OKUTSU FRAMES OF IRREDUCIBLE POLYNOMIALS OVER HENSELIAN FIELDS

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Abstract. For a henselian valued field (K,v) we establish a complete parallelism between the arithmetic properties of irreducible polynomials $F \in K[x]$, encoded by their Okutsu frames, and the valuation-theoretic properties of their induced valuations v_F on $K[x]$, encoded by their Mac Lane-Vaquié chains. This parallelism was only known for defectless irreducible polynomials.

Introduction

The pioneering work of Mac Lane on valuations on a polynomial ring [13], was inspired in a question of Ore about the design of an algorithm to compute prime ideal decomposition in number fields [22]. To solve this question, Mac Lane used the methods of [13] to develop a polynomial factorization algorithm over the completion K_v of any discrete rank-one valued field (K,v) [14]. The algorithm finds *key polynomials* of certain valuations on $K[x]$, as approximations to the irreducible factors in $K_v[x]$ of any given separable polynomial in $K[x]$.

Motivated by the computation of integral bases in finite extensions of local fields, Okutsu constructed similar approximations without using valuations on $K[x]$, nor key polynomials [21]. Each irreducible polynomial $F \in K_v[x]$ admits an *Okutsu frame*, which is a finite list of polynomials which are best possible approximations to F among all polynomials with degree smaller than a certain bound.

Fernández, Guàrdia, Montes and Nart showed that the approaches of Ore-Mac Lane and Okutsu are essentially equivalent in the discrete rank-one case [5, 6].

The techniques of Mac Lane in [13] were extended to arbitrary valued fields independently by Vaquié [24] and Herrera-Mahboub-Olalla-Spivakovsky [7, 8]. The extension of [14] to a polynomial factorization algorithm over arbitrary henselian fields is still an open problem.

Let (K,v) be a henselian valued field of an arbitrary rank. Let $\operatorname{Irr}(K)$ be the set of all monic irreducible polynomials in $K[x]$. Every $F \in \operatorname{Irr}(K)$ determines a valuation v_F on $K[x]$ given by

$$v_F(f) = v(f(\theta)), \quad \text{for all } f \in K[x],$$

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where θ is any root of F in an algebraic closure of K.

This valuation v_F is the end node of some finite *Mac Lane-Vaquié (MLV) chain* of valuations on $K(x)$

$$\mu_0 \longrightarrow \mu_1 \longrightarrow \cdots \longrightarrow \mu_r \longrightarrow v_F$$

where μ_0 admits key polynomials of degree one, and every step $\mu_i \to \mu_{i+1}$ may be either an ordinary or a limit augmentation.

We say that $F \in \operatorname{Irr}(K)$ is *defectless* if the unique extension w of v to $K[x]/(F)$ satisfies $\deg(F) = e(w/v)f(w/v)$; or, equivalently, if the MLV chain of v_F contains only ordinary augmentations [25, 15].

The results of [5, 6] were generalized in [15] to approximate defectless polynomials over arbitrary henselian fields. A certain *Okutsu equivalence relation* $\sim_{\rm ok}$ was defined on the set Dless of all defectless polynomials, so that the quotient set Dless $/\sim_{\rm ok}$ admits a parametrization by a *Mac Lane space* $\mathbb{M}$ described in terms of valuations on $K[x]$ [15, Thm. 5.14]. On the other hand, a complete parallelism was established between the arithmetic properties of any $F \in$ Dless, encoded by their Okutsu frames, and the valuation-theoretic properties of the valuation v_F, encoded by their MLV chains [15, Thms. 5.5, 5.6].

In this paper, we apply the methods of [2, Secs. 6,7] on valuative trees to generalize all these results to arbitrary irreducible polynomials in $K[x]$. The main obstacle is the presence of limit augmentations in the MLV chains of v_F. This implies that there are no longer "best possible" approximations to F of a given bounded degree, so that Okutsu frames need to be reformulated.

The structure of the paper is as follows. Section 1 collects preliminary results on key polynomials, valuative trees and their tangent spaces. Let $\mathcal{T}_{\mathbb{Q}}$ be the tree whose nodes are extensions of v to valuations on $K[x]$ taking values in the divisible hull of the group $v(K^*)$. This tree admits a certain *small-extensions closure* $\mathcal{T}_{\mathbb{Q}} \subset \mathcal{T}_{\rm sme}$ whose tangent space plays a key role.

In Section 2, we generalize [15, Thm. 5.14]. We introduce an Okutsu equivalence relation $\sim_{\rm ok}$ on the set $\mathcal{L}_{\rm fin}(\mathcal{T}_{\mathbb{Q}})$ of finite leaves of the tree $\mathcal{T}_{\mathbb{Q}}$. After a natural identification of $\mathcal{L}_{\rm fin}(\mathcal{T}_{\mathbb{Q}})$ with $\operatorname{Irr}(K)$, the quotient set $\operatorname{Irr}(K)/\sim_{\rm ok}$ may be parametrized by the set $\mathbb{T}^{\rm prim}$ of primitive tangent vectors in the tangent space of $\mathcal{T}_{\rm sme}$.

In Section 3 we show that two irreducible polynomials are Okutsu equivalent if and only if they are "sufficiently close" with respect to the classical ultrametric topology induced by v. Also, we see that the subset of $\mathbb{T}^{\rm prim}$ corresponding to defectless polynomials may be identified with the Mac Lane space $\mathbb{M}$.

In Section 4, an *Okutsu frame* of any $F \in \operatorname{Irr}(K)$ is defined as a list of sets of polynomials: $[\Phi_0, \dots, \Phi_r, \Phi_{r+1} = \{F\}]$. Each Φ_i contains monic polynomials $f \in K[x]$ of constant degree m_i, whose weighted values $v_F(f)/\deg(f)$ are cofinal in the set of all weighted v_F-values of polynomials of degree less than m_{i+1}. These degrees satisfy

$$1 = m_0 \mid m_1 \mid \cdots \mid m_r \mid m_{r+1} = \deg(F).$$

Theorems 4.4 and 4.5 show that the Okutsu frames of F and the MLV chains of v_F are essentially equivalent objects. As a consequence, we obtain still another interpretation of Okutsu frames: the set $\Phi_0 \cup \cdots \cup \Phi_r \cup \{F\}$ is a complete set of *abstract* key polynomials of v_F, in the terminology of [3, 19].

1. Valuative trees

In this section we recall some results on valuations on a polynomial ring, valuative trees and their tangent spaces.

Let (K, v) be a valued field. Let k be the residue class field, $\Gamma = v(K^*)$ the value group and $\Gamma_{\mathbb{Q}} = \Gamma \otimes \mathbb{Q}$ the divisible hull of Γ. The *rational rank* of Γ is defined as $\operatorname{rr}(\Gamma) = \dim_{\mathbb{Q}} \Gamma_{\mathbb{Q}}$.

Let $\Gamma \hookrightarrow \Lambda$ be an extension of ordered abelian groups. We write simply $\Lambda\infty$ instead of $\Lambda \cup \{\infty\}$. Consider a valuation on the polynomial ring $K[x]$

$$\mu\colon K[x] \longrightarrow \Lambda\infty$$

whose restriction to K is v.

Let $\mathfrak{p} = \operatorname{supp}(\mu) := \mu^{-1}(\infty) \in \operatorname{Spec}(K[x])$ be the support of μ. The valuation μ induces in a natural way a valuation $\overline{\mu}$ on the field of fractions of $K[x]/\mathfrak{p}$, which is $K(x)$ if $\mathfrak{p} = 0$, or $K[x]/\mathfrak{p}$ if $\mathfrak{p} = fK[x]$ for some $f \in \operatorname{Irr}(K)$.

The residue field k_μ of μ is, by definition, the residue field of $\overline{\mu}$. The field $\kappa(\mu)$ of *algebraic residues* of μ is the relative algebraic closure of k in k_μ.

We say that μ is *commensurable* (over v) if Γ_μ/Γ is a torsion group; or equivalently, $\operatorname{rr}(\Gamma_\mu/\Gamma) = 0$. In this case, there is a canonical embedding $\Gamma_\mu \hookrightarrow \Gamma_{\mathbb{Q}}$.

All valuations on $K[x]$ satisfy *Abhyankar's inequality*

$$\operatorname{rr}(\Gamma_\mu/\Gamma) + \operatorname{trdeg}(k_\mu/k) \le 1.$$

This yields a basic classification of these valuations in three families:

- μ is *valuation-algebraic* if $\operatorname{rr}(\Gamma_\mu/\Gamma) = \operatorname{trdeg}(k_\mu/k) = 0$.
- μ is incommensurable if $\operatorname{rr}(\Gamma_\mu/\Gamma) = 1$.
- μ is *residue transcendental* if $\operatorname{trdeg}(k_\mu/k) = 1$.

