A Tale of Two Motion Problems

Federico Thomas[1] and Jaume Franch[1,2]

[1] Institut de Robòtica i Informàtica Industrial (CSIC-UPC)
ETSEIB, Diagonal 647, Pavelló E, planta 1, 08028 Barcelona, Spain
f.thomas@csic.es
[2] Department of Mathematics (UPC)
Pau Gargallo 14, 08028 Barcelona, Spain
jaume.franch@upc.edu

Abstract. This paper establishes a formal equivalence between the disk-on-sphere motion problem and the terrestrial brachistochrone problem by demonstrating that both are governed by the same differential equation. From a geometric perspective, the underlying problem reduces to interpolating a cusp-free arc of a hypocycloid between two points on a circle. This insight not only reveals a deep connection between seemingly unrelated problems but also offers a unifying framework for their analysis.

Keywords: Nonholonomic motion planning, disk-on-sphere motion problem, terrestrial brachistochrone problem, linear time-varying systems.

1 Introduction

The disk-on-sphere motion problem consists in driving a disk on a sphere without slipping —or, equivalently, driving a sphere in contact with a freely rotating disk, as illustrated in Fig. 1. In this setup, the disk restricts the sphere's rotation about the $z-$axis, yet any desired orientation can still be achieved by the sphere by following an appropriate path in the space of rotations. A naive approach, involving alternating rotations about the unconstrained x and y axes, yields simple but inefficient paths. A first attempt to generate coordinated rotations about the unconstrained rotation axes was presented in [1]. Recently, a complete general solution to this problem was presented in [2] where a comprehensive up-to-date overview and the practical significance of this problem is also presented.

The terrestrial brachistochrone problem is a variation of the classical brachistochrone problem, which seeks the path of fastest descent between two points under gravity. In the terrestrial version, the problem is adapted to the Earth's spherical shape and accounts for its gravitational field, which varies with radial distance from the center. Instead of assuming uniform gravity and a flat surface, the terrestrial brachistochrone seeks the curve of quickest descent through the Earth (neglecting friction and air resistance) between two surface points. The resulting optimal path is not a straight line or a simple arc, but a segment of a hipocycloid on the Earth's great circle containing the two surface points. In 1966, Edwards' article [3] ignited the interest in gravity-powered transportation and inspired the articles by Cooper [4, 5], Venezian [6], Mallett [7], Laslett [8], and Patel [9] on the terrestrial brachistochrone. Afterward, different authors showed that the terrestrial brachistochrones are also tautochrones [11–14]. After this series of papers, the interest on this problem rapidly decayed and the only later reference that probably deserves to be mentioned is due to Stalford and Garrett [15] who revisited the terrestrial brachistochrone problem using differential geometry and optimal control theory.

In this paper, we present the result that the disk-on-sphere problem can also be reduced to finding an epicycloid connecting two points on a circle. This elegant connection with the terrestrial brachistochrone problem not only simplifies the geometric interpretation of the problem but also offers fresh insights into its underlying structure and solution.

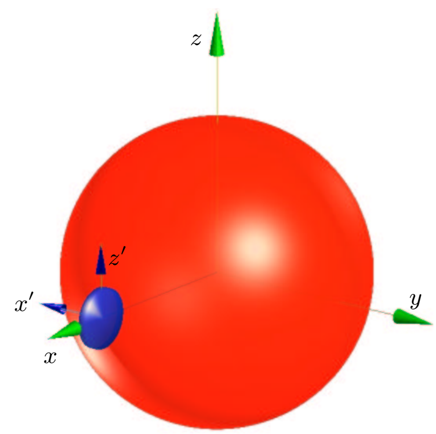

Fig. 1. A disk rolling on a sphere without slipping is kinematically equivalent to a sphere constrained to rotate about two fixed orthogonal axis (the x and the y axes in the figure). This equivalence can be established through a kinematic inversion.

