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On Driftless Systems with m controls and $2m$ or $2m - 1$ states that are Flat by Pure Prolongation

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It is widely recognized that no tractable necessary and sufficient conditions exist for determining whether a system is, in general, differentially flat. However, specific cases do provide such conditions. For instance, driftless systems with two inputs have known necessary and sufficient conditions. For driftless systems with three or more inputs, the available conditions are only sufficient. This paper presents new findings on determining whether a system with m inputs and $2m$ or $2m - 1$ states is flat by pure prolongation, a specific subclass of differential flatness. While this condition is more restrictive than general differential flatness, the algorithm for computing flat outputs remains remarkably simple, and the verification requirements are relatively lenient. Moreover, the conditions proposed in this work broaden the class of systems recognized as differentially flat, as our sufficient condition differs from existing criteria.

Keywords: Nonlinear control system; differential geometric control; Lie bracket; driftless system; differential flatness; pure prolongation.

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1. Introduction

For linear control systems, controllability, transformation into Brunovsky canonical form through a change of variables and a regular feedback, and controller design for trajectory planning are interchangeable concepts. However, this equivalence no longer holds for nonlinear systems. While a nonlinear system may exhibit controllability, it cannot necessarily be transformed into Brunovsky canonical form.

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To address this disparity, new criteria were established to ascertain a nonlinear system's capability to achieve its Brunovsky canonical form. These criteria encompass static feedback linearizable systems, as detailed in [11–13].

Subsequently, dynamically feedback linearizable systems were introduced in works like [1, 2]. Such systems also present the possibility of transformation into Brunovsky form by augmenting the system with additional state variables, each possessing its own dynamics and through the application of a diffeomorphism and a regular state feedback to the overall system. Unfortunately, there lacks a definitive necessary and sufficient condition for determining whether a given system is dynamically feedback linearizable.

Significant advances were achieved in the final decade of the previous century with the introduction of differentially flat systems [4]. Notably, for driftless systems, more sophisticated outcomes emerged for two-input systems [19] and systems of codimension two [20], where both necessary and sufficient conditions were given for a system to exhibit differential flatness. For systems with more than two inputs, a sufficient condition was also given in [19]. The book [15] summarizes all the mentioned results and gives algorithms for the computation of the flat outputs. More recently, algorithms to compute the flat outputs for driftless systems with two inputs were given in [10, 18].

A specific category of dynamic feedback is pure prolongation, wherein certain inputs and their derivatives up to a specified order are introduced as new states. In the context of this dynamic feedback type, both necessary and sufficient conditions for the linearization of a control system through prolongations have been presented in [6, 8]. In particular, a bound on the number of integrators needed to linearize a control system was given. In a recent development, an algorithm for determining the possibility to linearize system through pure prolongations has been presented in [17]. The algorithm, designed for computational feasibility, efficiently identifies the minimal prolongation in a finite number of steps, relying solely on Lie brackets and linear algebra.

For controllable driftless systems with m inputs and $m+1$ states, it is straightforward to demonstrate their linearizability through pure prolongations, as discussed in [9]. Additionally, comprehensive conditions outlining the necessity and sufficiency for a driftless system with two inputs to undergo linearization via pure prolongations were elucidated in [7].

This paper generalizes these results by providing sufficient conditions for driftless systems with m ($m > 2$) inputs and either $2m$ or $2m-1$ states to be linearizable by prolongations. In most of the cases, these conditions differ from the sufficient conditions for differential flatness provided in [19]. Consequently, our findings also broaden the set of systems recognized as differentially flat for m inputs, $2m$ or $2m-1$ states. The three-input case is studied first because it offers a simpler setting that helps to convey the main ideas and intuition before presenting the general construction. Moreover, for $m = 3$ we systematically apply the algorithm presented in [17].

The paper is structured as follows: Section 2 introduces the necessary mathematical background and reviews relevant existing results. Section 3 explores the study of three-input driftless systems, with Sec. 3.1 presenting specific results for the case of six states using the algorithm from [17]; the proof of the main result is detailed in Appendiix A. Section 3.2 extends these findings to systems with five states, building on the results from the previous section. Section 4 generalizes the analysis to m-input systems with $2m$ or $2m-1$ states. Finally, the paper concludes with a summary of findings and potential directions for future research.

2. Mathematical Background

A control system in affine form over a smooth n-dimensional manifold X is given by

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \tag{2.1}$$

where $x \in X$ is the n-dimensional state vector, with $m < n$ (otherwise the problem of feedback linearization is trivial) and f and g_i are smooth vector fields in the tangent bundle TX of X for each $u = (u_1, \dots, u_m)$. We stress that, although all the results of this paper are done in suitably chosen local charts, all the results are intrinsic.

$f(x)$ is called the drift vector field, while $g_i(x)$ are the input vector fields. These input vector fields are assumed to be independent since, otherwise, one can reduce the number of inputs by removing redundant directions until the remaining input vector fields are linearly independent.

The system is static feedback linearizable if there exists a diffeomorphism $z = \phi(x)$ and a regular feedback law $u = a(x) + b(x)v$ such that, in the new variables, the system becomes linear. A necessary and sufficient condition for a control system to be linearizable by static feedback was given in [11, 13]:

The distributions

$$D_0 = \{g_1, \dots, g_m\} \quad D_i = \{g_1, \dots, g_m, \dots, ad_f^i g_1, \dots, ad_f^i g_m\}$$

where we use the notation $\{g_1, \dots, g_m\}$ for the distribution generated by the vector fields $g_1, g_2 \dots, g_m$, must be of constant rank and involutive. Moreover, there exists i such that the rank of D_i equals n.[a] Throughout the paper it is assumed that all the distributions have constant rank as the Frobenius Theorem (see, for instance, [3, 12, 21]), applies only for regular distributions. Several studies have investigated control systems with singularities [14], which can be either apparent (in which case an appropriate change of the set of flat outputs resolves the issue) or intrinsic. Intrinsic singularities include the set of points where the system fails to satisfy the strong accessibility rank condition. The analysis of systems with singularities lies beyond the scope of this paper.

[a]We have used the classical notations $ad_f^0 g = g$ and $ad_f^k g = [f, ad_f^{k-1} g]$ for all $k \geq 1$.

A driftless control system is

$$\dot{x} = \sum_{i=1}^{m} g_i(x)u_i. \tag{2.2}$$

Note that this system is not static feedback linearizable since it would require the involutivity of the distribution $\{g_1, \dots, g_m\}$ that would in turn imply that the brackets $\mathrm{ad}^k_{(u_1g_1+\cdots+u_mg_m)}g_i$, $i = 1, \dots, m$, $k \geq 1$ would all remain in $\{g_1, \dots, g_m\}$, of rank $m < n$, thus preventing the system from being controllable.

A dynamic system

$$\dot{x} = f(x, u)$$

is said to be differentially flat if and only if there exists m functions $y = (y_1, \dots, y_m)$ (called flat outputs) such that

$$y = y(x, u, \dot{u}, \dots) \quad x = x(y, \dot{y}, \dots) \quad u = u(y, \dot{y}, \dots) \tag{2.3}$$

where the dependence is up to a finite number of derivatives.

Differential flatness was introduced in [4]. It has been proven that a system is differentially flat if, and only if, it is dynamic feedback linearizable by *endogenous dynamic feedback* (see [4, 5]). The first of these papers introduces differential flatness in the framework of differential algebra, while the second one uses infinite jets. Although necessary and sufficient conditions for verifying differential flatness exist [16], they do not yield a finite set of criteria for determining whether a given system is differentially flat. There are some easy conditions for driftless systems. More precisely, we have the following theorem.

Theorem 1. *A two-input driftless system*

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2 \tag{2.4}$$

is differentially flat if, and only if, the ranks of the following distributions

$$D_0 = \{g_1, g_2\} \qquad D_{i+1} = D_i + [D_i, D_i]$$

are $d_i = \mathrm{rank}\ D_i = i + 2$, $\forall\, i = 0, \dots, n-2$, *with the usual notation* $\{g_1, g_2\}$ *for the distribution generated by* g_1 *and* g_2 *and with* $[D, D] = \{[\alpha, \beta] \mid \alpha, \beta \in D\}$, $[\alpha, \beta]$ *denoting the Lie bracket of the vector fields* α *and* β.

The details can be found in [19]. In the same paper, it is shown that the above condition is a sufficient condition for flatness when the number of inputs is strictly greater than two.

On the other hand, for systems with $n - 2$ inputs, flatness has been proven to be equivalent to controllability [20].