The valuations for which equality holds in Abhyankar's inequality, corresponding to the two latter families, are said to be *valuation-transcendental.*

All valuations with nontrivial support are valuation-algebraic, but the converse statement is false.

1.1. Key polynomials.

For any $\alpha \in \Gamma_\mu$, consider the abelian groups:

$$\mathcal{P}_\alpha = \{g \in K[x] \mid \mu(g) \ge \alpha\} \supset \mathcal{P}_\alpha^+ = \{g \in K[x] \mid \mu(g) > \alpha\}.$$

The *graded algebra of* μ is the integral domain:

$$\mathcal{G}_\mu := \operatorname{gr}_\mu(K[x]) = \bigoplus\nolimits_{\alpha\in\Gamma_\mu} \mathcal{P}_\alpha/\mathcal{P}_\alpha^+.$$

There is an *initial coefficient* mapping $\operatorname{in}_\mu\colon K[x] \to \mathcal{G}_\mu$, given by $\operatorname{in}_\mu \mathfrak{p} = 0$ and

$$\operatorname{in}_\mu g = g + \mathcal{P}^+_{\mu(g)} \in \mathcal{P}_{\mu(g)}/\mathcal{P}^+_{\mu(g)}, \qquad \text{if } g \in K[x] \setminus \mathfrak{p}.$$

The following definitions translate properties of the action of μ on $K[x]$ into algebraic relationships in the graded algebra $\mathcal{G}_\mu$.

Definition. Let $g,\, h \in K[x]$.

We say that g, h are *μ-equivalent*, and we write $g \sim_\mu h$, if $\operatorname{in}_\mu g = \operatorname{in}_\mu h$.

We say that g is *μ-divisible* by h, and we write $h \mid_\mu g$, if $\operatorname{in}_\mu h \mid \operatorname{in}_\mu g$ in $\mathcal{G}_\mu$.

We say that g is μ-irreducible if $\operatorname{in}_\mu g$ is a prime element; that is, it generates a nonzero homogeneus prime ideal.

We say that g is μ-minimal if $g \nmid_\mu f$ for all nonzero $f \in K[x]$ with $\deg(f) < \deg(g)$.

For all $g \in K[x] \setminus K$ we define the *truncation* μ_g as follows:

$$f = \sum\nolimits_{0 \le s} a_s g^s, \quad \deg(a_s) < \deg(g) \quad \Longrightarrow \quad \mu_g(f) = \min\left\{\mu\left(a_s g^s\right) \mid 0 \le s\right\}.$$

This function μ_g is not necessarily a valuation, but it is useful to characterize the μ-minimality of g. Let us recall [16, Prop. 2.3].

Lemma 1.1. *A polynomial $g \in K[x] \setminus K$ is μ-minimal if and only if $\mu_g = \mu$.*

Definition. A *(Mac Lane-Vaquié) key polynomial* for μ is a monic polynomial in $K[x]$ which is simultaneously μ-minimal and μ-irreducible.

The set of key polynomials for μ is denoted $\mathrm{KP}(\mu)$.

All $\phi \in \mathrm{KP}(\mu)$ are irreducible in $K[x]$. A *tangent direction* of μ is a μ-equivalence class $[\,\phi\,]_\mu \subset \mathrm{KP}(\mu)$ determined by all key polynomials having the same initial coefficient in $\mathcal{G}_\mu$. We denote the set of all tangent directions of μ by:

$$\mathbf{td}(\mu) = \mathrm{KP}(\mu)/\!\sim_\mu .$$

Since all polynomials in $[\,\phi\,]_\mu$ have the same degree [16, Prop. 6.6], it makes sense to consider the degree $\deg\,[\phi]_\mu$ of a tangent direction.

The basic families of valuations can be characterized as follows by their tangent directions [2, Thms. 1.2, 1.4].

- If μ is valuation-algebraic, then $\mathbf{td}(\mu) = \emptyset$. That is, $\mathrm{KP}(\mu) = \emptyset$.
- If μ is incommensurable, then $\mathbf{td}(\mu)$ is a one-element set.
- If μ is residue transcendental, then $\mathbf{td}(\mu)$ is in bijection with $\mathrm{Irr}(\kappa(\mu))$.

In the latter case, the bijection is determined by the choice of a key polynomial of minimal degree and a certain homogeneous unit in $\mathcal{G}_\mu$ [16, Sec. 6].

Definition. Suppose that $\mathrm{KP}(\mu) \ne \emptyset$ and take $\phi \in \mathrm{KP}(\mu)$ of minimal degree. The following data are independent of the choice of ϕ:

$$\deg(\mu) = \deg(\phi), \qquad \mathrm{sv}(\mu) = \mu(\phi), \qquad \mathrm{wt}(\mu) = \mathrm{sv}(\mu)/\deg(\mu).$$

They are called the *degree*, the *singular value* and the *weight* of μ, respectively.

Theorem 1.2. [16, Thm. 3.9] *If $\mathrm{KP}(\mu) \ne \emptyset$, then for all monic $f \in K[x] \setminus K$, we have $\mu(f)/\deg(f) \le \mathrm{wt}(\mu)$. Equality holds if and only if f is μ-minimal.*

1.2. **Tangent space of a valuative tree.** Let $\mathcal{T} = \mathcal{T}(\Lambda)$ be the set of all valuations

$$\mu\colon K[x] \longrightarrow \Lambda\infty,$$

whose restriction to K is v. This set admits a partial ordering. For $\mu, \nu \in \mathcal{T}$ we define

$$\mu \le \nu \iff \mu(f) \le \nu(f), \quad \forall\, f \in K[x].$$

As usual, we write $\mu < \nu$ to indicate that $\mu \le \nu$ and $\mu \ne \nu$.

This poset $\mathcal{T}$ has the structure of a tree. By this, we simply mean that the intervals

$$(-\infty, \mu\,] := \{\rho \in \mathcal{T} \mid \rho \le \mu\}$$

are totally ordered for all $\mu \in \mathcal{T}$ [17, Thm. 2.4].

A node $\mu \in \mathcal{T}$ is a *leaf* if it is a maximal element with respect to the ordering $\le$. Otherwise, we say that μ is an *inner node*.

Theorem 1.3. [17, Thm. 2.3] *A node $\mu \in \mathcal{T}$ is a leaf if and only it* $\mathrm{KP}(\mu) = \emptyset$.

All valuations with nontrivial support are leaves of $\mathcal{T}$. We call them *finite leaves.* The leaves of $\mathcal{T}$ having trivial support are called *infinite leaves.*

A finite leaf $\nu \in \mathcal{T}$ has $\mathrm{supp}(\nu) = FK[x]$ for some $F \in \mathrm{Irr}(K)$. We define

$$\deg(\nu) = \deg(F), \qquad \mathrm{sv}(\nu) = \mathrm{wt}(\nu) = \infty.$$

The following result recalls some fundamental properties of tangent directions. It follows from [24, Thm. 1.15], [17, Prop. 2.2] and [2, Prop. 2.5].

Lemma 1.4. *Let $\mu < \nu$ be two nodes in $\mathcal{T}$. Let $\mathbf{t}(\mu, \nu)$ be the set of all monic polynomials $\phi \in K[x]$ of minimal degree satisfying $\mu(\phi) < \nu(\phi)$.*

(i) If ν is an inner node or a finite leaf, then $\deg(\mu) \le \deg(\nu)$ and $\mathrm{wt}(\mu) < \mathrm{wt}(\nu)$.

(ii) The set $\mathbf{t}(\mu, \nu)$ is a tangent direction of μ. Moreover, for all $\phi \in \mathbf{t}(\mu, \nu)$, $f \in K[x]$, the equality $\mu(f) = \nu(f)$ holds if and only if $\phi \nmid_\mu f$.

(iii) If $\mu < \nu'$ for some $\nu' \in \mathcal{T}$, then

$$\mathbf{t}(\mu, \nu) = \mathbf{t}(\mu, \nu') \iff (\mu, \nu\,] \cap (\mu, \nu'\,] \neq \emptyset.$$

Definition. The *tangent space* of $\mathcal{T}$ is the set

$$\mathbb{T}(\mathcal{T}) = \{(\mu, t) \mid \mu \text{ inner node in } \mathcal{T},\ t \in \mathbf{td}(\mu)\}.$$

1.3. **Mac Lane–Vaquié chains.** For any $\phi \in \mathrm{KP}(\mu)$ and any $\gamma \in \Lambda\infty$ such that $\mu(\phi) < \gamma$, one can construct [13, Theorem 4.4] the *ordinary augmented valuation* $\nu = [\mu; \phi, \gamma] \in \mathcal{T}$, defined in terms of ϕ-expansions as

$$f = \sum\nolimits_{0 \le s} a_s \phi^s, \quad \deg(a_s) < \deg(\phi) \implies \nu(f) = \min\nolimits_{0 \le s}\{\mu(a_s) + s\gamma\}.$$

Note that $\nu(\phi) = \gamma$, $\mu < \nu$ and $\mathbf{t}(\mu, \nu) = [\,\phi\,]_\mu$.