This paper is organized as follows. Section 2 presents the mathematical formulation of the disk-on-sphere motion problem and shows how it can be reduced to solving a system of two nonlinear equations in two unknowns. In Section 3, this system is analyzed and shown to correspond to the same differential equation that governs the terrestrial brachistochrone problem. It is therefore concluded that both problems can be solved by interpolating a hypocycloid between two points on a circle. Section 4 provides an illustrative example that clarifies and validates the theoretical results. Finally, Section 5 summarizes the main contributions and outlines several open problems that warrant further investigation.

2 Disk-on-sphere motion problem formulation

By fixing the location of the disk, we only need to keep the reference frame located in the center of the sphere which is now considered the moving object. To avoid representation singularities, the orientation of the sphere can be described using Euler parameters. If we arrange these parameters as the elements of a quaternion, say $\mathbf{q}$, the rate of change of $\mathbf{q}$ is described by the well-known relation

$$\dot{\mathbf{q}} = \frac{1}{2}\begin{pmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & -\omega_z & \omega_y \\ \omega_y & \omega_z & 0 & -\omega_x \\ \omega_z & -\omega_y & \omega_x & 0 \end{pmatrix}\mathbf{q} = \frac{1}{2}\left(\mathbf{A}_x\omega_x + \mathbf{A}_y\omega_y + \mathbf{A}_z\omega_z\right)\mathbf{q}, \tag{1}$$

where the scalar elements above correspond to the angular velocity vector components $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$ and

$$\mathbf{A}_x = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{A}_y = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \mathbf{A}_z = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

If the disk imposes the constraint ω_z=0 (as in Fig. 1), the equation of motion of the sphere is reduced to the standard equation of a time-varying linear system $\dot{\mathbf{q}} = \mathbf{A}(t)\mathbf{q}$ with

$$\mathbf{A}(t) = \frac{1}{2}\left(\mathbf{A}_x\omega_x + \mathbf{A}_y\omega_y\right). \tag{3}$$

Observe that $\mathbf{A}_x$, $\mathbf{A}_y$, and $\mathbf{A}_z$ are antisymmetric and orthogonal. In other words, $\mathbf{A} = -\mathbf{A}^T$ and $\mathbf{A}\mathbf{A}^T = \mathbf{I}$. Thus, it is easy to verify that $\mathbf{A}_x^2 = \mathbf{A}_y^2 = \mathbf{A}_z^2 = -\mathbf{I}$ and $\mathbf{A}_x\mathbf{A}_y\mathbf{A}_z = -\mathbf{I}$. This allows us to conclude that the three matrices in (2) and the identity matrix $\mathbf{I}$ behave as the four units that define a quaternion.

Consider an arbitrary linear combination of the elements of $\{\mathbf{A}_x, \mathbf{A}_y, \mathbf{A}_z\}$. That is,

$$\mathbf{C} = \begin{pmatrix} 0 & -l_1 & -l_2 & -l_3 \\ l_1 & 0 & -l_3 & l_2 \\ l_2 & l_3 & 0 & -l_1 \\ l_3 & -l_2 & l_1 & 0 \end{pmatrix} = l_1\mathbf{A}_x + l_2\mathbf{A}_y + l_3\mathbf{A}_z. \tag{4}$$

Then, it can be verified that $\mathbf{C}^2 = -(l_1^2+l_2^2+l_3^2)\,\mathbf{I}$. Therefore, since $\sqrt{-l_1^2 - l_2^2 - l_3^2}$ is a pure imaginary magnitude, we have that

$$\mathbf{C}^2 = -\,\theta^2\,\mathbf{I}, \;\; \mathbf{C}^3 = -\,\theta^3\,\mathbf{C}, \;\; \mathbf{C}^4 = \theta^4\,\mathbf{I}, \;\; \mathbf{C}^5 = \theta^5\,\mathbf{C}, \ldots \tag{5}$$