Consider the m-input system (2.1) with $x \in X$, a n-dimensional smooth manifold. A pure prolongation of the system of order $\mathbf{j} \triangleq (j_1, \dots, j_m)$ with $j_1, \dots, j_m \in \mathbb{N}$, is given by the associated prolonged vector field

$$g^{(j)} \doteq g + \sum_{i=1}^{m} \sum_{p=0}^{j_i-1} u_i^{(p+1)} \frac{\partial}{\partial u_i^{(p)}},$$

where the prolonged states are $u_i^{(p)}$, $p = 0, \dots, j_i - 1$, and the control inputs $u_i^{(j_i)}$. In other words, the original states remain the same, while the new states added to the system are inputs and derivatives of these inputs up to the order $\mathbf{j-1}$. The prolonged state thus belongs to the prolonged manifold $X \times \mathbb{R}^{|\mathbf{j}|}$, with the notation $|\mathbf{j}| \triangleq \sum_{i=1}^{m} j_i$. Throughout the paper, the notation g will denote the vector field $\sum_{j=1}^{n} g_j \frac{\partial}{\partial x_j}$ for both the vector field $g \in TX$ and the natural embedding $g \in TX \times \mathbb{R}^{|\mathbf{j}|}$.[b]

Definition 2.1. System (2.1) is *flat by pure prolongation or linearizable by pure prolongation* (P^2-flat) at a point $(x_0, \overline{u}_0) \in X \times \mathbb{R}^{|\mathbf{j}|}$ if, and only if, there exist finite $\mathbf{j}$ such that the prolonged system of order $\mathbf{j}$ is equivalent by diffeomorphism and feedback to a system in Brunovský form.

In the coordinates $(x, \overline{u})$, the notation $\overline{u}$ stands for

$$\overline{u} \triangleq (u, \dot{u}, \ddot{u}, \dots, u^{(\mathbf{j-1})}) \triangleq \left(u_i^{(k)} \; ; \; i = 1, \dots, m \; ; 0 \le k \le j_i - 1\right) \in \mathbb{R}^{|\mathbf{j}|}.$$

More on this construction can be found in [15, 16].

Regarding pure prolongations, a nonlinear driftless system with $n-1$ inputs and n states is flat by pure prolongations if, and only if, it is controllable [9]. For driftless systems with two inputs, the condition for a system to be flat by pure prolongations [7] is shown in the following theorem:

The two-input driftless system

$$\dot{x} = g_1 u_1 + g_2 u_2$$

is flat by pure prolongations if and only if

(i) $ad^2_{\left(ad^k_{g_2} g_1\right)} g_2 \in \{g_1, ad_{g_2} g_1, \dots, ad^k_{g_2} g_1\}$, $\forall k \in \{1, \dots, n-3\}$
(ii) rank $\{g_1, ad_{g_2} g_1, \dots, ad^{n-2}_{g_2} g_1, g_2\} = n$

or

(i) $ad^2_{\left(ad^k_{g_1} g_2\right)} g_1 \in \{g_2, ad_{g_1} g_2, \dots, ad^k_{g_1} g_2\}$, $\forall k \in \{1, \dots, n-3\}$
(ii) rank $\{g_2, ad_{g_1} g_2, \dots, ad^{n-2}_{g_1} g_2, g_1\} = n$

[b]This is not the usual notation in differential geometry, when typically the superscripts are used for coordinates. In the control community it is more common to use the subscripts for the coordinates and superscripts for the order of derivative with respect to time, see for instance [12, 21].

In order to illustrate all the above definitions and results, let us introduce a very simple example based on a simplified (kinematic) model of a car:

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2$$

which is a two-input system defined on the 3-dimensional manifold $X = \mathbb{R}^2 \times S^1$. This system is not static feedback linearizable since the Lie bracket $[g_1, g_2]$ does not belong to the distribution spanned by g_1 and g_2. On the other hand, the system is flat since the distributions

$$D_0 = \{g_1, g_2\} \quad D_1 = \{g_1, g_2, [g_1, g_2]\}$$

have rank 2 and 3, respectively. Actually, the system is flat by pure prolongation since a one-order prolongation of the input u_1 leads to the prolonged system:

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{u}_1^{(0)} \end{pmatrix} = \begin{pmatrix} \cos x_3 \\ \sin x_3 \\ 0 \\ 0 \end{pmatrix} u_1^{(0)} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} u_2 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u_1^{(1)}$$

which is static feedback linearizable. The flat outputs, that can be computed through this prolongation, are $y_1 = x_1$ and $y_2 = x_2$. It is easy to see that all the states and inputs can be written as functions of the flat outputs and their derivatives (see (2.3)):

$$x_1 = y_1, \quad x_2 = y_2, \quad x_3 = \arctan \frac{\dot{x}_2}{\dot{x}_1} = \arctan \frac{\dot{y}_2}{\dot{y}_1},$$

$$u_1^{(0)} = u_1 = \sqrt{\dot{x}_1^2 + \dot{x}_2^2} = \sqrt{\dot{y}_1^2 + \dot{y}_2^2}, \;\; u_1^{(1)} = \dot{u}_1^{(0)}, \;\; u_2^{(0)} = u_2 = \dot{x}_3 = \frac{\ddot{y}_2\dot{y}_1 - \dot{y}_1\ddot{y}_2}{\dot{y}_1^2 + \dot{y}_2^2}.$$

3. Linearization by Pure Prolongations of driftless Systems with Three Inputs

In this section, we will study driftless three-input systems:

$$\dot{x} = u_1^{(0)} g_1(x) + u_2^{(0)} g_2(x) + u_3^{(0)} g_3(x) \tag{3.1}$$

with $x \in X$, a n-dimensional smooth manifold, and rank $g_1(x), g_2(x), g_3(x) = 3$ for all x in a suitable open dense subset of X. The superscript 0 means that no prolongation has yet been added to the system.

We aim at obtaining conditions for which this system is flat by pure prolongation.

In order to study the P^2-flatness of (3.1), we apply the algorithm described in [17, Sec. 4.3], based on Theorem 2 recalled here below.

The unknown prolongation order[c] is denoted by $\mathbf{j} = (0, j_2, j_3)$. Recall that we also denote by $\mid \mathbf{j} \mid = \sum_{i=1}^{m} j_i$. The prolonged vector fields are

$$g_0^{(\mathbf{j})} = \Big(u_1^{(0)} g_1(x) + u_2^{(0)} g_2(x) + u_3^{(0)} g_3(x)\Big) + \sum_{p=0}^{j_2-1} u_2^{(p+1)} \frac{\partial}{\partial u_2^{(p)}} + \sum_{p=0}^{j_3-1} u_3^{(p+1)} \frac{\partial}{\partial u_3^{(p)}},$$

$$g_1^{(\mathbf{j})} = \frac{\partial}{\partial u_1^{(0)}}, \qquad g_2^{(\mathbf{j})} = \frac{\partial}{\partial u_2^{(j_2)}}, \qquad g_3^{(\mathbf{j})} = \frac{\partial}{\partial u_3^{(j_3)}}. \tag{3.2}$$

According to [17], we consider the distributions $\Delta_k^{(\mathbf{j})}$ and $\Gamma_k^{(\mathbf{j})}$ given by

$$\begin{aligned} \Delta_k^{(\mathbf{j})} &\triangleq \sum_{p=1}^{m} \left\{ \mathrm{ad}_{g_0^{(\mathbf{j})}}^{l-j_p} \frac{\partial}{\partial u_p^{(0)}} \mid l = j_p, \ldots, k \right\}, \\ \Gamma_k^{(\mathbf{j})} &\triangleq \bigoplus_{i=1,\ldots,m} \{ \frac{\partial}{\partial u_i^{(j_i - r)}} \mid r = 0, \ldots, k \vee (j_i - 1))\} \end{aligned} \tag{3.3}$$

and

$$G_k^{(\mathbf{j})} = \Gamma_k^{(\mathbf{j})} \oplus \Delta_k^{(\mathbf{j})}, \quad \forall k \geq 0. \tag{3.4}$$

Theorem 2. *The necessary and sufficient conditions for P^2-flatness are*:

(i) $[\Delta_k^{(\mathbf{j})}, \Delta_k^{(\mathbf{j})}] \subset \Delta_k^{(\mathbf{j})}$ *and* rank $\Delta_k^{(\mathbf{j})}$ *locally constant for all* k,

(ii) $[\Gamma_k^{(\mathbf{j})}, \Delta_k^{(\mathbf{j})}] \subset \Delta_k^{(\mathbf{j})}$ *for all* k,

(iii) $\exists k_\star^{(\mathbf{j})} \leq n + \mid \mathbf{j} \mid$ *such that* rank $\Delta_k^{(\mathbf{j})} = n + m$ *for* $k \geq k_\star^{(\mathbf{j})}$.