If $\gamma < \infty$, then ϕ is a key polynomial for ν of minimal degree [16, Cor. 7.3].

Let $\mathcal{A} = (\rho_i)_{i \in A} \subset \mathcal{T}$ be a totally ordered family not admitting a maximal element. Assume that $\mathcal{A}$ is parametrized by a totally ordered set A of indices such that the mapping $A \to \mathcal{A}$ determined by $i \mapsto \rho_i$ is an isomorphism of ordered sets.

If $\deg(\rho_i)$ is stable for all sufficiently large $i \in A$, we say that $\mathcal{A}$ has *stable degree*, and we denote this stable degree by $\deg(\mathcal{A})$.

We say that $f \in K[x]$ is $\mathcal{A}$-*stable* if for some index $i \in A$, it satisfies

$$\rho_i(f) = \rho_j(f), \quad \text{for all } \ j > i.$$

We obtain a *stability function* $\rho_{\mathcal{A}}$, defined on the set of all $\mathcal{A}$-stable polynomials by $\rho_{\mathcal{A}}(f) = \max\{\rho_i(f) \mid i \in A\}$.

A *limit key polynomial* for $\mathcal{A}$ is a monic $\mathcal{A}$-unstable polynomial of minimal degree. Let $\mathrm{KP}_\infty(\mathcal{A})$ be the set of all these limit key polynomials. Since the product of stable polynomials is stable, all limit key polynomials are irreducible in $K[x]$.

The *limit degree* of $\mathcal{A}$, denoted $\deg_\infty(\mathcal{A})$, is the degree of any limit key polynomial. If $\mathrm{KP}_\infty(\mathcal{A}) = \emptyset$, we agree that $\deg_\infty(\mathcal{A}) = \infty$.

Definition. We say that $\mathcal{A}$ is an *essential continuous family* of valuations in $\mathcal{T}$ if it has stable degree and $\deg(\mathcal{A}) < \deg_\infty(\mathcal{A}) < \infty$.

Take any limit key polynomial $\phi \in \mathrm{KP}_\infty(\mathcal{A})$, and any $\gamma \in \Lambda\infty$ such that $\rho_i(\phi) < \gamma$ for all $i \in A$. We define the *limit augmentation* $\nu = [\mathcal{A}; \phi, \gamma] \in \mathcal{T}$ as the following mapping, defined in terms of ϕ-expansions:

$$f=\sum\nolimits_{0\le s} a_s\phi^s \quad \deg(a_s)<\deg(\phi) \quad \Longrightarrow \quad \nu(f)=\min_{0\le s}\{\rho_{\mathcal{A}}(a_s)+s\gamma\}.$$

Since $\deg(a_s)<\deg_\infty(\mathcal{A})$, all coefficients a_s are $\mathcal{A}$-stable.

By [24, Proposition 1.22], for every essential continuous family of valuations, ν is a valuation. Note that $\nu(\phi)=\gamma$ and $\rho_i<\nu$ for all $i\in A$. If $\gamma<\infty$, then ϕ is a key polynomial for ν of minimal degree [16, Cor. 7.13].

Consider a finite chain of valuations in $\mathcal{T}$

$$\mu_0 \ \stackrel{\phi_1,\gamma_1}{\longrightarrow}\ \mu_1 \ \stackrel{\phi_2,\gamma_2}{\longrightarrow}\ \cdots \ \longrightarrow\ \mu_{r-1} \ \stackrel{\phi_r,\gamma_r}{\longrightarrow}\ \mu_r=\nu$$

in which every node is an augmentation of the previous node, of one of the two types:

Ordinary augmentation: $\mu_{n+1}=[\mu_n;\,\phi_{n+1},\gamma_{n+1}]$, for some $\phi_{n+1}\in\mathrm{KP}(\mu_n)$.

Limit augmentation: $\mu_{n+1}=[\mathcal{A}_n;\,\phi_{n+1},\gamma_{n+1}]$, for some $\phi_{n+1}\in\mathrm{KP}_\infty(\mathcal{A}_n)$, where $\mathcal{A}_n$ is an essential continuous family containing μ_n as its first valuation.

We always consider an implicit choice of a key polynomial $\phi_0\in\mathrm{KP}(\mu_0)$ of minimal degree, and we denote $\gamma_0=\mu_0(\phi_0)$.

Therefore, for all n such that $\gamma_n<\infty$, the polynomial ϕ_n is a key polynomial for μ_n of minimal degree, and we have

$$m_n:=\deg(\mu_n)=\deg(\phi_n),\qquad \mathrm{sv}(\mu_n)=\mu_n(\phi_n)=\gamma_n.$$

Definition. A chain of mixed augmentations as above is said to be a *Mac Lane–Vaquié (MLV) chain* if $\deg(\mu_0)=1$ and every augmentation step satisfies:

- If $\mu_n\to\mu_{n+1}$ is ordinary, then $\deg(\mu_n)<\deg(\mu_{n+1})$.
- If $\mu_n\to\mu_{n+1}$ is limit, then $\deg(\mu_n)=\deg(\mathcal{A}_n)$ and $\phi_n\not\in\mathbf{t}(\mu_n,\mu_{n+1})$.

In an MLV chain, all nodes μ_n, with $n<r$, are residue transcendental valuations and satisfy $\phi_n\not\in\mathbf{t}(\mu_n,\mu_{n+1})$. In particular, $\nu(\phi_n)=\gamma_n$ for all n, by Lemma 1.4.

Theorem 1.5. [17, Thm. 4.3] *Let $\nu\in\mathcal{T}$ be an inner node or a finite leaf. Then, ν is the end node of a finite MLV chain.*

These valuations are "bien-specifiées" in Vaquié's terminology.

The main advantage of imposing the technical condition of MLV chain is that the nodes of the chain are essentially unique [17, Thm. 4.7]. Thus, we may read in the chain several data intrinsically associated to the valuation ν.

Definition. The *depth* of ν is the length of any MLV chain with end node ν. The *limit-depth* of ν is the number of limit augmentations in this MLV chain.

An inner node or a finite leaf $\nu\in\mathcal{T}$ is said to be an *inductive* valuation if it has limit-depth equal to zero.

The depth-zero valuations take the form $\nu=\omega_{a,\delta}$, for some $a\in K$ and $\delta\in\Lambda\infty$. They act as follows on $(x-a)$-expansions:

$$\nu\left(\sum\nolimits_{0\le s} a_s(x-a)^s\right)=\min\left\{v(a_s)+s\delta\mid 0\le s\right\}.$$

Clearly, $\omega_{a,\infty}$ is a finite leave of $\mathcal{T}$ with support $(x-a)K[x]$, while for $\delta<\infty$ the valuation $\omega_{a,\delta}$ is an inner node admitting $x-a$ as a key polynomial. The valuation $\omega_{a,\delta}$ is commensurable if and only if $\delta\in\Gamma_{\mathbb{Q}}\infty$. We clearly have

$$\omega_{a,\delta}\le\omega_{b,\epsilon}\iff\delta\le\min\{v(b-a),\epsilon\}. \tag{1}$$

1.4. **Parametrization of equivalence classes of extensions of v to $K[x]$.** Let $\Gamma\hookrightarrow\Lambda$ be an extension of ordered abelian groups, and let $\Delta\subset\Lambda$ be the relative divisible hull of Γ in Λ. We say that the extension $\Gamma\hookrightarrow\Lambda$ is *small* if Λ/Δ is a cyclic group. For instance, for every valuation μ extending v to $K[x]$, the extension $\Gamma\hookrightarrow\Gamma_\mu$ is small [10, Theorem 1.5].

In [11], a totally ordered real vector space $\mathbb{R}_{\text{sme}}$ was constructed, which contains all small extensions of Γ up to Γ-equivalence. This object is canonical, depending only on the set of nonzero principal convex subgroups of Γ. However, the order-preserving embeddings $\Gamma\hookrightarrow\mathbb{R}_{\text{sme}}$ are non-canonical, because they are obtained as an application of Hahn's embedding theorem [11, Section 4].

