (x, y, 1) \cdot \boldsymbol{H} \cdot (x, y, 1)^\top = 0 \right\} . \tag{9}$$

The equation expands to the component form[5]

$$F(x, y) = a_{\mathsf{xx}} \cdot x^2 + a_{\mathsf{yy}} \cdot y^2 + 2 \cdot a_{\mathsf{xy}} \cdot xy + 2 \cdot b_{\mathsf{xo}} \cdot x + 2 \cdot b_{\mathsf{yo}} \cdot y + c_{\mathsf{oo}} . \tag{10}$$

The ellipse is invariant to scalar multiplications as $\hat{F}(x, y) = F(x, y)/\varDelta$. An example is sketched in Figure 2.

Diagonal Form. Yet another definition for $\mathcal{E}$ is given via its diagonals. The diagonal directions $\boldsymbol{d}_+$ and $\boldsymbol{d}_-$ and their lines d_+ and d_- are defined by

$$\begin{aligned} \boldsymbol{d}_+ &:= (\boldsymbol{a} + \boldsymbol{b})/\|\boldsymbol{a} + \boldsymbol{b}\| \qquad & \boldsymbol{d}_- &:= (\boldsymbol{b} - \boldsymbol{a})/\|\boldsymbol{b} - \boldsymbol{a}\| \\ d_+ &:= \boldsymbol{d}_+ + \epsilon \cdot (\boldsymbol{c} \times \boldsymbol{d}_+) \qquad & d_- &:= \boldsymbol{d}_- + \epsilon \cdot (\boldsymbol{c} \times \boldsymbol{d}_-) . \end{aligned} \tag{11}$$

It turns out [27,16] that the points of an ellipse can be characterized by the constant value of the sum of quadrances to d_+ and d_-, the diagonal lines. As an arbitrary point of $\mathcal{E}$, the point $\boldsymbol{e} := \boldsymbol{c} + \boldsymbol{a}$ is introduced for convenience. With $\boldsymbol{e}$, and the definitions $A := \mathrm{Q}(\boldsymbol{a}) = a^2$ and $B := \mathrm{Q}(\boldsymbol{b}) = b^2$, it can be shown that

[4] The infinite lines of the major and minor axis vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ with $\|\boldsymbol{a}\| = a$ and $\|\boldsymbol{b}\| = b$ are also called *transverse axis* and *conjugate axis* [17].

[5] The relation of the CAB matrix form in (7) and the homogeneous matrix form in (9) is addressed in Appendix A.

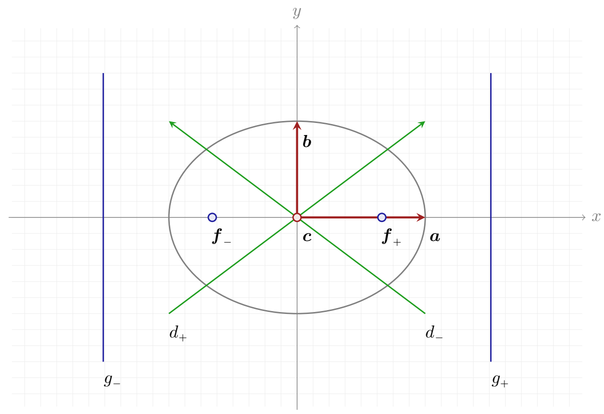

Fig. 1. Illustration of distinct representations of an ellipse. Its focal points $\boldsymbol{f}_+$, $\boldsymbol{f}_-$ and directrices g_+, g_- are drawn in blue. Its center, major vector, and minor vector, $\boldsymbol{c}$, $\boldsymbol{a}$, and $\boldsymbol{b}$, are drawn in green. The diagonals d_+ and d_- are drawn in red. The ellipse with $a^2 = 16$ and $b^2 = 9$ has the radiance (12) with value $R_{\mathsf{E}} = \mathrm{Harm}(16, 9) = 11.52$.

the *'ellipse radiance'* R_{E} is characterized[6] by the harmonic mean

$$R_{\mathsf{E}} := \mathrm{Q}(\boldsymbol{e}, d_+) + \mathrm{Q}(\boldsymbol{e}, d_-) = 2 \cdot \frac{A \cdot B}{A + B} = \mathrm{Harm}(A, B)\ . \tag{12}$$