We also consider the sequence of integers

$$\rho_k^{(\mathbf{j})} \triangleq \text{rank } G_k^{(\mathbf{j})} / G_{k-1}^{(\mathbf{j})} \quad \forall k \geq 1, \qquad \rho_0^{(\mathbf{j})} \triangleq \text{rank } G_0^{(\mathbf{j})} = m$$

and, with the notation $\#A$ for the number of elements of a set A, the Brunovský controllability indices of order $\mathbf{j}$

$$\kappa_k^{(\mathbf{j})} \triangleq \#\{l \mid \rho_l^{(\mathbf{j})} \geq k\}, \quad k = 1, \ldots, m.$$

The equivalent linear system is thus

$$y_1^{(\kappa_1^{(\mathbf{j})})} = v_1, \quad y_2^{(\kappa_2^{(\mathbf{j})})} = v_2, \quad y_3^{(\kappa_3^{(\mathbf{j})})} = v_3,$$

where (y_1, y_2, y_3) is a flat output, obtained as a locally non-trivial solution of the system of PDE's

$$\left\langle G_k^{(\mathbf{j})}, dy_i \right\rangle = 0, \ k = 0, \ldots, \kappa_i^{(\mathbf{j})} - 2, \quad \text{with} \quad \left\langle G_{\kappa_i^{(\mathbf{j})} - 1}^{(\mathbf{j})}, dy_i \right\rangle \neq 0, \quad i = 1, \ldots, m. \tag{3.5}$$

[c]It is proven in [17, Lemma 4.4], that considering $j_1 = 0$ suffices for our purpose. It was also proven in [23].

Here, dy stands for the differential of the function y and, hence, it is a one form that annihilates $G_k^{(\mathbf{j})}$.

3.1. *Linearization by prolongations of driftless systems with three inputs and six states*

We set

$$\begin{aligned}
&H_{0,3} \triangleq \{g_1, g_2, g_3\}\,,\\
&H_{0,2} \triangleq \{g_1, g_2\}\,, \quad H'_{0,2} \triangleq \{g_1, g_3\}\,, \quad H''_{0,2} \triangleq \{g_2, g_3\}\,,\\
&H_{0,1} \triangleq \{g_1\}\,,
\end{aligned} \tag{3.6}$$

where we have denoted by g_i the vector field $g_i\frac{\partial}{\partial x}$, $i = 1, 2, 3$, for simplicity's sake. The first index i of the subscripts (i,j) corresponds to the maximum number of Lie brackets involved in the distribution, while the second subscript j is related to the number of vector fields that we have at the initialization.

We further introduce

$$\begin{aligned}
&H_{2,3} \triangleq \{g_1, g_2, g_3, [g_2, g_1], [g_3, g_2], [g_3, [g_3, g_1]], [g_3, [g_3, g_2]]\}\,,\\
&H_{1,3} \triangleq \{g_1, g_2, g_3, [g_3, g_1], [g_3, g_2]\}\,,\\
&H_{1,2} \triangleq \{g_1, g_2, [g_3, g_1], [g_3, g_2]\}\,,\\
&H'_{1,2} \triangleq \{g_1, g_2, [g_2, g_1], [g_3, g_1]\}\,,\\
&H_{1,1} \triangleq \{g_1, [g_2, g_1], [g_3, g_1]\}\,.
\end{aligned} \tag{3.7}$$

There are two possible initializations of the algorithm described in [17, Sec. 4.3], whether the largest involutive subdistribution of $H_{0,3}$ is, up to a suitable input permutation, $H_{0,2}$ or $H_{0,1}$ *i.e.*

$$H_{0,2} = \overline{H_{0,2}}, \quad H_{0,3} \neq \overline{H_{0,3}} \tag{3.8}$$

or

$$H_{0,2} \neq \overline{H_{0,2}}, \quad H'_{0,2} \neq \overline{H'_{0,2}}, \quad H''_{0,2} \neq \overline{H''_{0,2}}, \quad H_{0,3} \neq \overline{H_{0,3}}, \tag{3.9}$$

where we have denoted by $\overline{H}$ the involutive closure of a distribution H.

Note that the initialization (3.8) implies that $j_1 = j_2 = 0$, whereas (3.9) implies that $j_2 \geq 1$. In both cases we assume, without loss of generality, that $j_3 \geq j_2$.

Theorem 3. *Assume that $\{g_1, g_2, g_3\}$ is not involutive. Two, and only two, cases are possible: either the largest involutive subdistribution of $\{g_1, g_2, g_3\}$ is, up to a suitable permutation of the inputs, $\{g_1, g_2\}$, or $\{g_1\}$.*

(1) *If the largest involutive subdistribution of $\{g_1, g_2, g_3\}$ is $\{g_1, g_2\}$, a necessary and sufficient condition for the system* (3.1) *to be P^2-flat is*

$$H_{1,2} = \{g_1, g_2, [g_3, g_1], [g_3, g_2]\} \text{ involutive with rank } H_{1,2} = 4,$$

and the minimal prolongation order is $\mathbf{j} = (0, 0, 2)$.

Moreover, the flat outputs y_1, y_2, y_3 are such that their differentials annihilate $H_{1,2}$.

(2) *If the largest involutive subdistribution of $\{g_1, g_2, g_3\}$ is $\{g_1\}$, a necessary and sufficient condition for the system* (3.1) *to be P^2-flat is*

$$H'_{1,2} = \{g_1, g_2, [g_2, g_1], [g_3, g_1]\} \text{ involutive, with rank } H'_{1,2} = 3,$$

$$\text{rank } H_{2,3} = 6$$

$$\text{with } H_{2,3} = \{g_1, g_2, g_3, [g_2, g_1], [g_3, g_2], [g_3, [g_3, g_1]], [g_3, [g_3, g_2]]\},$$

and the minimal prolongation order is $\mathbf{j} = (0, 1, 2)$.

Moreover, the flat outputs y_1, y_2, y_3 are such that their differentials annihilate $H'_{1,2}$.

The proof of this theorem can be found in Appendix A.

Remark. As mentioned earlier in Theorem 1 ([19]), the known sufficient conditions for a controllable driftless system with six states and three inputs to be differentially flat, are $d_i = \text{rank } D_i = 3 + i$, where

$$\begin{aligned} D_0 &= \{g_1, g_2, g_3\}, \\ D_1 &= \{D_0, [D_0, D_0]\} = \{g_1, g_2, g_3, [g_1, g_2], [g_1, g_3], [g_2, g_3]\}, \\ D_2 &= \{D_1, [D_1, D_1]\}. \end{aligned} \tag{3.10}$$

So, assuming that the input vector fields are independent, these conditions reduce to check $d_1 = 4$ and $d_2 = 5$. Recall that the conditions found in this paper are

- $H_{1,2}$ is involutive and of rank 4 if the largest involutive subdistribution of $\{g_1, g_2, g_3\}$ is $\{g_1, g_2\}$, or
- $H'_{1,2}$ is involutive and of rank 3, and $H_{2,3}$ is of rank 6 if the largest involutive subdistribution of $\{g_1, g_2, g_3\}$ is $\{g_1\}$.

Hence, our conditions differ from the ones mentioned above. As a result, in addition to identifying which systems are flat by pure prolongation, we establish a new sufficient condition for differential flatness in driftless systems with three inputs and six states.

3.2. *Linearization by prolongations of driftless systems with three inputs and five states*

Regarding controllable driftless systems with three inputs and five states, let us remind, again, that the known conditions for flatness (3.10), reduce to check if $d_0 = 3$ (which is usually assumed as long as the input vector fields are independent), $d_1 = 4$, and $d_2 = 5$ (this last condition is trivial if the controllability of the system is assumed). Hence, the only condition to be checked is $d_1 = 4$. There are, indeed, two excluding possibilities for d_1, namely $d_1 = 4$ and $d_1 = 5$. $d_1 = 4$ has been proven to be a sufficient condition for flatness in [19].

Building on the results from Sec. 3.1, this section demonstrates that for the complementary case $d_1 = 5$, the system is flat by pure prolongation under an additional assumption. The precise statement is as follows:

We consider the driftless linear-in-the-control system

$$\dot{x} = u_1^{(0)} g_1(x) + u_2^{(0)} g_2(x) + u_3^{(0)} g_3(x) \tag{3.11}$$

with $x \in X$, a smooth manifold of dimension 5, and rank $g_1(x), g_2(x), g_3(x) = 3$ for all x in a suitable open dense subset of X. We set

$$\begin{aligned} &H_{0,3} \triangleq \{g_1, g_2, g_3\}, \\ &H_{0,2} \triangleq \{g_1, g_2\}, \quad H_{1,3} \triangleq \{g_1, g_2, g_3, [g_3, g_1], [g_3, g_2]\} \end{aligned} \tag{3.12}$$

where, again, we have denoted by g_i the vector field $\sum_{j=1}^{5} g_{i,j} \frac{\partial}{\partial x_j}$, $i = 1, 2, 3$, for simplicity's sake.

Theorem 4. *Assume that $\{g_1, g_2, g_3\}$ is not involutive.*

(1) *If the largest involutive subdistribution of $\{g_1, g_2, g_3\}$ is $\{g_1, g_2\}$, a necessary and sufficient condition for the system* (3.11) *to be flat by pure prolongation is*

$$H_{1,3} = \{g_1, g_2, g_3, [g_3, g_1], [g_3, g_2]\} \text{ has rank } 5$$

and the minimal prolongation order is $\mathbf{j} = (0, 0, 1)$. Moreover, the flat outputs are y_1, y_2, y_3 such that their differentials annihilate g_1 and g_2.