From now on, we fix any such embedding $\Gamma\hookrightarrow\mathbb{R}_{\text{sme}}$ and we identify Γ with its image in $\mathbb{R}_{\text{sme}}$. By the universal property of $\mathbb{R}_{\text{sme}}$, for any small extension $\Gamma\hookrightarrow\Lambda$, there exists an embedding $\Lambda\hookrightarrow\mathbb{R}_{\text{sme}}$ fitting into a commutative diagram

$$\begin{array}{ccc}\Lambda & & \\ \uparrow & \searrow & \\ \Gamma & \hookrightarrow & \mathbb{R}_{\text{sme}}.\end{array}$$

In particular, every extension of v to $K[x]$ is equivalent to some extension of v taking values in $\mathbb{R}_{\text{sme}}$.

Consider two elements $x,y\in\mathbb{R}_{\text{sme}}$ to be equivalent if there exists an isomorphism of ordered groups between the subgroups $\langle\,\Gamma,x\,\rangle$ and $\langle\,\Gamma,y\,\rangle$, which maps x to y and acts as the identity on Γ. There is a canonical set of representatives $\Gamma_{\text{sme}}\subset\mathbb{R}_{\text{sme}}$ of this equivalence relation. We have $\Gamma\subset\Gamma_{\mathbb{Q}}\subset\Gamma_{\text{sme}}\subset\mathbb{R}_{\text{sme}}$.

For each $x\in\Gamma_{\text{sme}}$ let us denote

$$(\Gamma_{\mathbb{Q}})_{\le x}=\{\gamma\in\Gamma_{\mathbb{Q}}\mid\gamma\le x\},\qquad(\Gamma_{\mathbb{Q}})_{\ge x}=\{\gamma\in\Gamma_{\mathbb{Q}}\mid\gamma\ge x\}$$

The set Γ_{sme} satisfies the following property, easily deduced from [11, Lemma 5.4].

Proposition 1.6. *The mapping $x\mapsto\left((\Gamma_{\mathbb{Q}})_{\le x},(\Gamma_{\mathbb{Q}})_{\ge x}\right)$ establishes an order-preserving isomorphism between Γ_{sme} and the set of all quasicuts in $\Gamma_{\mathbb{Q}}$.*

A *quasicut* in $\Gamma_{\mathbb{Q}}$ is a pair of subsets $D=(D^L,D^R)$ such that $D^L\le D^R$ (every element in D^L is less than or equal to every element in D^R), $D^L\cup D^R=\Gamma_{\mathbb{Q}}$ and $D^L\cap D^R$ contains at most one element. The set of quasicuts admits a total ordering:

$$\left(D^L,D^R\right)\le\left(E^L,E^R\right)\iff D^L\subset E^L\quad\text{and}\quad D^R\supset E^R.$$

The following result follows immediately from Proposition 1.6.

Corollary 1.7. *Every subset $S\subset\Gamma_{\mathbb{Q}}$ admits a supremum in Γ_{sme}, which we simply denote by $\sup(S)$. If S does not contain a maximal element, then $\sup(S)\notin\Gamma_{\mathbb{Q}}$.*

Indeed, if S contains a maximal element γ, then $\sup(S)=\gamma\in\Gamma_{\mathbb{Q}}$. Otherwise, $\sup(S)$ is the cut $(I,\Gamma_{\mathbb{Q}}\setminus I)$, where I is the initial segment of $\Gamma_{\mathbb{Q}}$ generated by S. Since I contains no maximal element, necessarily $I\ne(\Gamma_{\mathbb{Q}})_{\le\gamma}$ for all $\gamma\in\Gamma_{\mathbb{Q}}$.

Denote $\mathcal{T}_{\mathbb{Q}} = \mathcal{T}(\Gamma_{\mathbb{Q}})$. Consider the intermediate tree $\mathcal{T}_{\mathbb{Q}} \subset \mathcal{T}_{\rm sme} \subset \mathcal{T}(\mathbb{R}_{\rm sme})$, defined as follows

$$\mathcal{T}_{\rm sme} = \mathcal{T}_{\mathbb{Q}} \cup \{\rho \in \mathcal{T}(\mathbb{R}_{\rm sme}) \mid \rho \text{ inner node with } \operatorname{sv}(\rho) \in \Gamma_{\rm sme}\}\,.$$

Note that $\mathcal{T}_{\mathbb{Q}}$ and $\mathcal{T}_{\rm sme}$ have the same finite leaves.

The nodes of $\mathcal{T}_{\rm sme}$ parametrize the equivalence classes of valuations on $K[x]$ whose restriction to K is equivalent to v [2, Thm. 7.1]. We need to recall some particular properties of the tree $\mathcal{T}_{\rm sme}$ which will be useful in the sequel.

1.4.1. *Inner depth-zero nodes.* The inner depth-zero nodes in $\mathcal{T}_{\rm sme}$ are of the form $\omega_{a,\gamma}$ for $a \in K$ and $\gamma \in \Gamma_{\rm sme}$.

Let $-\infty = \min(\Gamma_{\rm sme})$ be the absolute minimal element in $\Gamma_{\rm sme}$, corresponding to the quasicut $(\emptyset, \Gamma_{\mathbb{Q}})$. By (1), we have

$$\omega_{a,-\infty} = \omega_{b,-\infty} \le \omega_{c,\gamma} \quad \text{for all } \ a,b,c \in K,\ \gamma \in \Gamma_{\rm sme}. \tag{2}$$

Definition. Let us denote by $\omega_{-\infty} = \omega_{a,-\infty}$ this minimal depth-zero valuation, which is independent of $a \in K$. By Theorem 1.5 and (2), $\omega_{-\infty}$ is an absolute minimal node of $\mathcal{T}_{\rm sme}$. We say that $\omega_{-\infty}$ is the *root node* of $\mathcal{T}_{\rm sme}$.

Since $\mathcal{T}_{\mathbb{Q}}$ has no minimal node, the root node $\omega_{-\infty}$ must be incommensurable. Hence, it has a unique tangent direction; actually,

$$\operatorname{KP}(\omega_{-\infty}) = \{x - a \mid a \in K\} = [x]_{\omega_{-\infty}}.$$

All inner depth-zero nodes in $\mathcal{T}_{\rm sme}$ are obtained as a single ordinary augmentation of the root node:

$$\omega_{a,\gamma} = [\omega_{-\infty};\, x - a, \gamma] \quad \text{for all } a \in K,\ \gamma \in \Gamma_{\rm sme},\ \gamma > -\infty.$$

1.4.2. *Limit augmentations.* Let $\mathcal{A} = (\rho_i)_{i\in A}$ be an essential continuous family in $\mathcal{T}_{\mathbb{Q}}$, and let $\phi \in \operatorname{KP}_\infty(\mathcal{A})$ be a limit key polynomial. By Corollary 1.7, there exists a minimal limit augmentation of $\mathcal{A}$ in $\mathcal{T}_{\rm sme}$ with respect to ϕ; namely

$$\mu_{\mathcal{A}} := [\mathcal{A};\, \phi, \gamma_{\mathcal{A}}], \qquad \gamma_{\mathcal{A}} := \sup\{\rho_i(\phi) \mid i \in A\} \in \Gamma_{\rm sme}.$$

Since $\mathcal{A}$ has no maximal element, the family $\{\rho_i(\phi) \mid i \in A\}$ has no maximal element either. By Corollary 1.7, $\gamma_{\mathcal{A}}$ does not belong to $\Gamma_{\mathbb{Q}}$.

Lemma 1.8. [2, Lem. 7.2] *The value $\gamma_{\mathcal{A}} \in \Gamma_{\rm sme} \setminus \Gamma_{\mathbb{Q}}$ and the valuation $\mu_{\mathcal{A}} \in \mathcal{T}_{\rm sme}$ are independent of the choice of the limit key polynomial ϕ in* $\operatorname{KP}_\infty(\mathcal{A})$.

Since $\mu_{\mathcal{A}}$ is incommensurable, it has a unique tangent direction. Actually,

$$\operatorname{KP}(\mu_{\mathcal{A}}) = [\phi]_{\mu_{\mathcal{A}}} = \{\phi + a \mid a \in K[x],\ \deg(a) < \deg(\phi),\ \rho_{\mathcal{A}}(a) > \gamma_{\mathcal{A}}\} = \operatorname{KP}_\infty(\mathcal{A}).$$

Also, all limit augmentations $[\mathcal{A};\, \phi, \gamma]$ are ordinary augmentations of $\mu_{\mathcal{A}}$:

$$[\mathcal{A};\, \phi, \gamma] = [\mu_{\mathcal{A}};\, \phi, \gamma] \quad \text{for all } \gamma \in \Gamma_{\rm sme},\ \gamma > \gamma_{\mathcal{A}}.$$

1.4.3. *Primitive nodes of* $\mathcal{T}_{\rm sme}$. For the ease of the reader we shall consider the depth-zero valuations as a special case of limit augmentations.