where $\theta = \sqrt{l_1^2 + l_2^2 + l_3^3}$. Therefore, the exponential of $\mathbf{C}$ can be expressed as

$$\begin{aligned} e^{\mathbf{C}} &= \mathbf{I} + \mathbf{C} - \frac{\theta^2}{2!}\mathbf{I} - \frac{\theta^2}{3!}\mathbf{C} + \frac{\theta^4}{4!}\mathbf{I} + \frac{\theta^4}{5!}\mathbf{C} + \ldots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right)\mathbf{I} + \frac{1}{\theta}\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \ldots\right)\mathbf{C} = \cos\theta\,\mathbf{I} + \operatorname{sinc}\theta\mathbf{C}, \end{aligned} \tag{6}$$

where $\operatorname{sinc}\theta = \dfrac{\sin\theta}{\theta}$ (i.e., the standard sinc function).

In the 1970s, M.-Y. Wu showed that the solution to the linear time-varying system $\dot{\mathbf{q}} = \mathbf{A}(t)\mathbf{q}$ can be expressed as [16]

$$\mathbf{q}(t) = e^{\mathbf{A}_1 t} e^{\mathbf{A}_2 t}\mathbf{q}(0), \tag{7}$$

if and only if there exists a constant matrix $\mathbf{A}_1$ which satisfies

$$\mathbf{A}_1\mathbf{A}(t) - \mathbf{A}(t)\mathbf{A}_1 = \dot{\mathbf{A}}(t), \tag{8}$$

and $\mathbf{A}_2$ is a constant matrix obtained as follows:

$$\mathbf{A}_2 = \mathbf{A}(0) - \mathbf{A}_1. \tag{9}$$

In our case, it can be proved that $\mathbf{A}_1$ must be proportional to $\mathbf{A}_z$ (see the appendix in [2]). Then, if we take

$$\mathbf{A}_1 = \frac{\alpha}{2}\mathbf{A}_z, \tag{10}$$

it can be concluded that

$$\mathbf{A}(t) = \frac{a}{2}\left[\mathbf{A}_x \cos(\alpha t + \alpha_0) + \mathbf{A}_y \sin(\alpha t + \alpha_0)\right], \tag{11}$$

where α, α_0 and a have to be found to attain the desired sphere's orientation. and, as a consequence,

$$\mathbf{A}_2 = \frac{1}{2}\left(a \cos\alpha_0 \mathbf{A}_x + a \sin\alpha_0 \mathbf{A}_y - \alpha \mathbf{A}_z\right). \tag{12}$$

Then, $\mathbf{A}_1$ and $\mathbf{A}_2$, as given in (10) and (12), satisfy (8). Thus,

$$\mathbf{q}(t) = e^{\mathbf{A}_1 t} e^{\mathbf{A}_2 t} \mathbf{q}(0). \tag{13}$$

Without loss of generality, by properly locating the reference frame of the moving sphere, we can set $\mathbf{q}(0) = (1, 0, 0, 0)$ and $\mathbf{q}(t) = (q_1, q_2, q_3, q_4)$. Then, substituting the exponentials in (13) according to (6), expanding the products, and grouping terms, we can organize the four resulting scalar equations as follows:

$$\begin{pmatrix} q_1 \\ q_4 \end{pmatrix} = \begin{pmatrix} \cos(\theta_1 t) & -\sin(\theta_1 t) \\ \sin(\theta_1 t) & \cos(\theta_1 t) \end{pmatrix} \begin{pmatrix} \cos(\theta_2 t) \\ -\rho \sin(\theta_2 t) \end{pmatrix}, \tag{14}$$

$$\begin{pmatrix} q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \cos(\theta_1 t) & -\sin(\theta_1 t) \\ \sin(\theta_1 t) & \cos(\theta_1 t) \end{pmatrix} \begin{pmatrix} \sqrt{1-\rho^2}\, \sin(\theta_2 t)\, \cos\alpha_0 \\ \sqrt{1-\rho^2}\, \sin(\theta_2 t)\, \sin\alpha_0 \end{pmatrix}. \tag{15}$$

where

$$\theta_1 = \frac{\alpha}{2}, \quad (16) \qquad \theta_2 = \frac{1}{2}\sqrt{a^2 + \alpha^2}, \quad (17) \qquad \rho = \frac{\theta_1}{\theta_2} = \frac{\alpha}{\sqrt{a^2 + \alpha^2}}. \quad (18)$$