(2) *If the largest involutive subdistribution of $\{g_1, g_2, g_3\}$ is $\{g_1\}$, a necessary and sufficient condition for the system* (3.11) *to be flat by pure prolongation is*

$$H'_{1,2} = \{g_1, g_2, [g_2, g_1], [g_3, g_1]\} \text{ involutive, with rank } H'_{1,2} = 3,$$

$$\text{rank } H_{2,3} = 5$$

$$\text{with } H_{2,3} = \{g_1, g_2, g_3, [g_2, g_1], [g_3, g_2], [g_3, [g_3, g_1]], [g_3, [g_3, g_2]]\},$$

and the minimal prolongation order is $\mathbf{j} = (0, 1, 2)$.

Moreover, the flat outputs are y_1, y_2, y_3 such that y_1, y_2 are independent of the inputs and their differentials annihilate $H'_{1,2}$, while y_3 is independent of u_1 and u_2 and its differential annihilates g_1.

Proof.

(1) After adding an order one prolongation of u_3, the drift and the input vector fields of the prolonged system are

$$\begin{aligned} g_0^{(\mathbf{j})} &= \left(u_1^{(0)} g_1(x) + u_2^{(0)} g_2(x) + u_3^{(0)} g_3(x)\right) + u_3^{(1)} \frac{\partial}{\partial u_3^{(0)}}, \\ g_1^{(\mathbf{j})} &= \frac{\partial}{\partial u_1^{(0)}}, \qquad g_2^{(\mathbf{j})} = \frac{\partial}{\partial u_2^{(0)}}, \qquad g_3^{(\mathbf{j})} = \frac{\partial}{\partial u_3^{(1)}}. \end{aligned} \tag{3.13}$$

We compute the distributions (3.3) for $k = 0, 1, 2$:

$$\begin{aligned} \Gamma_0^{(\mathbf{j})} &\triangleq \left\{ \frac{\partial}{\partial u_3^{(1)}} \right\}, \\ \Delta_0^{(\mathbf{j})} &\triangleq \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}} \right\}, \\ \Gamma_1^{(\mathbf{j})} &\triangleq \left\{ \frac{\partial}{\partial u_3^{(1)}} \right\}, \\ \Delta_1^{(\mathbf{j})} &\triangleq \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_3^{(0)}}, g_1, g_2 \right\}, \\ \Gamma_2^{(\mathbf{j})} &\triangleq \left\{ \frac{\partial}{\partial u_3^{(1)}} \right\}, \\ \Delta_2^{(\mathbf{j})} &\triangleq \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_3^{(0)}}, g_1, g_2, g_3, [g_0^{(\mathbf{j})}, g_1], [g_0^{(\mathbf{j})}, g_2] \right\}. \end{aligned} \tag{3.14}$$

But, due to the involutivity of $H_{0,2}$,

$$\Delta_2^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_3^{(0)}}, g_1, g_2, g_3, [g_3, g_1], [g_3, g_2] \right\}$$

which has rank 8 thanks to the hypothesis that the rank of $H_{1,3}$ is 5.

It is a straightforward computation to check the necessary and sufficient conditions for P^2-flatness given in Theorem 2:

(i) $[\Delta_k^{(\mathbf{j})}, \Delta_k^{(\mathbf{j})}] \subset \Delta_k^{(\mathbf{j})}$ and rank $\Delta_k^{(\mathbf{j})}$ locally constant for all $k = 0, 1, 2$,
(ii) $[\Gamma_k^{(\mathbf{j})}, \Delta_k^{(\mathbf{j})}] \subset \Delta_k^{(\mathbf{j})}$ for all $k = 0, 1, 2$,
(iii) rank $\Delta_2^{(\mathbf{j})} = n + m = 8$.

Therefore, the system is P^2-flat with an order one prolongation of u_3 and the flat outputs y_1, y_2, y_3 are the solutions of the system of PDE's:

$$\left\langle G_1^{(\mathbf{j})}, dy_i \right\rangle = 0$$

or, equivalently, the flat outputs must be independent of $u_1^{(0)}, u_2^{(0)}, u_3^{(0)}$ and must be the solution of the system of PDE's

$$\langle H_{0,2}, dy_i \rangle = 0.$$

Remark that, by the Frobenius theorem, there exists y_1, y_2, y_3 independent solutions of this system of linear partial differential equations since, by assumption, $H_{0,2}$ is an involutive distribution.

(2) Let us consider now the case $\mathbf{j} = (0,1,2)$. The distributions (3.3) for $k = 0,1,2$ are

$$\begin{aligned}
\Gamma_0^{(\mathbf{j})} &= \left\{ \frac{\partial}{\partial u_2^{(1)}}, \frac{\partial}{\partial u_3^{(2)}} \right\}, \\
\Delta_0^{(\mathbf{j})} &= \left\{ \frac{\partial}{\partial u_1^{(0)}} \right\}, \\
\Gamma_1^{(\mathbf{j})} &= \left\{ \frac{\partial}{\partial u_2^{(1)}}, \frac{\partial}{\partial u_3^{(2)}}, \frac{\partial}{\partial u_3^{(1)}} \right\}, \\
\Delta_1^{(\mathbf{j})} &= \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, g_1 \right\}, \\
\Gamma_2^{(\mathbf{j})} &= \Gamma_1^{(\mathbf{j})}, \\
\Delta_2^{(\mathbf{j})} &= \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_3^{(0)}}, g_1, g_2, [g_2, g_1] \right\}.
\end{aligned} \tag{3.15}$$

are all involutive if, and only if, $H'_{1,2}$ is involutive with rank $H'_{1,2} = 3$. Moreover, we have rank $\Delta_2^{(\mathbf{j})} = 6$ and we have $\Gamma_3^{(\mathbf{j})} = \Gamma_1^{(\mathbf{j})}$ and $\Delta_3^{(\mathbf{j})} =$

$$\left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_3^{(0)}}, g_1, g_2, g_3, [g_2, g_1], [g_3, g_2], [g_3, [g_3, g_1]], [g_3, [g_3, g_2]] \right\}.$$

Thus, rank $H'_{1,2} = 3$ implies that rank $\Delta_3^{(\mathbf{j})} = 8 = n + m$ if rank $H_{2,3} = 5$.

Hence, if $H'_{1,2}$ is involutive, with rank $H'_{1,2} = 3$, and rank $H_{2,3} = 5$, the system (3.1) is P^2-flat with minimal prolongation $\mathbf{j} = (0,1,2)$. The flat outputs y_1, y_2 are the solutions of the system of PDE's:

$$\left\langle G_2^{(\mathbf{j})}, dy_i \right\rangle = 0$$

or, equivalently, the flat outputs y_1, y_2 must be independent of $u_1^{(0)}, u_2^{(0)}, u_3^{(0)}$ and must be the solution of the system of PDE's

$$\left\langle H'_{1,2}, dy_i \right\rangle = 0.$$

Again, by the Frobenius theorem, there exists y_1, y_2 independent solutions of this system of linear partial differential equations since, by assumption, $H'_{1,2}$

is an involutive distribution. Finally, the third flat output y_3 can be obtained as an independent function of y_1, y_2 satisfying the system of PDE's:

$$\left\langle G_1^{(\mathbf{j})}, dy_3 \right\rangle = 0$$

or, equivalently, y_3 must be independent of $u_1^{(0)}, u_2^{(0)}$ and must be the solution of the system of PDE's

$$\langle g_1, dy_i \rangle = 0. \qquad \square$$

Remark. It was also proven in [20] that any controllable driftless system of codimension 2 (that is to say, with m inputs and $m+2$ states) is differentially flat if, and only if, the system is controllable. Hence, the importance of the above mentioned results relies on the fact that we prove that the system is flat by pure prolongation to deduce the differential flatness of the system.

3.3. *Example*

Consider the following example, borrowed from [22]:

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \\ \dot{x_5} \end{pmatrix} = u_1 \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ \frac{-1}{2} \\ 0 \\ 1 \\ 0 \end{pmatrix} + u_3 \begin{pmatrix} -\frac{1}{2}x_2 \\ \frac{1}{2}x_1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

It is easy to check that this example fits into the hypothesis of Theorem 4 since $[g_1, g_2] \in H_{0,2} = \{g_1, g_2\}$ and the rank of $H_{1,3}$ is 5. Therefore, the system is flat by pure prolongation by adding an order one prolongation of u_3. The flat outputs y_1, y_2, y_3 are the solution of the system of PDEs:

$$\langle H_{0,2}, dy_i \rangle = 0.$$

A possible solution for this system is

$$y_1 = x_5; \quad y_2 = 2x_1 + x_3; \quad y_3 = 2x_2 + x_4.$$

To illustrate the definition of differential flatness, it is a straightforward computation that, after differentiating the above equations, one can write:

$$\begin{aligned} x_1 &= \frac{\dot{y}_3}{\dot{y}_1}, \\ x_2 &= -\frac{\dot{y}_2}{\dot{y}_1}, \\ x_3 &= y_2 - 2\frac{\dot{y}_3}{\dot{y}_1}, \\ x_4 &= y_3 + \frac{\dot{y}_2}{\dot{y}_1}, \\ x_5 &= y_1 \end{aligned}$$

which proves that all the states can be written as function of the flat outputs and their derivatives. The inputs u_1, u_2, u_3 are obtained by differentiating the above equations for x_3, x_4 and x_5 once. Hence, the system fulfills the definition of differential flatness given in the background section (see (2.3)).