Convention. We admit the empty set $\mathcal{A}=\emptyset$ as an essential continuous family in $\mathcal{T}_{\mathbb{Q}}$. We agree that this family has

$$\gamma_{\mathcal{A}}=-\infty,\quad \mu_{\mathcal{A}}=\omega_{-\infty},\quad \mathrm{KP}_\infty(\mathcal{A})=\mathrm{KP}(\mu_{\mathcal{A}})=\{x-a\mid a\in K\}.$$

Consider the subset $\mathrm{KP}_{\rm str}(\mu)\subset\mathrm{KP}(\mu)$ of *strong* key polynomials, defined as

$$\mathrm{KP}_{\rm str}(\mu)=\{\phi\in\mathrm{KP}(\mu)\mid \deg(\phi)>\deg(\mu)\}.$$

If $\mathrm{KP}_{\rm str}(\mu)\neq\emptyset$, then μ admits more than one tangent direction; thus, it is necessarily a residue transcendental valuation. In particular, $\mathrm{sv}(\mu)\in\Gamma_{\mathbb{Q}}$ and $\mu\in\mathcal{T}_{\mathbb{Q}}$.

Definition. A *limit-primitive* node in $\mathcal{T}_{\rm sme}$ is the inner limit node $\mu_{\mathcal{A}}$ associated to an essential continuous family $\mathcal{A}$ in $\mathcal{T}_{\mathbb{Q}}$.

An *ordinary-primitive* node in $\mathcal{T}_{\rm sme}$ is an inner node $\mu\in\mathcal{T}_{\mathbb{Q}}$ such that $\mathrm{KP}_{\rm str}(\mu)\neq\emptyset$.

A *primitive* node in $\mathcal{T}_{\rm sme}$ is a node which is either limit-primitive or ordinary-primitive. Let us denote by $\mathrm{Prim}\,(\mathcal{T}_{\rm sme})$ the set of all primitive nodes.

For any inner node $\mu\in\mathcal{T}_{\rm sme}$ and any $\phi\in\mathrm{KP}(\mu)$, consider the set of all ordinary augmentations of μ with respect to ϕ:

$$\mathcal{P}_\mu(\phi)=\{[\mu;\,\phi,\gamma]\mid \gamma\in\Gamma_{\rm sme}\infty,\quad \mu(\phi)<\gamma\le\infty\}\subset\mathcal{T}_{\rm sme}.$$

Definition. Let $\rho\in\mathcal{T}_{\rm sme}$ be a primitive node. Then, we define

$$\mathcal{P}(\rho)=\begin{cases}\bigcup_{\phi\in\mathrm{KP}_{\rm str}(\rho)}\mathcal{P}_\rho(\phi), & \text{if } \rho \text{ is ordinary-primitive},\\ \{\rho\}\cup\bigcup_{\phi\in\mathrm{KP}(\rho)}\mathcal{P}_\rho(\phi), & \text{if } \rho \text{ is limit-primitive.}\end{cases}$$

Theorem 1.9. [2, Thm. 7.3] *Let* $\nu\in\mathcal{T}_{\rm sme}$ *be either an inner node, or a finite leaf. There exists a unique primitive node* $\rho\in\mathrm{Prim}(\mathcal{T}_{\rm sme})$ *such that* $\nu\in\mathcal{P}(\rho)$.

2. Okutsu equivalence of finite leaves

Let (K,v) be a valued field. Consider an algebraic closure of K and the corresponding separable closure: $K\subset K^{\rm sep}\subset\overline{K}$. Let $\overline{v}$ be an extension of v to $\overline{K}$, and let $(K^h,\overline{v})$ be the henselization of (K,v) determined by $\overline{v}$. Thus, $K^h\subset K^{\rm sep}$ is the fixed field of the decomposition group of $\overline{v}$ in the Galois group $\mathrm{Gal}(K^{\rm sep}/K)$.

In this section, we introduce a certain *Okutsu equivalence* $\sim_{\rm ok}$ on the set $\mathcal{L}_{\rm fin}(\mathcal{T}_{\mathbb{Q}})$ of finite leaves of $\mathcal{T}_{\mathbb{Q}}$ and we obtain a natural parametrization of the quotient set $\mathcal{L}_{\rm fin}(\mathcal{T}_{\mathbb{Q}})/\sim_{\rm ok}$ by a certain subset of the tangent space of $\mathcal{T}_{\rm sme}$. Also, we use the tree structure of $\mathcal{T}_{\mathbb{Q}}$ to define an ultrametric topology on the set $\mathcal{L}_{\rm fin}(\mathcal{T}_{\mathbb{Q}})$.

For any $\phi\in\mathrm{Irr}(K)$, we denote by K_ϕ the simple field extension $K[x]/\phi K[x]$.

2.1. **Finite leaves of** $\mathcal{T}_{\mathbb{Q}}$**.** Any $F\in\mathrm{Irr}(K^h)$ determines a valuation v_F on $K^h[x]$ defined as:

$$v_F(f)=v\left(f(\theta)\right)\quad\text{for all}\quad f\in K^h[x],$$

where $\theta\in\overline{K}$ is a root of F. By the henselian property, the value $v\left(f(\theta)\right)$ does not depend on the choice of θ. The support of v_F is the prime ideal $FK^h[x]$.

Let us denote by w_F the restriction of v_F to $K[x]$. The support of w_F is the prime ideal $\phi K[x]$, where the "norm" polynomial $\phi=N(F)\in\mathrm{Irr}(K)$ is uniquely determined by the equality $\big(FK^h[x]\big) \cap K[x] = \phi K[x]$. Let $\overline{w}_F$ be the valuation on K_ϕ naturally induced by w_F.

As a consequence of the results in [4, Sec. 17], the following mapping is bijective:

$$\mathrm{Irr}(K^h) \longrightarrow \mathcal{L}_{\mathrm{fin}}(\mathcal{T}_{\mathbb{Q}}), \qquad F \longmapsto w_F. \tag{3}$$

Let us now recall how to describe the extensions of v to the field K_ϕ, for an arbitrary $\phi \in \mathrm{Irr}(K)$. Since K^h/K is a separable extension, the factorization of ϕ into a product of monic irreducible polynomials in $K^h[x]$ takes the form

$$\phi = F_1 \cdots F_r,$$

with pairwise different $F_1, \dots, F_r \in \mathrm{Irr}(K^h)$, whose norm is $N(F_i) = \phi$ for all i.

Theorem 2.1. [18, Cor. 3.2] *The extensions of v to K_ϕ are precisely $\overline{w}_{F_1}, \dots, \overline{w}_{F_r}$.*

2.2. **Primitive tangent space of $\mathcal{T}_{\mathrm{sme}}$ and finite leaves.** By Theorem 1.9, each finite leaf $\mathfrak{f} \in \mathcal{L}_{\mathrm{fin}}(\mathcal{T}_{\mathbb{Q}})$ belongs to $\mathcal{P}(\rho)$ for a unique primitive node $\rho \in \mathcal{T}_{\mathrm{sme}}$. We say that ρ is the *previous primitive node* of $\mathfrak{f}$, and we denote it by $\rho = \rho(\mathfrak{f}) \in \mathcal{T}_{\mathrm{sme}}$.

Since $\mathfrak{f}$ is commensurable and has nontrivial support, we have $\rho(\mathfrak{f}) \neq \mathfrak{f}$. Indeed, the ordinary-primitive nodes have trivial support, while the limit-primitive nodes are incommensurable. Consider a finite MLV chain whose end node is $\mathfrak{f}$:

$$\mu_0 \ \stackrel{\phi_1,\gamma_1}{\longrightarrow}\ \mu_1 \ \longrightarrow\ \cdots\ \longrightarrow\ \mu_{r-1} \ \stackrel{\phi_r,\infty}{\longrightarrow}\ \mu_r = \mathfrak{f}.$$

If $\mathfrak{f}$ has depth zero, then $\mathfrak{f} = \mu_0 = \omega_{a,\infty}$ for some $a \in K$. Then,

$$\rho(\mathfrak{f}) = \omega_{-\infty}, \qquad \deg(\mathfrak{f}) = \deg(\rho(\mathfrak{f})) = 1.$$

Suppose that $\mathfrak{f}$ has a positive depth and the last augmentation $\mu_{r-1} \to \mathfrak{f}$ is ordinary; that is, $\mathfrak{f} = [\mu_{r-1};\, \phi_r, \infty]$. By the definition of a MLV chain, $\phi_r \in \mathrm{KP}(\mu_{r-1})$ and $\deg(\mathfrak{f}) = \deg(\phi_r) > \deg(\mu_{r-1})$; thus, μ_{r-1} is an ordinary-primitive node and

$$\rho(\mathfrak{f}) = \mu_{r-1}, \qquad \deg(\mathfrak{f}) = \deg(\phi_r) > \deg(\rho(\mathfrak{f})). \tag{4}$$