If we solve (14) for θ_1, then α_0 directly follows from (15) as follows:

$$\alpha_0 = \operatorname{atan2}(q_3, q_2) - \theta_1 t. \tag{19}$$

Thus, the problem can be reduced to solving system (14). This is treated in the next section. Finally, observe that, when $q_2 = q_3 = 0$, α_0 is undefined. In Section 4, we will return to this singularity.

3 Deriving a fundamental equation

Expressing (q_1, q_4) in polar form, the system of equations (14) can be rewritten as:

$$\left.\begin{aligned} r_1^2 &= \cos^2\theta + \rho^2 \sin^2\theta = (1-\rho^2)\cos^2\theta + \rho^2 \\ \eta_1 &= \rho\,\theta - \tan^{-1}(\rho \tan\theta) \end{aligned}\right\} \tag{20}$$

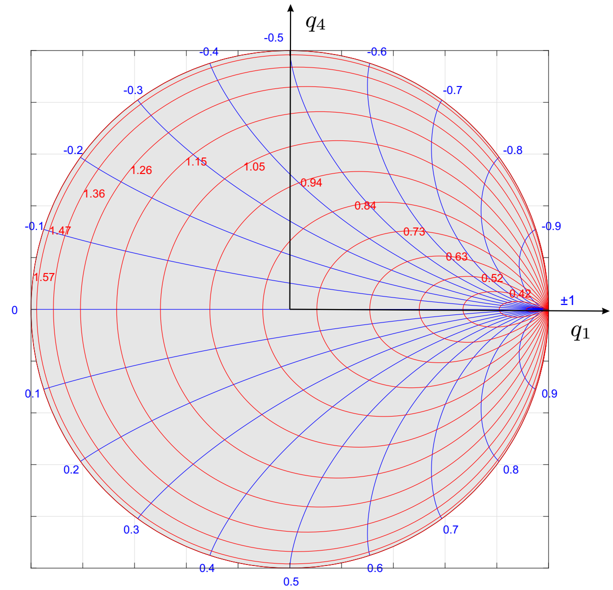

Fig. 2. Curves for constant values of $\theta = \theta_2 t$ (in red) and constant values of ρ (in blue). Each curved blue segment is the lobe of a hypocycloid.

where $\theta = \theta_2 t$. It can be verified that these are the parametric equations of a hypocycloid traced by a fixed point on the circumference of a circle of radius $R = (1-\rho)/2$ rolling with angle θ around the inside of the unit circle [17, p. 171]. The corresponding curves, taking either ρ or θ as a parameter, appear in Fig. 2. It can be verified that the mappings $(q_1, q_4) \leftrightarrow (r_1^2, \eta_1) \leftrightarrow (\rho, \theta)$, where $q_1^2 + q_4^2 < 1$, $r_1 \in [0,1]$, $\eta_1 \in [-\pi, \pi]$, $\rho \in [-1,1]$ and $\theta \in [0, \pi/2]$, are one-to-one.