4. Generalization to Driftless Systems with m Inputs and $2m-1$ or $2m$ States

Consider the m-input system

$$\dot{x} = \sum_{i=1}^{m} g_i u_i \tag{4.1}$$

with $x \in X$ a smooth manifold of dimension $2m-1$ or $2m$.

We set

$$\begin{aligned}
H_{0,2} &\triangleq \{g_1, \ldots, g_{m-1}\}, \\
H_{2,3} &\triangleq \{g_1, \ldots, g_m, [g_m, g_1], \ldots, [g_m, g_{m-1}], [g_m, [g_m, g_1]], \ldots [g_m, [g_m, g_{m-1}]]\}, \\
H_{1,2} &\triangleq \{g_1, \ldots, g_{m-1}, [g_m, g_1], \ldots, [g_m, g_{m-1}]\}, \\
H'_{1,2} &\triangleq \{g_1, \ldots, g_{m-1}, g_m, [g_m, g_1], \ldots, [g_m, g_{m-1}]\}.
\end{aligned} \tag{4.2}$$

The results from the previous section for $m = 3$ can be extended to systems with an arbitrary number of inputs and $2m$ or $2m-1$ states. Based on the findings in Theorems 3 and 4, we propose and prove the following theorem:

Theorem 5.

(1) *For the case $n = 2m-1$, if $H_{0,2}$ is involutive and the rank of $H'_{1,2}$ is $2m-1$, the system (4.1) is P^2-flat and the minimal prolongation order is $\mathbf{j} = (0, \ldots, 0, 1)$. Moreover, the flat outputs $y_1, \ldots y_m$ can be computed such that their differentials annihilate $H_{0,2}$.*

(2) *For the case $n = 2m$, if $H_{0,2}$ is involutive, $H_{1,2}$ is involutive and of rank $2m-2$ and $H_{2,3}$ is the full space, the system (4.1) is P^2-flat and the minimal prolongation order is $\mathbf{j} = (0, \ldots, 0, 2)$. Moreover, the flat outputs $y_1, \ldots, y_m$ are m functions differentially independent such that the differentials of y_1, y_2 annihilate $H_{1,2}$ and the differentials of $y_3, \ldots, y_m$ annihilate $H_{0,2}$.*

Proof.

(1) After adding an order one prolongation of u_m, the drift and the input vector fields of the prolonged system are:

$$\begin{aligned}
g_0^{(\mathbf{j})} &= \left(u_1^{(0)} g_1(x) + \cdots + u_m^{(0)} g_m(x)\right) + u_m^{(1)} \frac{\partial}{\partial u_m^{(0)}}, \\
g_i^{(\mathbf{j})} &= \frac{\partial}{\partial u_i^{(0)}}, \quad \forall\, i = 1, \ldots, m-1 \qquad g_m^{(\mathbf{j})} = \frac{\partial}{\partial u_m^{(1)}}.
\end{aligned} \tag{4.3}$$

We compute the distributions (3.3) for $k = 0, 1, 2$:

$$\begin{aligned}
\Gamma_0^{(\mathbf{j})} &= \left\{ \frac{\partial}{\partial u_m^{(1)}} \right\} \quad \Gamma_i^{(\mathbf{j})} = \Gamma_0^{(\mathbf{j})},\ i = 1, 2, \\
\Delta_0^{(\mathbf{j})} &= \left\{ \frac{\partial}{\partial u_1^{(0)}}, \dots, \frac{\partial}{\partial u_{m-1}^{(0)}} \right\}, \\
\Delta_1^{(\mathbf{j})} &= \left\{ \frac{\partial}{\partial u_1^{(0)}}, \dots, \frac{\partial}{\partial u_m^{(0)}}, g_1, \dots, g_{m-1} \right\}, \\
\Delta_2^{(\mathbf{j})} &= \left\{ \frac{\partial}{\partial u_1^{(0)}}, \dots, \frac{\partial}{\partial u_m^{(0)}}, g_1, \dots, g_m, [g_m, g_1], \dots, [g_m, g_{m-1}] \right\}.
\end{aligned} \tag{4.4}$$

Note that, due to the involutivity of $H_{0,2}$, $\Delta_1^{(\mathbf{j})}$ is also involutive, while $\Delta_2^{(\mathbf{j})}$ has rank $3m - 1$ given the hypothesis that $H'_{1,2}$ has rank $2m - 1$. Therefore, the sufficient conditions for P^2-flatness stated in Theorem 2 are satisfied:

(i) $[\Delta_k^{(\mathbf{j})}, \Delta_k^{(\mathbf{j})}] \subset \Delta_k^{(\mathbf{j})}$ and rank $\Delta_k^{(\mathbf{j})}$ locally constant for all $k = 0, 1, 2$,

(ii) $[\Gamma_k^{(\mathbf{j})}, \Delta_k^{(\mathbf{j})}] \subset \Delta_k^{(\mathbf{j})}$ for all $k = 0, 1, 2$,

(iii) rank $\Delta_2^{(\mathbf{j})} = n + m = 3m - 1$.

Hence, the system is P^2-flat with an order one prolongation of u_m. In order to compute the flat outputs, the Brunovský controllability indices are computed:

$$\kappa_k^{(\mathbf{j})} \triangleq \#\{l \mid \rho_l^{(\mathbf{j})} \geq k\} = 3, \quad \forall\, k = 1, \dots, m.$$

and, therefore, the flat outputs $y_1, \dots, y_m$ are the solutions of the system of PDE's:

$$\left\langle G_1^{(\mathbf{j})}, dy_i \right\rangle = 0$$

or, equivalently, the flat outputs $y_1, \dots, y_m$ must be independent of $u_1^{(0)}, \dots, u_m^{(0)}$ and must be the solution of the system of PDE's

$$\langle H_{0,2}, dy_i \rangle = 0$$

which can be solved thanks to the involutivity of $H_{0,2}$ by application of the Frobenius theorem.

(2) Let us consider now the case $n = 2m$. The distributions (3.3) for $k = 0, 1, 2, 3$ are

$$\begin{aligned}
\Gamma_0^{(\mathbf{j})} &= \left\{ \frac{\partial}{\partial u_m^{(2)}} \right\}, \\
\Gamma_1^{(\mathbf{j})} &= \left\{ \frac{\partial}{\partial u_m^{(2)}}, \frac{\partial}{\partial u_m^{(1)}} \right\}, \\
\Gamma_i^{(\mathbf{j})} &= \Gamma_1^{(\mathbf{j})},\ i = 2, 3, \\
\Delta_0^{(\mathbf{j})} &= \left\{ \frac{\partial}{\partial u_1^{(0)}}, \dots, \frac{\partial}{\partial u_{m-1}^{(0)}} \right\}, \\
\Delta_1^{(\mathbf{j})} &= \left\{ \frac{\partial}{\partial u_1^{(0)}}, \dots, \frac{\partial}{\partial u_{m-1}^{(0)}}, g_1, \dots, g_{m-1} \right\}, \\
\Delta_2^{(\mathbf{j})} &= \left\{ \frac{\partial}{\partial u_1^{(0)}}, \dots, \frac{\partial}{\partial u_m^{(0)}}, g_1, \dots, g_{m-1}, [g_m, g_1], \dots, [g_m, g_{m-1}] \right\}, \\
\Delta_3^{(\mathbf{j})} &= \left\{ \frac{\partial}{\partial u_1^{(0)}}, \dots, \frac{\partial}{\partial u_m^{(0)}}, g_1, \dots, g_m, \right. \\
&\qquad \left. [g_m, g_1], \dots, [g_m, g_{m-1}], [g_m, [g_m, g_1]], \dots, [g_m, [g_m, g_{m-1}]] \right\}.
\end{aligned} \tag{4.5}$$

Hence, $\Delta_1^{(\mathbf{j})}$ is involutive thanks to the involutivity of $H_{0,2}$, $\Delta_2^{(\mathbf{j})}$ is involutive due to the involutivity of $H_{1,2}$, and $\Delta_3^{(\mathbf{j})}$ has rank $3m$ since, by hypothesis, $H_{2,3}$ has rank $2m$. Summarizing, the conditions for P^2-flatness given in Theorem 2 are fulfilled.