Suppose that $\mathfrak{f}$ has a positive depth and $\mu_{r-1} \to \mathfrak{f}$ is a limit augmentation; that is, $\mathfrak{f} = [\mathcal{A};\, \phi_r, \infty]$ for some essential continuous family $\mathcal{A}$ in $\mathcal{T}_{\mathbb{Q}}$ whose first valuation is μ_{r-1}. Then,

$$\rho(\mathfrak{f}) = \mu_{\mathcal{A}}, \qquad \deg(\mathfrak{f}) = \deg(\phi_r) = \deg(\rho(\mathfrak{f})).$$

Definition. On the set $\mathcal{L}_{\mathrm{fin}}(\mathcal{T}_{\mathbb{Q}})$, we define the equivalence relation

$$\mathfrak{f} \sim_{\mathrm{ok}} \mathfrak{f}' \iff \rho(\mathfrak{f}) = \rho(\mathfrak{f}') \ \text{ and } \ \mathbf{t}(\rho(\mathfrak{f}), \mathfrak{f}) = \mathbf{t}(\rho(\mathfrak{f}), \mathfrak{f}').$$

In this case, we say that the finite leaves $\mathfrak{f}$ and $\mathfrak{f}'$ are *Okutsu equivalent.*

We denote by $[\,\mathfrak{f}\,]_{\mathrm{ok}}$ the Okutsu equivalence class of $\mathfrak{f}$.

Definition. Let $\mathbb{T}^{\mathrm{prim}}$ be the subset of the tangent space of $\mathcal{T}_{\mathrm{sme}}$ consisting of all tangent vectors based on primitive nodes:

$$\mathbb{T}^{\mathrm{prim}} = \{(\rho, t) \mid \rho \in \mathrm{Prim}(\mathcal{T}_{\mathrm{sme}}),\ t \in \mathbf{td}(\rho)\}\,.$$

The next result follows immediately from Theorem 1.9 and the definition of $\sim_{\mathrm{ok}}$.

Theorem 2.2. *There is a canonical bijective mapping*

$$\mathbb{T}^{\mathrm{prim}} \longrightarrow \mathcal{L}_{\mathrm{fin}}(\mathcal{T}_{\mathbb{Q}})/\sim_{\mathrm{ok}}, \qquad (\rho, [\phi]_\rho) \longmapsto [\,[\rho;\, \phi, \infty]\,]_{\mathrm{ok}}\,,$$

whose inverse mapping is: $[\,\mathfrak{f}\,]_{\mathrm{ok}} \mapsto (\rho(\mathfrak{f}), \mathbf{t}(\rho(\mathfrak{f}), \mathfrak{f}))$.

Combined with the identification $\mathcal{L}_{\rm fin}(\mathcal{T}_{\mathbb{Q}})=\operatorname{Irr}(K)$ given by (3), this result generalizes [15, Thm. 5.14] to the most general case, where no assumption at all is made on the base valued field (K,v).

Let us quote some basic properties of the equivalence relation $\sim_{\rm ok}$.

Lemma 2.3. *Let $\mathfrak{f}$, $\mathfrak{f}'$ be two finite leaves in $\mathcal{T}_{\mathbb{Q}}$, and let* $\operatorname{supp}(\mathfrak{f})=\phi K[x]$.

(i) *The polynomial ϕ belongs to* $\operatorname{KP}(\rho(\mathfrak{f}))$ *and* $\mathbf{t}(\rho(\mathfrak{f}),\mathfrak{f})=[\phi]_{\rho(\mathfrak{f})}$.

(ii) *Let ρ be a primitive node. Then, $\rho=\rho(\mathfrak{f})$ if and only if $\rho<\mathfrak{f}$, $\phi\in\operatorname{KP}(\rho)$ and*

$$\deg(\phi)>\deg(\rho),\quad \textit{if } \rho \textit{ is ordinary-primitive.} \tag{5}$$

(iii) *If $\rho(\mathfrak{f})$ is a limit-primitive node, then $\mathfrak{f}\sim_{\rm ok}\mathfrak{f}'$ if and only if $\rho(\mathfrak{f})=\rho(\mathfrak{f}')$.*

(iv) *If $\mathfrak{f}\sim_{\rm ok}\mathfrak{f}'$, then* $\deg(\mathfrak{f})=\deg(\mathfrak{f}')$.

Proof. By the definition of $\rho(\mathfrak{f})$, the valuation $\mathfrak{f}=[\rho(\mathfrak{f});\,\varphi,\gamma]$ is an ordinary augmentation of $\rho(\mathfrak{f})$, for some $\varphi\in\operatorname{KP}(\rho(\mathfrak{f}))$, $\gamma\in\Gamma_{\rm sme}\infty$. Since $\mathfrak{f}$ has nontrivial support, we have necessarily $\gamma=\infty$. Then, $\varphi\in K[x]$ is a monic irreducible polynomial such that $\operatorname{supp}(\mathfrak{f})=\varphi K[x]$. This implies $\varphi=\phi$, and this proves (i).

Let ρ be a primitive node. If $\rho=\rho(\mathfrak{f})$, then $\rho<\mathfrak{f}$ and $\phi\in\operatorname{KP}(\rho)$ by (i). Also, (5) follows from (4).

Conversely, suppose that $\rho<\mathfrak{f}$ and $\phi\in\operatorname{KP}(\rho)$. Since $\rho(\phi)<\infty=\mathfrak{f}(\phi)$, Lemma 1.4 shows that $\mathbf{t}(\rho,\mathfrak{f})=[\varphi]_\rho$ for some $\varphi\in\operatorname{KP}(\rho)$ such that $\varphi\mid_\rho\phi$. By [16, Prop. 6.6], we have $\varphi\sim_\rho\phi$, so that $\mathbf{t}(\rho,\mathfrak{f})=[\phi]_\rho$. This implies, $[\rho;\,\phi,\infty]=\mathfrak{f}$ because both valuations coincide on ϕ-expansions, again by Lemma 1.4.

If ρ is limit-primitive, or ρ is ordinary-primitive and (5) holds, then $\mathfrak{f}$ belongs to $\mathcal{P}(\rho)$ and this implies $\rho=\rho(\mathfrak{f})$ by Theorem 1.9. This ends the proof of (ii).

If $\rho(\mathfrak{f})$ is a limit-primitive node, then it has a unique tangent direction; thus, for all $\mathfrak{f}'$ such that $\rho(\mathfrak{f})<\mathfrak{f}'$ we have necessarily $\mathbf{t}(\rho(\mathfrak{f}),\mathfrak{f})=\mathbf{t}(\rho(\mathfrak{f}),\mathfrak{f}')$. This proves (iii).

Finally, if $\mathfrak{f}\sim_{\rm ok}\mathfrak{f}'$, then (iv) follows from (i), because $\deg(\mathfrak{f})=\deg\,\mathbf{t}(\rho(\mathfrak{f}),\mathfrak{f})=\deg\,\mathbf{t}(\rho(\mathfrak{f}),\mathfrak{f}')=\deg(\mathfrak{f}')$. ☐

2.3. **Ultrametric topology on the set of finite leaves.** Let us introduce an ultrametric topology on the set $\mathcal{L}_{\rm fin}(\mathcal{T}_{\mathbb{Q}})$.

Definition. For all $\mathfrak{f},\mathfrak{g}\in\mathcal{L}_{\rm fin}(\mathcal{T}_{\mathbb{Q}})$, we define

$$u(\mathfrak{f},\mathfrak{g})=\begin{cases}\operatorname{wt}(\mathfrak{f}\wedge\mathfrak{g}), & \text{if } \mathfrak{f}\neq\mathfrak{g},\\ \infty, & \text{if } \mathfrak{f}=\mathfrak{g},\end{cases}$$

where $\mathfrak{f}\wedge\mathfrak{g}$ is the greatest common lower node in the tree $\mathcal{T}_{\mathbb{Q}}$ [2, Sec. 5.6].

Lemma 2.4. *The function $u\colon\mathcal{L}_{\rm fin}(\mathcal{T}_{\mathbb{Q}})\times\mathcal{L}_{\rm fin}(\mathcal{T}_{\mathbb{Q}})\to\Gamma_{\mathbb{Q}}\infty$ has the following properties:*

(i) $u(\mathfrak{f},\mathfrak{g})=\infty\iff\mathfrak{f}=\mathfrak{g}$.