Based on this invariant, the ellipse is characterized by the formula

$$\mathcal{E} = \left\{ \boldsymbol{p} \in \mathbb{R}^2 \ :\ \mathrm{Q}(\boldsymbol{z}, d_+) + \mathrm{Q}(\boldsymbol{z}, d_-) = R_{\mathsf{E}} \right\}\ . \tag{13}$$

With the point-line quadrance in (3), the constraint in (13) is reformed to

$$\|(\boldsymbol{a} + \boldsymbol{b})^{\circledcirc} \cdot (\boldsymbol{x} - \boldsymbol{c})\|^2 + \|(\boldsymbol{b} - \boldsymbol{a})^{\circledcirc} \cdot (\boldsymbol{x} - \boldsymbol{c})\|^2 = 2 \cdot (A \cdot B) \cdot (A + B)$$

The sketch in Figure 1 includes the diagonals d_+ and d_-.

For the particular case of an ellipse centered to the origin, with $\boldsymbol{c} = \boldsymbol{0}$, and aligned with the coordinate axes $\boldsymbol{a} \parallel \hat{\boldsymbol{e}}_{\mathsf{x}}$ and $\boldsymbol{b} \parallel \hat{\boldsymbol{e}}_{\mathsf{y}}$, the equation simplifies to

$$\|(\boldsymbol{a} + \boldsymbol{b})^{\circledcirc} \cdot \boldsymbol{x}\|^2 + \|(\boldsymbol{b} - \boldsymbol{a})^{\circledcirc} \cdot \boldsymbol{x}\|^2 = 2 \cdot A \cdot B \cdot (A + B)$$

and, with $\boldsymbol{a} = (a, 0)^\top$ and $\boldsymbol{b} = (0, b)^\top$, and

$$(\boldsymbol{a} + \boldsymbol{b})^{\circledcirc} = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)^{\circledcirc} = \begin{pmatrix} B & -ab \\ -ab & A \end{pmatrix} \qquad (\boldsymbol{b} - \boldsymbol{a})^{\circledcirc} = \left(\begin{smallmatrix} -a \\ b \end{smallmatrix}\right)^{\circledcirc} = \begin{pmatrix} B & +ab \\ +ab & A \end{pmatrix}\ ,$$

further to

$$\begin{aligned}(B \cdot x - ab \cdot y)^2 + (-ab \cdot x + A \cdot y)^2 + (B \cdot x + ab \cdot y)^2 + (ab \cdot x + A \cdot y)^2 \\ = 2B^2 \cdot x^2 + 2(ab \cdot x)^2 + 2(ab \cdot y)^2 + 2A^2 \cdot y^2 = 2 \cdot A \cdot B \cdot (A + B)\ .\end{aligned}$$

The equation can be simplified to $x^2 \cdot \dfrac{1}{A} + y^2 \cdot \dfrac{1}{B} = 1$, the standard ellipse equation.

[6] In terms of f and h, the invariant R_{E} reads $R_{\mathsf{E}} = (2 \cdot (h^2 f - h f^2)/(2h - f)$.