Let us compute again the Brunovský controllability indices:

$$\kappa_k^{(\mathbf{j})} = 4, \quad \forall\, k = 1, 2 \quad \kappa_k^{(\mathbf{j})} = 3 \quad \forall\, k = 3, \dots, m.$$

So, the flat outputs y_1, y_2 are the solutions of the system of PDE's:

$$\left\langle G_2^{(\mathbf{j})}, dy_i \right\rangle = 0$$

or, equivalently, they must be independent of the inputs and their differentials must annihilate $H_{1,2}$. On the other hand, the flat outputs $y_3, \dots, y_m$ must be independent of the inputs $u_1, \dots, u_{m-1}$ and their differentials must annihilate $H_{0,2}$. Again, this system of equations can be solved due to the involutivity of $H_{0,2}$ and $H_{1,2}$. □

5. Conclusions

A sufficient condition has been established for determining whether a three-input driftless system with five or six states is flat by pure prolongation. This condition requires verifying the involutivity of some distributions along with certain rank conditions. For the six dimension case, since these sufficient conditions differ from existing ones, our results not only provide criteria for a system to be flat by pure prolongation but also expand the class of systems known to be differentially flat.

The results derived for the three-input case have been extended to m-input systems with $2m$ or $2m-1$ states. Once again, the conditions remain relatively lenient, requiring only the verification of the involutivity of certain distributions and specific rank conditions. These findings provide sufficient conditions for an m-input system to be flat by pure prolongation, and consequently, they also serve as a sufficient condition for differential flatness.

Future work may explore general systems with drift or systems with an arbitrary number of inputs and states.

Appendix A: Proof of Theorem 3

Proof. For every k, we compute the distributions (3.3) for all $\mathbf{j}$ up to $k_\star^{(\mathbf{j})}$, the index at which the rank of $\Delta_k^{(\mathbf{j})}$ reaches $n+m$.

We start proving the result in the first case (3.8), namely if the largest involutive subdistribution of $H_{0,3}$ is, up to a suitable input permutation, $H_{0,2}$ *i.e.*

$$H_{0,2} = \overline{H_{0,2}}, \quad H_{0,3} \neq \overline{H_{0,3}}$$

that corresponds to $j_1 = j_2 = 0$ and $j_3 \geq 1$.

$\mathbf{k = 0}$: For all $\mathbf{j_3 \geq 1}$,

$$\Gamma_0^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_3^{(j_3)}} \right\}, \quad \Delta_0^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}} \right\}.$$

$\Delta_0^{(\mathbf{j})}$ is involutive, rank $\Delta_0^{(\mathbf{j})} = 2$ and $[\Gamma_0^{(\mathbf{j})}, \Delta_0^{(\mathbf{j})}] = \{0\} \subset \Delta_0^{(\mathbf{j})}$.

$\mathbf{k = 1}$: If $\mathbf{j_3 = 1}$

$$\Gamma_1^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_3^{(1)}} \right\}, \quad \Delta_1^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_3^{(0)}}, g_1, g_2 \right\}.$$

$\Delta_1^{(\mathbf{j})}$ is involutive if, and only if, $H_{0,2}$ is involutive, rank $\Delta_1^{(\mathbf{j})} = 5$ and $[\Gamma_1^{(\mathbf{j})}, \Delta_1^{(\mathbf{j})}] = \{0\} \subset \Delta_1^{(\mathbf{j})}$.

If $\mathbf{j_3 \geq 2}$

$$\Gamma_1^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_3^{(j_3)}}, \frac{\partial}{\partial u_3^{(j_3-1)}} \right\}, \quad \Delta_1^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, g_1, g_2 \right\}.$$

Again, $\Delta_1^{(\mathbf{j})}$ is involutive if, and only if, $H_{0,2}$ is involutive, rank $\Delta_1^{(\mathbf{j})} = 4$ and $[\Gamma_1^{(\mathbf{j})}, \Delta_1^{(\mathbf{j})}] = \{0\} \subset \Delta_1^{(\mathbf{j})}$.

k = 2: If $\mathbf{j_3 = 1}$

$$\Gamma_2^{(\mathbf{j})} = \left\{\frac{\partial}{\partial u_3^{(1)}}\right\}, \qquad \Delta_2^{(\mathbf{j})} =$$
$$\left\{\frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_3^{(0)}}, g_1, g_2, g_3, u_2^{(0)}[g_2, g_1] + u_3^{(0)}[g_3, g_1], u_1^{(0)}[g_1, g_2] + u_3^{(0)}[g_3, g_2]\right\}.$$

$\Delta_2^{(\mathbf{j})}$ is involutive if, and only if, $H_{1,3}$ is involutive with $4 \leq \text{rank } H_{1,3} \leq 5$ (condition (3.8) implies that rank $H_{1,3} \geq$ rank $H_{0,3} \geq 4$), in which case

$$\Delta_2^{(\mathbf{j})} = \left\{\frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_3^{(0)}}, g_1, g_2, g_3, [g_3, g_1], [g_3, g_2]\right\} = \overline{\Delta_2^{(\mathbf{j})}}, \quad 7 \leq \text{rank } \Delta_2^{(\mathbf{j})} \leq 8,$$

and $[\Gamma_2^{(\mathbf{j})}, \Delta_2^{(\mathbf{j})}] = \{0\} \subset \Delta_2^{(\mathbf{j})}$.

If $\mathbf{j_3 = 2}$

$$\Gamma_2^{(\mathbf{j})} = \left\{\frac{\partial}{\partial u_3^{(2)}}, \frac{\partial}{\partial u_3^{(1)}}\right\},$$
$$\Delta_2^{(\mathbf{j})} = \left\{\frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_3^{(0)}}, g_1, g_2, u_2^{(0)}[g_2, g_1] + u_3^{(0)}[g_3, g_1], u_1^{(0)}[g_1, g_2] + u_3^{(0)}[g_3, g_2]\right\}.$$

$\Delta_2^{(\mathbf{j})}$ is involutive if, and only if, $H_{1,2}$ is involutive with $3 \leq \text{rank } H_{1,2} \leq 4$ since, $H_{0,2}$ being involutive, rank $H_{1,2} = 2$ would yield $[g_3, g_1], [g_3, g_2] \in H_{0,2}$ and then every other bracket generated by g_1, g_2, g_3 would also belong to $H_{0,2}$, which would contradict the strong controllability condition (iii). We thus get

$$\Delta_2^{(\mathbf{j})} = \left\{\frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_3^{(0)}}, g_1, g_2, [g_3, g_1], [g_3, g_2]\right\} = \overline{\Delta_2^{(\mathbf{j})}},$$
$$6 \leq \text{rank } \Delta_2^{(\mathbf{j})} \leq 7, \ [\Gamma_2^{(\mathbf{j})}, \Delta_2^{(\mathbf{j})}] = \{0\} \subset \Delta_2^{(\mathbf{j})}.$$

If $\mathbf{j_3 \geq 3}$

$$\Gamma_2^{(\mathbf{j})} = \left\{\frac{\partial}{\partial u_3^{(j_3)}}, \frac{\partial}{\partial u_3^{(j_3-1)}}, \frac{\partial}{\partial u_3^{(j_3-2)}}\right\},$$
$$\Delta_2^{(\mathbf{j})} = \left\{\frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, g_1, g_2, u_2^{(0)}[g_2, g_1] + u_3^{(0)}[g_3, g_1], u_1^{(0)}[g_1, g_2] + u_3^{(0)}[g_3, g_2]\right\}.$$

Again, $\Delta_2^{(\mathbf{j})}$ is involutive if, and only if, $H_{1,2}$ is involutive with $3 \le \operatorname{rank} H_{1,2} \le 4$, in which case

$$\Delta_2^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, g_1, g_2, [g_3, g_1], [g_3, g_2] \right\} = \overline{\Delta_2^{(\mathbf{j})}}, \quad 5 \le \operatorname{rank} \Delta_2^{(\mathbf{j})} \le 6$$

$$[\Gamma_2^{(\mathbf{j})}, \Delta_2^{(\mathbf{j})}] = \{0\} \subset \Delta_2^{(\mathbf{j})}.$$

$\mathbf{k = 3}$: If $\mathbf{j_3 = 1}$ we have proven so far that, for $\mathbf{j} = (0,0,1)$, $\Delta_1^{(\mathbf{j})}$ is involutive if, and only if, $H_{0,2}$ is involutive, with $\operatorname{rank} \Delta_1^{(\mathbf{j})} = 5$, and that $\Delta_2^{(\mathbf{j})}$ is involutive if, and only if, $H_{1,3}$ is involutive with $4 \le \operatorname{rank} H_{1,3} \le 5$.

$$\Delta_2^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_3^{(0)}}, g_1, g_2, g_3, [g_3, g_1], [g_3, g_2] \right\} = \overline{\Delta_2^{(\mathbf{j})}},$$

$7 \le \operatorname{rank} \Delta_2^{(\mathbf{j})} \le 8$, and $[\Gamma_2^{(\mathbf{j})}, \Delta_2^{(\mathbf{j})}] = \{0\} \subset \Delta_2^{(\mathbf{j})}$.