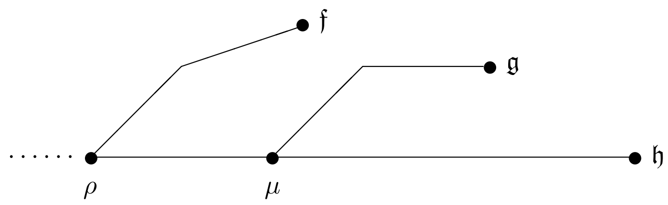
(ii) $u(\mathfrak{f},\mathfrak{g})\ge\min\{u(\mathfrak{f},\mathfrak{h}),u(\mathfrak{h},\mathfrak{g})\}$ *for all* $\mathfrak{h}\in\mathcal{L}_{\rm fin}(\mathcal{T}_{\mathbb{Q}})$.

(iii) $u(\mathfrak{f},\mathfrak{g})=u(\mathfrak{g},\mathfrak{f})$.

Proof. Conditions (i) and (iii) follow immediately from the definition of u.

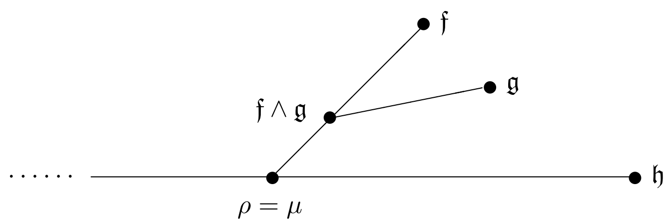
Let us prove (ii). If there is any coincidence between the three leaves $\mathfrak{f},\mathfrak{g},\mathfrak{h}$, then the statement of (ii) is obvious. Let us suppose that these leaves are pairwise different.

Let $\rho=\mathfrak{h}\wedge\mathfrak{f}$, $\mu=\mathfrak{h}\wedge\mathfrak{g}$. Since $u(\mathfrak{f},\mathfrak{g})=u(\mathfrak{g},\mathfrak{f})$, we may assume that $\rho\le\mu$. The relative position of all these nodes in the tree $\mathcal{T}_{\mathbb{Q}}$ is the following:



We have $u(\mathfrak{f},\mathfrak{g})=u(\mathfrak{f},\mathfrak{h})=\operatorname{wt}(\rho)$ and $u(\mathfrak{g},\mathfrak{h})=\operatorname{wt}(\mu)$. On the other hand, Lemma 1.4 shows that $\operatorname{wt}(\rho)<\operatorname{wt}(\mu)$, if $\rho<\mu$. Thus, condition (ii) holds. ☐

If $u(\mathfrak{f},\mathfrak{h})\ne u(\mathfrak{h},\mathfrak{g})$, then $\rho<\mu$ and $u(\mathfrak{f},\mathfrak{g})=\operatorname{wt}(\rho)=\min\{u(\mathfrak{f},\mathfrak{h}),u(\mathfrak{h},\mathfrak{g})\}$. The inequality $u(\mathfrak{f},\mathfrak{g})>\min\{u(\mathfrak{f},\mathfrak{h}),u(\mathfrak{h},\mathfrak{g})\}$ holds in the following situation:



As a consequence, the function u provides an structure of *ultrametric space* on the set $\mathcal{L}_{\rm fin}(\mathcal{T}_\mathbb{Q})$, and a corresponding topology [9, Sec. 1.3]. A basis for the topology is formed by the balls

$$B_\gamma(\mathfrak{f}):=\{\mathfrak{g}\in\mathcal{L}_{\rm fin}(\mathcal{T}_\mathbb{Q})\mid u(\mathfrak{f},\mathfrak{g})>\gamma\}, \quad \mathfrak{f}\in\mathcal{L}_{\rm fin}(\mathcal{T}_\mathbb{Q}), \quad \gamma\in\Gamma_\mathbb{Q}.$$

In this topology, two finite leaves $\mathfrak{f},\mathfrak{g}$ are "close" if the value $u(\mathfrak{f},\mathfrak{g})$ is "large".

Finally, let us remark that two finite leaves in $\mathcal{L}_{\rm fin}(\mathcal{T}_\mathbb{Q})$ are Okutsu equivalent if and only if they are "close enough" with respect to the ultrametric topology.

Lemma 2.5. *Let $\mathfrak{f},\mathfrak{g}\in\mathcal{L}_{\rm fin}(\mathcal{T}_\mathbb{Q})$. Then, the following conditions are equivalent.*

(i) $\mathfrak{f}\sim_{\rm ok}\mathfrak{g}$,
(ii) $\rho(\mathfrak{f}),\rho(\mathfrak{g})<\mathfrak{f}\wedge\mathfrak{g}$,
(iii) $u(\mathfrak{f},\mathfrak{g})>\max\{\operatorname{wt}(\rho(\mathfrak{f})),\operatorname{wt}(\rho(\mathfrak{g}))\}$.

Proof. The equivalence between (i) and (ii) follows from Theorem 1.9. The equivalence between (ii) and (iii) follows from Lemma 1.4. ☐

3. Okutsu equivalence in the henselian case

In this section, we suppose that (K,v) is henselian. We fix an algebraic closure $\overline{K}$ of K and we still denote by v the unique extension of v to $\overline{K}$.

Recall the canonical bijection between $\operatorname{Irr}(K)$ and $\mathcal{L}_{\rm fin}(\mathcal{T}_\mathbb{Q})$ given in (3):

$$\operatorname{Irr}(K)\longrightarrow\mathcal{L}_{\rm fin}(\mathcal{T}_\mathbb{Q}), \qquad F\longmapsto v_F. \tag{6}$$

Our first aim is to show that, under the identification $\mathcal{L}_{\rm fin}(\mathcal{T}_\mathbb{Q})=\operatorname{Irr}(K)$, the ultrametric topology on the set $\mathcal{L}_{\rm fin}(\mathcal{T}_\mathbb{Q})$ introduced in Section 2 coincides with the classical topology induced by v on $\operatorname{Irr}(K)\subset K[x]$.

The following result was proved in [15] for defectless polynomials and extended to arbitrary irreducible polynimials in [18, Thm. 4.4].

Theorem 3.1. *Let $F \in \operatorname{Irr}(K)$ and $\phi \in \operatorname{KP}(\mu)$ for some valuation μ on $K[x]$. Then,*

$$\phi \mid_\mu F \iff \mu < v_F \quad \text{and} \quad \mathbf{t}(\mu, v_F) = [\,\phi\,]_\mu .$$

In this case, $F \sim_\mu \phi^\ell$ with $\ell = \deg(F)/\deg(\phi)$.

Corollary 3.2. *Let $F \in \operatorname{Irr}(K)$. Then, F is μ-minimal for all valuations $\mu < v_F$.*

Proof. Let $\mathbf{t}(\mu, v_F) = [\phi]_\mu$. Since $\mu(F) < \infty = v_F(F)$, Lemma 1.4 shows that $\phi \mid_\mu F$. By Theorem 3.1, $F \sim_\mu \phi^\ell$ with $\ell = \deg(F)/\deg(\phi)$. Since ϕ is μ-minimal, Theorem 1.2 shows that

$$\frac{\mu(F)}{\deg(F)} = \frac{\mu(\phi)}{\deg(\phi)} = \operatorname{wt}(\mu).$$

Hence, F is μ-minimal, again by Theorem 1.2. □

As another consequence of Theorem 3.1, the ultrametric distance u on $\operatorname{Irr}(K) = \mathcal{L}_{\mathrm{fin}}(\mathcal{T}_{\mathrm{sme}})$ is given by a classical formula.

Corollary 3.3. *For all $F, G \in \operatorname{Irr}(K)$ we have*

$$u(F,G) := u(v_F, v_G) = v_F(G)/\deg(G) = v\left(\operatorname{Res}(F,G)\right),$$

where $\operatorname{Res}(F,G)$ is the resultant of the two polynomials.

Proof. If $F = G$, we have $v_F(F) = \infty = u(F,F)$. If $F \neq G$, then $\mu = v_F \wedge v_G$ is an inner node of $\mathcal{T}_{\mathbb{Q}}$ satisfying $\mu < v_F$, $\mu < v_G$ and $\mathbf{t}(\mu, v_F) \neq \mathbf{t}(\mu, v_G)$.

Let $\mathbf{t}(\mu, v_F) = [\phi]_\mu$, $\mathbf{t}(\mu, v_G) = [\varphi]_\mu$, so that $\phi \not\sim_\mu \varphi$. By Theorem 3.1,

$$G \sim_\mu \varphi^\ell, \quad \ell = \deg(G)/\deg(\varphi).$$

Hence, $\phi \nmid_\mu G$ and this implies $\mu(G) = v_F(G)$ by Lemma 1.4. On the other hand, G is μ-minimal by Corollary 3.2. By Theorem 1.2,

$$\frac{v_F(G)}{\deg(G)} = \frac{\mu(G)}{\deg(G)} = \operatorname{wt}(\mu) = u(F,G).$$

The equality $v_F(G)/\deg(G) = v\left(\operatorname{Res}(F,G)\right)$ is well-known. □

In particular, the ultrametric topology on $\operatorname{Irr}(K)$ coincides with the classical topology induced by the valuation v (see [23, Chapter 3], [9]).