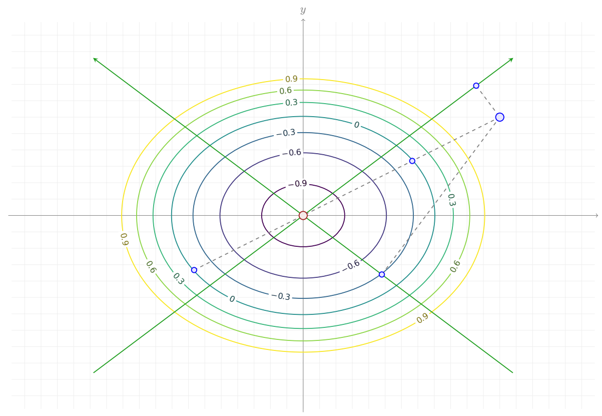

Fig. 2. The ellipse $\mathcal{E}$ of Figure 1 matches the zero element of the level sets $F(x, y) = \frac{1}{16} \cdot x^2 + \frac{1}{9} \cdot y^2 - 1$. The polynomial $F(x, y)$, as given in (9) and (10), matches with the Maley power (16) of the points with respect to $\mathcal{E}$. The center point $\boldsymbol{c}$ has the normalized power $F(c_\mathsf{x}, c_\mathsf{y}) = \hat{\mathrm{W}}(\boldsymbol{c}; \mathcal{E}) = -1$ and the Steiner power of $\boldsymbol{c}$ and $\mathcal{E}$ equals $W(\boldsymbol{c}; \mathcal{E}) = R_\mathsf{E} \cdot \hat{\mathrm{W}}(\boldsymbol{c}; \mathcal{E}) = -11.52$. The point P with coordinates $\boldsymbol{p} = (6, 3)^\top$ has the normalized power $\hat{\mathrm{W}}(\boldsymbol{p}; \mathcal{E}) = (1.2^2 + 6^2 - 11.52)/11.52 = (6^2 + 3^2 - 180/13)/(180/13) = 2.25$.

3 Power Concepts

The concept of the power of a point with respect to a circle was introduced by Steiner [23] in 1826. The idea was extended by Laguerre [12] to the power of a point with respect to an algebraic curve of degree n in 1865.[7] Distinct definitions for the power of a point with respect to an ellipse have been introduced later [7,15,14,6]. This section reflects some of these definitions. A more detailed treatment can be found in [24].

We start with the quadrance of a point P and an ellipse $\mathcal{E}$ defined by

$$Q_\mathsf{PE} = \mathrm{Q}(\boldsymbol{p}, \mathcal{E}) := \mathrm{Q}(\boldsymbol{p}, d_+) + \mathrm{Q}(\boldsymbol{p}, d_-) \ , \tag{14}$$

evaluating the quadrance (3) sum of diagonals of (12) to the point P. By means of Q_PE and the radiance R_E of (12), the *Steiner power* W_PE of a point P and an ellipse $\mathcal{E}$ is given as the difference

$$W_\mathsf{PE} = \mathrm{W}(\boldsymbol{p}, \mathcal{E}) := P * \mathcal{E} = Q_\mathsf{PE} - R_\mathsf{E} \ . \tag{15}$$

In addition, the normalized version of the Steiner power is defined by

$$\hat{W}_\mathsf{PE} = \hat{\mathrm{W}}(\boldsymbol{p}, \mathcal{E}) := \frac{\mathrm{W}(\boldsymbol{p}, \mathcal{E})}{\mathrm{W}(\boldsymbol{c}, \mathcal{E})} = \frac{Q_\mathsf{PE} - R_\mathsf{E}}{R_\mathsf{E}} \ ,$$

adjusted such that the ellipse center $\boldsymbol{c}$ attains a power of $\hat{\mathrm{W}}(\boldsymbol{c}, \mathcal{E}) = -1$. The example in Figure 2 includes the evaluations for the center and another point.

[7] As the product of distances from the point to the intersections of a circle through the point with the curve, divided by the nth power of the circle's diameter.

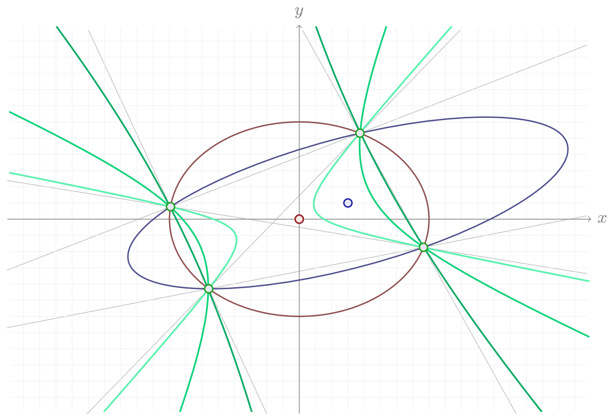

Fig. 3. Ellipse pair with four intersection points, radical conics, and degenerate conic.