The variable θ in (20) can be eliminated to obtain a single equation in ρ. To this end, first observe that the expression for r_1^2 in (20) can be rewritten as:

$$\frac{1}{\cos^2\theta} = 1 + \tan^2\theta = \frac{1-\rho^2}{r_1^2 - \rho^2}. \tag{21}$$

Then, isolating θ and substituting the result in the expression for η_1 in (20), we conclude that:

$$\boxed{\eta_1 = \rho \tan^{-1}\sqrt{\frac{1-r_1^2}{r_1^2-\rho^2}} - \tan^{-1}\left(\rho\sqrt{\frac{1-r_1^2}{r_1^2-\rho^2}}\right)} \tag{22}$$

Thus, the disk-on-sphere problem has been reduced to solving the above equation in ρ for given values of r_1 and η_1. Unfortunately, it is a transcendental equation that cannot be solved explicitly for ρ using elementary functions or standard special functions. However, since the solution is known to be unique and its derivative with respect to ρ admits the neat expression

$$\frac{d\eta_1}{d\rho} = \tan^{-1}\left(\sqrt{\frac{1-r_1^2}{r_1^2-\rho^2}}\right) - \sqrt{\frac{1-r_1^2}{r_1^2-\rho^2}}, \tag{23}$$

a Newton-Raphson method should provide the sought solution. For an approximate solution, we can simply use the plot in Fig. 2.

Now, by computing the derivatives of η_1 with respect to r_1, it is possible to eliminate ρ to obtain the differential equation that governs the disk-on-sphere motion problem. After some elementary algebraic manipulations, the first derivative can be expressed as:

$$\frac{d\eta_1}{dr_1} = \frac{\rho}{r_1}\sqrt{\frac{1-r_1^2}{r_1^2-\rho^2}}. \quad (24)$$

To simplify the computation of the second derivative, it is better to first square (24), That is,

$$\left(\frac{dr_1}{d\eta_1}\right)^2 = \frac{r_1^2(r_1^2-\rho^2)}{\rho^2(1-r_1^2)}, \quad (25)$$

and then derive it with respect to r_1, to obtain

$$2\frac{d\eta_1}{dr_1}\frac{d^2\eta_1}{dr_1^2} = \frac{2r_1(-r_1^4+2r_1^2-\rho^2)}{\rho^2(1-r_1^2)^2}\frac{d\eta_1}{dr_1}. \quad (26)$$

As a consequence,

$$\frac{d^2\eta_1}{dr_1^2} = \frac{r_1(-r_1^4+2r_1^2-\rho^2)}{\rho^2(1-r_1^2)^2}. \quad (27)$$

To eliminate ρ^2 from the system of equations formed by (25) and (27), we can isolate ρ^2 from both equations to obtain

$$\frac{1}{\rho^2} = \frac{(1-r_1^2)\left(\frac{dr_1}{d\eta_1}\right)^2 + r_1^2}{r_1^4} \qquad \text{and} \qquad \frac{1}{\rho^2} = \frac{(1-r_1^2)^2\left(\frac{d^2\eta_1}{dr_1^2}\right) + r_1}{r_1^3(2-r_1^2)}.$$

Finally, by equating them and rearranging terms, we obtain:

$$\frac{d^2r_1}{d\eta_1^2}r_1(r_1^2-1) + \left(\frac{dr_1}{d\eta_1}\right)^2(2-r_1^2) + r_1^2 = 0. \quad (28)$$

This differential equation also emerges in the analysis of the terrestrial brachistochrone problem [6, 9]. Consequently, the terrestrial brachistochrone and disk-on-sphere problems are mathematically equivalent, differing solely in the boundary conditions. In the former, both the initial and final points lie on the boundary of the unit circle, whereas in the latter, the final point is an arbitrary location within the unit circle. Thus, these problems represent two specific cases of the broader problem of interpolating a cusp-free arc of a hypocycloid between two points within the unit circle.

4 Example

Let us suppose that we want to drive the sphere from the initial orientation $\mathbf{q}(0) = (1, 0, 0, 0)$ to the target orientation given by $\mathbf{q}(1) = (0, 0, 0, 1)$. In this case, $r_1^2 = 1$, $\eta_1 = \pi/4$, and $\theta = \pi/2$. As a consequence, this problem is equivalent to find the brachistochrone connecting points $(1, 0)$ and $(0, 1)$ on the boundary of the unit circle. Although the corresponding value of ρ should be obtained numerically, for this simple case and according to Fig. 2, the sought hypocycloid corresponds to $\rho = 1$.