But the involutivity of $H_{0,2}$ and $H_{1,3}$ clearly implies that all the brackets $[g_p, [g_q, g_r]]$, $p, q, r = 1, 2, 3$, are in $\overline{\Delta_2^{(\mathbf{j})}} = \Delta_2^{(\mathbf{j})}$. Therefore $\Delta_k^{(\mathbf{j})} = \Delta_2^{(\mathbf{j})}$ for all $k \ge 3$ and $7 \le \operatorname{rank} \Delta_k^{(\mathbf{j})} \le 8 < n + m = 9$ for all $k \ge 3$. Hence $j_3 > 1$.

If $\mathbf{j_3 = 2}$, we have proven so far that, for $\mathbf{j} = (0,0,2)$,

$$\Delta_0^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}} \right\},$$

$$\Delta_1^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, g_1, g_2 \right\},$$

$$\Delta_2^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_3^{(0)}}, g_1, g_2, [g_3, g_1], [g_3, g_2] \right\}.$$

are all involutive if, and only if, $H_{1,2}$ and $H_{0,2}$ are involutive with $3 \le \operatorname{rank} H_{1,2} \le 4$ and $\operatorname{rank} \Delta_2^{(\mathbf{j})} \le 7$.

Moreover, we have

$$\Gamma_3^{(\mathbf{j})} = \Gamma_2^{(\mathbf{j})},$$

$$\Delta_3^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_3^{(0)}}, g_1, g_2, g_3, [g_3, g_1], [g_3, g_2], [g_3, [g_3, g_1]], [g_3, [g_3, g_2]] \right\}.$$

Thus, if $\operatorname{rank} H_{1,2} = 4$, which implies that $\operatorname{rank} \Delta_2^{(\mathbf{j})} = 7$, we get $\operatorname{rank} \Delta_3^{(\mathbf{j})} = 9 = n + m$.

Hence, $\mathbf{j} = (0,0,2)$ is such that

$$\Gamma_k^{(\mathbf{j})} = \Gamma_3^{(\mathbf{j})}, \quad \Delta_k^{(\mathbf{j})} = \Delta_3^{(\mathbf{j})} \quad \forall k \ge 3 = k_\star^{(\mathbf{j})}$$

and $\Delta_k^{(\mathbf{j})}$ is involutive with constant rank and invariant by $\Gamma_k^{(\mathbf{j})}$ for all k, which proves that, if (3.8) holds true, $H_{1,2}$ and $H_{0,2}$ are involutive with rank $H_{1,2} = 4$, the system (3.1) is P^2-flat with minimal prolongation $\mathbf{j} = (0,0,2)$. Moreover, it is easily verified that

$$\rho_0^{(0,0,2)} = 3, \quad \rho_1^{(0,0,2)} = 3, \quad \rho_2^{(0,0,2)} = 3, \quad \rho_3^{(0,0,2)} = 2$$

and

$$\kappa_1^{(0,0,2)} = 4, \quad \kappa_2^{(0,0,2)} = 4, \quad \kappa_3^{(0,0,2)} = 3$$

which proves that the equivalent linear system is

$$y_1^{(4)} = v_1, \quad y_2^{(4)} = v_2, \quad y_3^{(3)} = v_3. \tag{A.1}$$

We now prove the result in the second case (3.9), namely if the largest involutive subdistribution of $H_{0,3}$ is, up to a suitable input permutation, $H_{0,1}$, that corresponds to $j_1 = j_2 = 0$ and $j_3 \geq 1$.

$\mathbf{k = 0}$: For all $\mathbf{j_2 \geq 1,\ j_3 \geq 1}$,

$$\Gamma_0^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_2^{(j_2)}}, \frac{\partial}{\partial u_3^{(j_3)}} \right\}, \quad \Delta_0^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_1^{(0)}} \right\}.$$

$\Delta_0^{(\mathbf{j})}$ is involutive, rank $\Delta_0^{(\mathbf{j})} = 1$ and $[\Gamma_0^{(\mathbf{j})}, \Delta_0^{(\mathbf{j})}] = \{0\} \subset \Delta_0^{(\mathbf{j})}$.

$\mathbf{k = 1}$: If $\mathbf{j_2 = 1,\ j_3 = 1}$

$$\Gamma_1^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_2^{(1)}}, \frac{\partial}{\partial u_3^{(1)}} \right\}, \quad \Delta_1^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_3^{(0)}}, g_1 \right\}.$$

$\Delta_1^{(\mathbf{j})}$ is involutive, rank $\Delta_1^{(\mathbf{j})} = 4$ and $[\Gamma_1^{(\mathbf{j})}, \Delta_1^{(\mathbf{j})}] = \{0\} \subset \Delta_1^{(\mathbf{j})}$.

If $\mathbf{j_2 = 1,\ j_3 \geq 2}$

$$\Gamma_1^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_2^{(1)}}, \frac{\partial}{\partial u_3^{(j_3)}}, \frac{\partial}{\partial u_3^{(j_3-1)}} \right\}, \quad \Delta_1^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, g_1 \right\}.$$

$\Delta_1^{(\mathbf{j})}$ is involutive, rank $\Delta_1^{(\mathbf{j})} = 3$ and $[\Gamma_1^{(\mathbf{j})}, \Delta_1^{(\mathbf{j})}] = \{0\} \subset \Delta_1^{(\mathbf{j})}$.

If $\mathbf{j_2 \geq 2,\ j_3 \geq 2}$

$$\Gamma_1^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_2^{(j_2)}}, \frac{\partial}{\partial u_2^{(j_2-1)}}, \frac{\partial}{\partial u_3^{(j_3)}}, \frac{\partial}{\partial u_3^{(j_3-1)}} \right\}, \quad \Delta_1^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, g_1 \right\}.$$

$\Delta_1^{(\mathbf{j})}$ is involutive, rank $\Delta_1^{(\mathbf{j})} = 2$ and $[\Gamma_1^{(\mathbf{j})}, \Delta_1^{(\mathbf{j})}] = \{0\} \subset \Delta_1^{(\mathbf{j})}$.

k = 2: If $\mathbf{j_2 = 1,\ j_3 = 1}$

$$\Gamma_2^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_2^{(1)}}, \frac{\partial}{\partial u_3^{(1)}} \right\},$$

$$\Delta_2^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_3^{(0)}}, g_1, g_2, g_3, u_2^{(0)}[g_2, g_1] + u_3^{(0)}[g_3, g_1] \right\}.$$

$\Delta_2^{(\mathbf{j})}$ is involutive if, and only if, $H_{1,3}$ is involutive, with rank $H_{1,3} = 4$. To prove the latter claim, recall that $j_2 = 1$ corresponds to the assumption (3.9), so that $[g_2, g_1]$ and $[g_3, g_1]$ are not elements of $H_{0,3}$. But since $H_{0,3}$ is not involutive and its rank is 3, at least one Lie bracket $[g_2, g_1]$ or $[g_3, g_1]$ must belong to $H_{1,3}$, which proves that rank $H_{1,3} = 4$.

Thus

$$\Delta_2^{(\mathbf{j})} = \begin{cases} \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_3^{(0)}}, g_1, g_2, g_3, [g_2, g_1] \right\} \\ \text{or} \\ \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_3^{(0)}}, g_1, g_2, g_3, [g_3, g_1] \right\} \end{cases} = \overline{\Delta_2^{(\mathbf{j})}},$$

and

$$\text{rank } \Delta_2^{(\mathbf{j})} = 7,\ [\Gamma_2^{(\mathbf{j})}, \Delta_2^{(\mathbf{j})}] = \{0\} \subset \Delta_2^{(\mathbf{j})}.$$

If $\mathbf{j_2 = 1,\ j_3 = 2}$

$$\Gamma_2^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_2^{(1)}}, \frac{\partial}{\partial u_3^{(2)}}, \frac{\partial}{\partial u_3^{(1)}} \right\},$$

$$\Delta_2^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_3^{(0)}}, g_1, g_2, u_2^{(0)}[g_2, g_1] + u_3^{(0)}[g_3, g_1] \right\}.$$

$\Delta_2^{(\mathbf{j})}$ is involutive if $H'_{1,2}$ is involutive, with rank $H'_{1,2} = 3$ (recall that $j_2 = 1$ corresponds to the assumption (3.9) with $H_{0,2}$ non-involutive which implies that g_1, g_2 and $[g_2, g_1]$ are independent and thus that $[g_3, g_1] \in \{g_1, g_2, [g_2, g_1]\}$), in which case

$$\Delta_2^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_3^{(0)}}, g_1, g_2, [g_2, g_1] \right\} = \overline{\Delta_2^{(\mathbf{j})}},\ \text{rank } \Delta_2^{(\mathbf{j})} = 6,$$

$$[\Gamma_2^{(\mathbf{j})}, \Delta_2^{(\mathbf{j})}] = \{0\} \subset \Delta_2^{(\mathbf{j})}.$$