Maley [14] has introduced the normalized power $\hat{\mathrm{M}}$ as the fraction

$$\hat{M}_{\mathsf{PE}} = \hat{\mathrm{M}}(\boldsymbol{p}, \mathcal{E}) := -\frac{F(P)}{F(C)} \tag{16}$$

of potentials F in (10). The center[8] attains a potential $F(C) = -\Delta/\delta$ in terms of the determinants Δ and δ in (8). The power of a point P with coordinates $\boldsymbol{p} = (x, y)^\top$ has been defined by de Biasi [6] with respect to focus $\boldsymbol{f}$, directrix g, and eccentricity ε as the function

$$\Gamma_{\mathsf{PE}} = \Gamma(x, y) := \mathrm{Q}(\boldsymbol{p}, \boldsymbol{f}) - \varepsilon^2 \cdot \mathrm{Q}(\boldsymbol{p}, g) \ . \tag{17}$$

The focal-quadrance and the directrix-quadrance in the definition of (4) appear for Γ in a difference. One can prove that all concepts correspond, expressed by

$$\hat{M}_{\mathsf{PE}} = \hat{W}_{\mathsf{PE}} = \frac{W_{\mathsf{PE}}}{R_{\mathsf{E}}} = \frac{Q_{\mathsf{PE}} - R_{\mathsf{E}}}{R_{\mathsf{E}}} = -\frac{F(P)}{F(C)} = \frac{\Gamma_{\mathsf{PE}}}{b^2} \ . \tag{18}$$

This observation permits in particular to compute the power of a point with respect to an ellipse efficiently without 'conversions' (Appendix A) independent of the form (Apollonian, homogeneous, CAB) an ellipse is given in. For the limit case of a circle $\mathcal{E} \to \mathcal{C}$, it holds that $\lim_{a\to b}(W_{\mathsf{PE}}) = \lim_{a\to b}(\Gamma_{\mathsf{PE}}) = W_{\mathsf{PC}} = Q_{\mathsf{PC}} - R_{\mathsf{C}}$.

4 Two Ellipses

As addressed in the introduction, the thorough analysis of the mutual geometry of two ellipses is crucial for solving constraint equations in relevant problems of computational kinematics. The geometry of ellipses pairs has been analyzed and classified, for example in [9,1]. In this context, the classification and computation

[8] In [14], it is further proven that $\hat{\mathrm{M}}_{\mathsf{PE}} = (d_{\mathsf{PE}}^2 - r_{\mathsf{PE}}^2)/r_{\mathsf{PE}}^2$ holds for distances $d_{\mathsf{PE}} := |PC|$ and $r_{\mathsf{PE}} := |CY_{\mathsf{l}}| = |CY_{\mathsf{r}}|$ with secant PC intersection points Y_{l} and Y_{r}.

Algorithm 1 Compute ellipse intersections (`Intersect`)

In: Two ellipses $\mathcal{E}_1$ and $\mathcal{E}_2$, by tuples $(\boldsymbol{c}_1, \boldsymbol{a}_1, \boldsymbol{b}_1)$ and $(\boldsymbol{c}_2, \boldsymbol{a}_2, \boldsymbol{b}_2)$

Out: Four intersection points $(\boldsymbol{s}_{++}, \boldsymbol{s}_{+-}, \boldsymbol{s}_{-+}, \boldsymbol{s}_{--})$