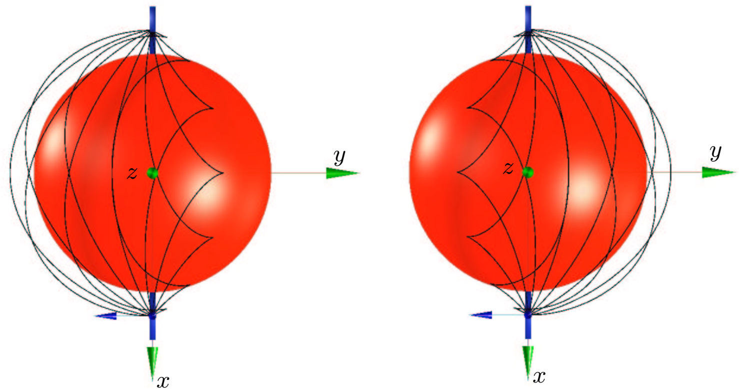

Fig. 3. Paths connecting the sphere's initial orientation $(1, 0, 0, 0)$ to the target orientation $(0, 0, 0, 1)$ (left), and to $(0, 0, 0, -1)$ (right). These paths are shown after the kinematic inversion that fixes the sphere and allows the disk to move.

With the obtained values for θ and ρ, we have that $\theta_2 = \pi$ and $\theta_1 = \rho\theta_2 = -\pi/2$. Therefore, $\alpha = 2\theta_1 = \pi$ and $a = \sqrt{4\theta_2^2 - \alpha^2} = \sqrt{3}\pi$. Finally, since $q_2 = q_3 = 0$, α_0 can take any value. Thus, the path to be followed by the sphere is given by $\mathbf{q}(t) = e^{\mathbf{A}_1 t} e^{\mathbf{A}_2 t}\mathbf{q}(0)$, $t \in [0, 1]$, where

$$\mathbf{A}_1 = -\frac{\pi}{2}\mathbf{A}_z \text{ and } \mathbf{A}_2 = \frac{\sqrt{3}}{2}\pi\mathbf{A}_x \cos\xi + \frac{\sqrt{3}}{2}\pi\mathbf{A}_y \sin\xi + \frac{\pi}{2}\mathbf{A}_z,\ \xi \in [-\pi, \pi].$$

Since we actually want to move the disk with respect to the sphere, we have to introduce a kinematic inversion. The generated paths, for $\xi = -\pi$ to $\xi = \pi$ in increments of $\pi/10$, appear in Fig. 3(left) after the inversion.

Euler parameters provide a double covering of $SO(3)$ because any vector of such parameters and its negative represent the same orientation. Then, $(0, 0, 0, 1)$ and $(0, 0, 0, -1)$ represent the same sphere's orientation. If we repeat the above procedure for $(0, 0, 0, -1)$ we obtain the paths shown in Fig. 3(right).

5 Conclusion

This paper has established a formal equivalence between the disk-on-sphere motion problem and the terrestrial brachistochrone problem by demonstrating that both are governed by the same fundamental differential equation. It was also shown that the solution to these problems reduces to interpolating a cusp-free hypocycloid arc between two points on a circle. The key contribution lies in connecting the nonholonomic constraints of the disk-on-sphere system to the variational principles underlying the terrestrial brachistochrone. This insight not only unifies two historically distinct challenges in kinematics and optimal control but also provides a powerful geometric framework for their interpretation and resolution.

Concerning future work, the integration of the presented geometric approach with numerical optimization techniques could enhance its applicability to real-world robotics and mechanical systems. The equivalence unveiled here invites cross-disciplinary applications.

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