If $\mathbf{j_2 = 1,\ j_3 \geq 3}$

$$\Gamma_2^{(\mathbf{j})} = \left\{\frac{\partial}{\partial u_2^{(1)}}, \frac{\partial}{\partial u_3^{(j_3)}}, \frac{\partial}{\partial u_3^{(j_3-1)}}, \frac{\partial}{\partial u_3^{(j_3-2)}}\right\},$$

$$\Delta_2^{(\mathbf{j})} = \left\{\frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, g_1, g_2, u_2^{(0)}[g_2, g_1] + u_3^{(0)}[g_3, g_1]\right\}.$$

As in the previous case, we deduce that

$$\Delta_2^{(\mathbf{j})} = \left\{\frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, g_1, g_2, [g_2, g_1]\right\} = \overline{\Delta_2^{(\mathbf{j})}},\ \text{rank}\ \Delta_2^{(\mathbf{j})} = 5,$$

$$[\Gamma_2^{(\mathbf{j})}, \Delta_2^{(\mathbf{j})}] = \{0\} \subset \Delta_2^{(\mathbf{j})}.$$

If $\mathbf{j_2 = 2,\ j_3 = 2}$

$$\Gamma_2^{(\mathbf{j})} = \left\{\frac{\partial}{\partial u_2^{(2)}}, \frac{\partial}{\partial u_2^{(1)}}, \frac{\partial}{\partial u_3^{(2)}}, \frac{\partial}{\partial u_3^{(1)}}\right\},$$

$$\Delta_2^{(\mathbf{j})} = \left\{\frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_3^{(0)}}, g_1, u_2^{(0)}[g_2, g_1] + u_3^{(0)}[g_3, g_1]\right\}.$$

$\Delta_2^{(\mathbf{j})}$ is involutive if, and only if, $H_{1,1}$ is involutive, with rank $H_{2,1} = 2$, in which case

$$\Delta_2^{(\mathbf{j})} = \left\{\frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_3^{(0)}}, g_1, [g_2, g_1], [g_3, g_1]\right\} = \overline{\Delta_2^{(\mathbf{j})}},\ \text{rank}\ \Delta_2^{(\mathbf{j})} = 5,$$

$$[\Gamma_2^{(\mathbf{j})}, \Delta_2^{(\mathbf{j})}] = \{0\} \subset \Delta_2^{(\mathbf{j})}.$$

If $\mathbf{j_2 = 2,\ j_3 \geq 3}$

$$\Gamma_2^{(\mathbf{j})} = \left\{\frac{\partial}{\partial u_2^{(2)}}, \frac{\partial}{\partial u_2^{(1)}}, \frac{\partial}{\partial u_3^{(j_3)}}, \frac{\partial}{\partial u_3^{(j_3-1)}}, \frac{\partial}{\partial u_3^{(j_3-2)}}\right\},$$

$$\Delta_2^{(\mathbf{j})} = \left\{\frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, g_1, u_2^{(0)}[g_2, g_1] + u_3^{(0)}[g_3, g_1]\right\}.$$

As in the previous case, we deduce that $\Delta_2^{(\mathbf{j})}$ is involutive if, and only if, $H_{1,1}$ is involutive, with rank $H_{1,1} = 2$, in which case

$$\Delta_2^{(\mathbf{j})} = \left\{\frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, g_1, [g_2, g_1], [g_3, g_1]\right\}, = \overline{\Delta_2^{(\mathbf{j})}}, \quad \text{rank}\ \Delta_2^{(\mathbf{j})} = 4,$$

$$[\Gamma_2^{(\mathbf{j})}, \Delta_2^{(\mathbf{j})}] = \{0\} \subset \Delta_2^{(\mathbf{j})}.$$

If $\mathbf{j_3 \geq j_2 \geq 3}$

$$\Gamma_2^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_2^{(j_2)}}, \frac{\partial}{\partial u_2^{(j_2-1)}}, \frac{\partial}{\partial u_2^{(j_2-2)}}, \frac{\partial}{\partial u_3^{(j_3)}}, \frac{\partial}{\partial u_3^{(j_3-1)}}, \frac{\partial}{\partial u_3^{(j_3-2)}} \right\},$$

$$\Delta_2^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, g_1, u_2^{(0)}[g_2, g_1] + u_3^{(0)}[g_3, g_1] \right\}.$$

As before, $\Delta_2^{(\mathbf{j})}$ is involutive if, and only if, $H_{1,1}$ is involutive, with rank $H_{1,1} = 2$, in which case

$$\Delta_2^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, g_1, [g_2, g_1], [g_3, g_1] \right\} = \overline{\Delta_2^{(\mathbf{j})}}, \quad \text{rank } \Delta_2^{(\mathbf{j})} = 3,$$

with $[\Gamma_2^{(\mathbf{j})}, \Delta_2^{(\mathbf{j})}] = \{0\} \subset \Delta_2^{(\mathbf{j})}$.

$\mathbf{k = 3}$: If $\mathbf{j_2 = 1}$, $\mathbf{j_3 = 1}$, we have proven so far that, for $\mathbf{j} = (0, 1, 1)$, and (3.9),

$$\Delta_0^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_1^{(0)}} \right\},$$

$$\Delta_1^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_3^{(0)}}, g_1 \right\},$$

$$\Delta_2^{(\mathbf{j})} = \begin{cases} \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_3^{(0)}}, g_1, g_2, g_3, [g_2, g_1] \right\} \\ \text{or} \\ \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_3^{(0)}}, g_1, g_2, g_3, [g_3, g_1] \right\} \end{cases}$$

are all involutive if, and only if, $H_{1,3}$ is involutive with rank $H_{1,3} = 4$ and we have rank $\Delta_2^{(\mathbf{j})} = 7$.

Since $H_{1,3}$ is involutive, it is easily verified that all the second-order brackets $[g_p, [g_q, g_r]] \in H_{1,3}$, for all $p, q, r = 1, 2, 3$, which proves that $\Delta_3^{(\mathbf{j})} = \Delta_2^{(\mathbf{j})}$, hence rank $\Delta_3^{(\mathbf{j})} =$ rank $\Delta_2^{(\mathbf{j})} = 7$. Taking account of the fact that $\Gamma_3^{(\mathbf{j})} = \Gamma_2^{(\mathbf{j})} = \Gamma_1^{(\mathbf{j})}$, we deduce that $\Delta_k^{(\mathbf{j})} = \Delta_2^{(\mathbf{j})}$ and rank $\Delta_k^{(\mathbf{j})} = 7 < n+m = 9$ for all $k \geq 3$, which contradicts the strong controllability condition (iii). Hence we must have $j_3 > 1$.

If $\mathbf{j_2 = 1,\ j_3 = 2,}$ we have proven so far that, for $\mathbf{j} = (0, 1, 2)$,

$$\Delta_0^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_1^{(0)}} \right\},$$
$$\Delta_1^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, g_1 \right\},$$
$$\Delta_2^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_3^{(0)}}, g_1, g_2, [g_2, g_1] \right\}$$

are all involutive if, and only if, $H'_{1,2}$ is involutive with rank $H'_{1,2} = 3$. Moreover, we have rank $\Delta_2^{(\mathbf{j})} = 6$ and we have

$$\Gamma_3^{(\mathbf{j})} = \Gamma_1^{(\mathbf{j})},$$
$$\Delta_3^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_3^{(0)}}, g_1, g_2, g_3, [g_2, g_1], [g_3, g_2], [g_3, [g_3, g_1]], [g_3, [g_3, g_2]] \right\}.$$

Thus, rank $H'_{1,2} = 3$ implies that rank $\Delta_3^{(\mathbf{j})} = 9 = n + m$ if rank $H_{2,3} = 6$. Hence, $\mathbf{j} = (0, 1, 2)$ is such that

$$\Gamma_k^{(\mathbf{j})} = \Gamma_3^{(\mathbf{j})}, \quad \Delta_k^{(\mathbf{j})} = \Delta_3^{(\mathbf{j})} \quad \forall k \geq 3 = k_\star^{(\mathbf{j})}$$

and $\Delta_k^{(\mathbf{j})}$ is involutive with constant rank and invariant by $\Gamma_k^{(\mathbf{j})}$ for all k, which proves that, if (3.9) holds true, if $H'_{1,2}$ is involutive, with rank $H'_{1,2} = 3$, and rank $H_{2,3} = 6$, the system (3.1) is P^2-flat with minimal prolongation $\mathbf{j} = (0, 1, 2)$.

Moreover, it is easily verified that

$$\rho_0^{(0,1,2)} = 3, \quad \rho_1^{(0,1,2)} = 3, \quad \rho_2^{(0,1,2)} = 3, \quad \rho_3^{(0,1,2)} = 3$$

and

$$\kappa_1^{(0,1,2)} = 4, \quad \kappa_2^{(0,1,2)} = 4, \quad \kappa_4^{(0,1,2)} = 3$$

which proves that the equivalent linear system is

$$y_1^{(4)} = v_1, \quad y_2^{(4)} = v_2, \quad y_3^{(3)} = v_3. \tag{A.2}$$

Since there are only two possible initialization cases (up to input permutation), the theorem is proven. □

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