1: $\boldsymbol{e}_2 \leftarrow \boldsymbol{c}_2 + \boldsymbol{a}_2$ ▷ point on $\mathcal{E}_2$ (Sec. 2)
2: $\boldsymbol{d}_{2+} \leftarrow (\boldsymbol{a}_2 + \boldsymbol{b}_2)/\|\boldsymbol{a}_2 + \boldsymbol{b}_2\|$, $\boldsymbol{d}_{2\text{-}} \leftarrow (\boldsymbol{b}_2 - \boldsymbol{a}_2)/\|\boldsymbol{b}_2 - \boldsymbol{a}_2\|$ ▷ diagonals (11)
3: $R_2 \leftarrow \mathrm{Q}(\boldsymbol{e}_2, \boldsymbol{c}_2, \boldsymbol{d}_{2+}) + \mathrm{Q}(\boldsymbol{e}_2, \boldsymbol{c}_2, \boldsymbol{d}_{2\text{-}})$ ▷ ellipse radiance (12)
4: $\boldsymbol{z}_1(\lambda) \leftarrow \boldsymbol{c}_1 + \frac{1-\lambda^2}{1+\lambda^2} \cdot \boldsymbol{a}_1 + \frac{2\lambda}{1+\lambda^2} \cdot \boldsymbol{b}_1$ ▷ rational parameter form (6)
5: $Q_{12}(\lambda) \leftarrow \mathrm{Q}(\boldsymbol{z}_1(\lambda), \boldsymbol{c}_2, \boldsymbol{d}_{2+}) + \mathrm{Q}(\boldsymbol{z}_1(\lambda), \boldsymbol{c}_2, \boldsymbol{d}_{2\text{-}})$ ▷ point-line quadrances (14)
6: $\tilde{W}_{12}(\lambda) \leftarrow (1 + \lambda^2)^2 \cdot (Q_{12}(\lambda) - R_2)$ ▷ quartic Steiner power (15)
7: $\lambda_{++}, \lambda_{+-}, \lambda_{-+}, \lambda_{--} \leftarrow \texttt{Quartic}\big(\tilde{W}_{12}(\lambda) = 0\big)$ ▷ quartic's solutions, Alg. 2
8: $\boldsymbol{s}_{++}, \boldsymbol{s}_{+-}, \boldsymbol{s}_{-+}, \boldsymbol{s}_{--} \leftarrow \boldsymbol{z}_1(\lambda_{++}), \boldsymbol{z}_1(\lambda_{+-}), \boldsymbol{z}_1(\lambda_{-+}), \boldsymbol{z}_1(\lambda_{--})$ ▷ $\boldsymbol{z}_1(\lambda)$ of line 4
9: **return** $\boldsymbol{s}_{++}$, $\boldsymbol{s}_{+-}$, $\boldsymbol{s}_{-+}$, $\boldsymbol{s}_{--}$

of intersections of an ellipse pair is of particular interest. For considering ellipse intersections from a geometric point of view, another concept is introduced based on the power concepts of the previous section: in accordance with de Biasi [6], we define the *power difference* Σ of two ellipses $\mathcal{E}_1$ and $\mathcal{E}_2$ as

$$\Sigma_i(\boldsymbol{p}) = \Sigma_i(\boldsymbol{p}; \mathcal{E}_1, \mathcal{E}_2) := \mathrm{W}_i(\boldsymbol{p}, \mathcal{E}_2) - \mathrm{W}_i(\boldsymbol{p}, \mathcal{E}_1) \ , \tag{19}$$

for $\mathrm{W}_i \in \{ \hat{\mathrm{M}}, \mathrm{W}, \Gamma \}$. A *radical conic* [6] is the zero set of Σ_i with

$$\mathcal{R}_i = \mathcal{R}_i(\mathcal{E}_1, \mathcal{E}_2) := \{ \boldsymbol{p} \in \mathbb{R}^2 \ : \ \mathrm{W}_i(\boldsymbol{p}, \mathcal{E}_2) - \mathrm{W}_i(\boldsymbol{p}, \mathcal{E}_1) = 0 \} \ . \tag{20}$$

A radical conic $\mathcal{R}_i$ is an element of the family of linear combinations

$$\mathcal{L}_{12} := \{ \{ \boldsymbol{p} \in \mathbb{R}^2 \ : \ F_{\alpha\beta}(x, y) = \alpha \cdot F_1(x, y) + \beta \cdot F_1(x, y) = 0 \} \ : \ \alpha, \beta \in \mathbb{R} \} \ .$$

Every conic in $\mathcal{L}_{12}$ contains the intersection points of $\mathcal{E}_1$ and $\mathcal{E}_2$. The geometry of an ellipse pair together with its degenerate and radical concics, as particular members of $\mathcal{L}_{12}$, and its intersection points is illustrated in Figure 3.

With these preparations, an efficient method to determine intersection points of an ellipse pair is outlined in Algorithm 1. A reference method for ellipse intersections is described in detail in [8]. The 'homogeneous approach' via the degenerate elements of the pencil of ellipses is outlined in [21, Sec. 5.2.1] in context with regional 3R manipulators and described more generally in [18, Sec. 11.4] for a pair of conics. A similar approach to the presented is given by method of Chrystal [4]: There, a quartic solution polynomial is obtained from matrix minors entirely, as recently reported in [25]. More information on the connection of quartic equations and ellipse geometry can be found in [10].

The input ellipses to `Intersect` in Algorithm 1 are given as two CAB triples $(\boldsymbol{c}_1, \boldsymbol{a}_1, \boldsymbol{b}_1)$ and $(\boldsymbol{c}_2, \boldsymbol{a}_2, \boldsymbol{b}_2)$. By means of the identities in (18) the algorithm can be

Algorithm 2 Solve quartic equation (`Quartic`)

In: Quartic $F(x) = Ax^4 + Bx^3 + Cx^2 + Dx + E$ by tuple (A, B, C, D, E)

Out: Complex roots of quartic equation with up to s real roots

if $A = B = C = D = E = 0$ **then** ▷ *coincident*:
  return $\{ \}$ ▷ $s = \infty$
else if $A = 0$ **then** ▷ *tangent*:
  if $B = 0$ **then** $B \leftarrow \epsilon$ ▷ $s \in \{1, 2, 3\}$
  $x_1, x_2, x_3 \leftarrow$ `cube`(B, C, D, E)
  return $\{ x_1, x_2, x_3 \}$
else if $B = D = 0$ **then** ▷ *aligned isoshape*: disjoint,
  $z_+, z_- \leftarrow$ `quad`(A, C, E) ▷ tangent, overlap, $\pi/2$-turn
  return $\{ \sqrt{z_+}, -\sqrt{z_+}, +\sqrt{z_-}, -\sqrt{z_-} \}$ ▷ $s \in \{0, 1, 2, 3, 4\}$
else if $A = E$ **and** $B = D$ **then** ▷ *shifted isoshape*: disjoint,
  $y_+, y_- \leftarrow$ `quad`$(A, B, C - 2A)$ ▷ tangent, overlap
  $x_{++}, x_{+-} \leftarrow$ `quad`$(+1, -y_+, +1)$ ▷ $s \in \{0, 1, 2\}$
  $x_{-+}, x_{--} \leftarrow$ `quad`$(+1, -y_-, +1)$
  return $\{ x_{++}, x_{+-}, x_{-+}, x_{--} \}$
else if $A = E$ **and** $|B| = |D|$ **then** ▷ *rotated isoshape*:
  $y_+, y_- \leftarrow$ `quad`$(A, B, C + 2A)$ ▷ skew-symm
  $x_{++}, x_{+-} \leftarrow$ `quad`$(+1, -y_+, -1)$ ▷ $s = 4$
  $x_{-+}, x_{--} \leftarrow$ `quad`$(+1, -y_-, -1)$
  return $\{ x_{++}, x_{+-}, x_{-+}, x_{--} \}$
else ▷ *generic*: disjoint, osculating,
  $x_{++}, x_{+-}, x_{-+}, x_{--} \leftarrow$ `quart`(A, B, C, D, E) ▷ overlap, proper
  return $\{ x_{++}, x_{+-}, x_{-+}, x_{--} \}$ ▷ $s \in \{0, 1, 2, 4\}$

adopted easily to a different input format of $\mathcal{E}_1$ without any conversion. Due to the chosen formulation, `Intersect` can be implemented in rational arithmetic. `Quartic` in Algorithm 2 is called as a subroutine to solve the quartic constraint equation. `Quartic` distinguishes several cases of mutual geometry [9,1] by conditions on the parameters of the quartic [13]. The methods `quad`, `cube`, and `quart` are called to compute the roots for the polynomials of respective degree.

From a geometric viewpoint, the method `Intersect` determines those points on the ellipse $\mathcal{E}_1$ that also feature zero Steiner power to $\mathcal{E}_2$: These are the intersection points of the ellipse pair and also elements of the radical conic in (20).

5 Conclusions

This paper considers the diagonal quadrance form of an ellipse in context with other established forms. Based on the distinct algebraic representations, several definitions for the power of a point and an ellipse are reviewed and their linear equivalence is outlined. By means of the point-ellipse power, an efficient method for computing the interesection points of two ellipses is formulated. The method can be implemeted in rational arithmetics and distinguishes several types of mutual ellipse geometry by a set of conditions on the algebraic coefficients.

A Conversions

CAB to Homogeneous. The entries of the homogeneous matrix $\boldsymbol{H}$ are obtained from the vectors $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b})$ via the angle $\gamma = \operatorname{atan2}(a_{\mathsf{y}}, a_{\mathsf{x}})$, the values $c_\gamma = \cos(\gamma)$ and $s_\gamma = \sin(\gamma)$, and the conversion formulae

$$
\begin{aligned}
a_{\mathsf{xx}} &= \frac{c_\gamma^2}{a^2} + \frac{s_\gamma^2}{b^2} & b_{\mathsf{xo}} &= -(c_{\mathsf{x}} \cdot a_{\mathsf{xx}} + c_{\mathsf{y}} \cdot a_{\mathsf{xy}}) \\
a_{\mathsf{yy}} &= \frac{s_\gamma^2}{a^2} + \frac{c_\gamma^2}{b^2} & b_{\mathsf{yo}} &= -(c_{\mathsf{x}} \cdot a_{\mathsf{xy}} + c_{\mathsf{y}} \cdot a_{\mathsf{yy}}) \\
a_{\mathsf{xy}} &= s_\gamma \cdot c_\gamma \cdot \left(\frac{1}{a^2} - \frac{1}{b^2}\right) & c_{\mathsf{oo}} &= \frac{(c_{\mathsf{x}} \cdot c_\gamma + c_{\mathsf{y}} \cdot s_\gamma)^2}{a^2} + \frac{(c_{\mathsf{x}} \cdot s_\gamma^2 - c_{\mathsf{y}} \cdot c_\gamma)^2}{b^2} - 1
\end{aligned}
$$

in terms of $a = \|\boldsymbol{a}\|$, $b = \|\boldsymbol{b}\|$, and $c_{\mathsf{x}}, c_{\mathsf{y}}$ as coordinates of $\boldsymbol{c} \in \mathbb{R}^2$.

Homogeneous to CAB. The three vectors $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b})$ are obtained from the homogeneous matrix $\boldsymbol{H}$ of (8) in closed formulae [17,14,2]. The center $\boldsymbol{c} = (c_{\mathsf{x}}, c_{\mathsf{y}})^\top$ is computed via the matrix minors

$$
c_{\mathsf{x}} = \frac{1}{\delta} \cdot \det \begin{pmatrix} b_{\mathsf{xo}} & a_{\mathsf{xy}} \\ b_{\mathsf{yo}} & a_{\mathsf{yy}} \end{pmatrix} \quad c_{\mathsf{y}} = \frac{1}{\delta} \cdot \det \begin{pmatrix} a_{\mathsf{xx}} & b_{\mathsf{xo}} \\ a_{\mathsf{xy}} & b_{\mathsf{yo}} \end{pmatrix} \quad \delta = -\det \begin{pmatrix} a_{\mathsf{xx}} & a_{\mathsf{xy}} \\ a_{\mathsf{xy}} & a_{\mathsf{yy}} \end{pmatrix} .
$$

The major and minor axes $\boldsymbol{a}$ and $\boldsymbol{b}$ are obtained via the ellipse attitude γ by

$$
\begin{aligned}
q &= \sqrt{(a_{\mathsf{xx}} - c_{\mathsf{oo}})^2 + (2b_{\mathsf{xo}})^2} & k &= \frac{1}{\delta} \cdot \det(\boldsymbol{H}) \\
\gamma &= \operatorname{atan2}(c_{\mathsf{oo}} - a_{\mathsf{xx}} + q, 2b_{\mathsf{xo}}) + \pi/2 & \boldsymbol{a} &= \sqrt{2k/(a_{\mathsf{xx}} + c_{\mathsf{oo}} - q)} \cdot \boldsymbol{R} \cdot \hat{\boldsymbol{e}}_{\mathsf{x}} \\
\boldsymbol{R} &= \exp(\gamma \cdot \left(\begin{smallmatrix} 0 & -1 \\ +1 & 0 \end{smallmatrix}\right)) & \boldsymbol{b} &= \sqrt{2k/(a_{\mathsf{xx}} + c_{\mathsf{oo}} + q)} \cdot \boldsymbol{R} \cdot \hat{\boldsymbol{e}}_{\mathsf{y}} .
\end{aligned}
$$

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