Definition. We denote by $\rho_F := \rho(\mathfrak{f})$ the previous primitive node of the leaf $\mathfrak{f} = v_F$.

Corollary 3.4. *If F, $G \in \operatorname{Irr}(K)$ have the same degree, then the following conditions are equivalent.*

(a) $v_F \sim_{\mathrm{ok}} v_G$.
(b) $u(F,G) > \max\{\operatorname{wt}(\rho_F), \operatorname{wt}(\rho_G)\}$.
(c) $F \sim_{\rho_F} G$.

Proof. Lemma 2.5 shows that conditions (a) and (b) are equivalent.

Since $\deg(F) = \deg(G)$, we have $\deg(F - G) < \deg(F)$, so that

$$\rho_F(F - G) = v_F(F - G) = v_F(G).$$

Thus, condition (c) is equivalent to $v_F(G) > \rho_F(F) = \deg(F)\operatorname{wt}(\rho_F)$, which is equivalent to (b) by Corollary 3.3 and the symmetry of the argument. □

An obvious comparison of Corollary 3.4 with [15, Lem. 5.13] shows that the restriction of the equivalence relation $\sim_{\text{ok}}$ to the subset $\text{Dless}(K) \subset \text{Irr}(K)$ of defectless polyomials, coincides with the Okutsu equivalence defined in [15].

Definition. An $F \in \text{Irr}(K)$ is said to be *defectless* if $\deg(F) = e(\overline{v}_F/v)f(\overline{v}_F/v)$, where $\overline{v}_F$ is the valuation on K_F induced by v_F.

Vaquié characterized this property as follows [25], [17, Cor. 6.1].

Theorem 3.5. *An $F \in \text{Irr}(K)$ is defectless if and only if v_F is inductive.*

Since v_F is an ordinary augmentation of its previous primitive node ρ_F, we see that F is defectless if and only if ρ_F is inductive.

Therefore, after identifying $\text{Irr}(K) = \mathcal{L}_{\text{fin}}(\mathcal{T}_{\mathbb{Q}})$ through (6), Theorem 2.2 yields a bijection between $\text{Dless}(K)/\sim_{\text{ok}}$ and the following subset of $\mathbb{T}^{\text{prim}}$:

$$\mathbb{T}^{\text{ind}} := \left\{(\rho, t) \in \mathbb{T}^{\text{prim}} \mid \rho \text{ is inductive}\right\}.$$

This subset $\mathbb{T}^{\text{ind}}$ may be easily identified with the Mac Lane space $\mathbb{M}$ of [15]. Therefore, even in the henselian case, Theorem 2.2 extends [15, Thm. 5.14] to a parametrization of Okutsu equivalence classes of arbitrary irreducible polynomials by a certain subset of the tangent space of $\mathcal{T}_{\text{sme}}$.

4. Okutsu frames over henselian fields

We keep assuming that our valued field (K, v) is henselian. Let us fix some $F \in \text{Irr}(K)$ of degree $n > 1$. We define the *weight* of a non-constant $g \in K[x]$ as

$$\text{wt}(g) := v_F(g)/\deg(g) \in \Gamma_{\mathbb{Q}}.$$

4.1. Definition of Okutsu frames.
For every integer $1 < m \le n$, consider the set

$$W_m(F) = \{\text{wt}(g) \mid g \in K[x] \text{ monic},\ 0 < \deg(g) < m\} \subset \Gamma_{\mathbb{Q}}.$$

The polynomial F is defectless if and only if all these subsets $W_m(F)$ contain a maximal value [25], [15, Thm. 5.7].

In Section 1.4, we defined a certain set Γ_{sme} containing $\Gamma_{\mathbb{Q}}$, and we showed the existence of the following supremums:

$$w_m(F) := \sup\left(W_m(F)\right) \in \Gamma_{\text{sme}}, \qquad 1 < m \le n.$$

By Corollary 1.7, if $W_m(F)$ does not contain a maximal element, then $w_m(F) \notin \Gamma_{\mathbb{Q}}$.

Lemma 4.1. *Let $\theta \in \overline{K}$ be a root of F in $\overline{K}$ and consider Krasner's constant*

$$\Omega(F) = \max\left\{v(\theta - \theta') \mid \theta' \text{ root of } F,\ \theta' \ne \theta\right\} \in \Gamma_{\mathbb{Q}}.$$

If F is separable, then $w_n(F) \le \Omega(F)$.

Proof. Suppose that $\text{wt}(g) > \Omega(F)$ for some monic $g \in K[x]$ such that $0 < \deg(g) < n$. Clearly, $\text{wt}(g) = v(g(\theta))/\deg(g)$ is the average of all $v(\theta - \alpha)$ for α running in the multiset $Z(g)$ of roots of g in $\overline{K}$, counting multiplicities. Hence, we must have $v(\theta - \alpha) > \Omega(F)$ for some $\alpha \in Z(g)$.

By Krasner's lemma, θ is purely inseparable over $K(\alpha)$. Since θ is separable over K, we must have $\theta \in K(\alpha)$ and this contradicts the fact that $\deg(g) < n$. ☐

If F is inseparable and has defect, then the set $W_n(F)$ may be unbounded in $\Gamma_{\mathbb{Q}}$ in which case, $w_n(F) = \max(\Gamma_{\text{sme}})$ corresponds to the quasicut $(\Gamma_{\mathbb{Q}}, \emptyset)$.

Lemma 4.2. *If there exists a maximal element in $W_n(F)$, then any monic $\phi \in K[x]$ of minimal degree satisfying* $\operatorname{wt}(\phi) = w_n(F)$ *is irreducible in $K[x]$.*

Proof. Suppose $\phi = ab$ for some monic $a, b \in K[x]$ with $\deg(a), \deg(b) < \deg(\phi)$. By the minimality of $\deg(\phi)$, we have $\operatorname{wt}(a),\ \operatorname{wt}(b) < w_n(F)$; thus,

$$w_n(F) = \operatorname{wt}(\phi) = \frac{v_F(a) + v_F(b)}{\deg(\phi)} < \frac{\deg(a)w_n(F) + \deg(b)w_n(F)}{\deg(\phi)} = w_n(F),$$

which is a contradiction. ☐

Suppose that $\max(W_n(F))$ does not exist. For all weighted values $\beta \in W_n$, let $\deg(\beta)$ be the minimal $\ell \in \mathbb{N} \cap [1, n)$ such that there exists a monic $g \in K[x]$ of degree ℓ such that $\beta = \operatorname{wt}(g)$. Consider the minimal $m \in \mathbb{N} \cap [1, n)$ such that there exists a totally ordered cofinal family of constant degree m:

$$\mathcal{B} = (\beta_i)_{i \in B} \subset W_n(F), \quad \deg(\beta_i) = m,\ \forall\, i \in B.$$

We may assume that the set of indices B is well-ordered and the mapping $i \mapsto \beta_i$ is an isomorphism of ordered sets between B and our family $(\beta_i)_{i \in B}$.

For all $i \in B$, choose a monic $\chi_i \in K[x]$ of degree m such that $\beta_i = \operatorname{wt}(\chi_i)$.

Let $A \subset B$ be the subset of all indices $i \in B$ such that χ_i is irreducible. Then the subfamily $(\beta_i)_{i \in A}$ is a final segment of $\mathcal{B}$.

Indeed, by the minimality of m, there exists $\beta_i \in \mathcal{B}$ such that $\operatorname{wt}(g) < \beta_i$ for all monic $g \in K[x]$ of degree less than m. Then, similar arguments to those used in the proof of Lemma 4.2 show that χ_j is irreducible for all $j \geq i$.

In particular, the family $(\beta_i)_{i \in A}$ is still cofinal in $W_n(F)$.

Consider the following set of monic irreducible polynomials of constant degree:

$$\Phi = \begin{cases} \{\chi_i \mid i \in A\}, & \text{if } \nexists \max\left(W_n(F)\right), \\ \{\phi\}, & \operatorname{wt}(\phi) = \max\left(W_n(F)\right), \text{ otherwise.} \end{cases}$$

Let m be the constant degree of the polynomials in the set Φ. We may apply this construction to find a set Φ' of monic irreducible polynomials of constant degree optimizing the weighted values in $W_m(F)$. An iteration of this procedure leads to a finite sequence of such sets of polynomials

$$[\Phi_0, \Phi_1, \ldots, \Phi_r, \Phi_{r+1} = \{F\}], \tag{7}$$

whose degrees grow strictly:

$$1 = m_0 < m_1 < \cdots < m_{r+1} = n, \qquad m_\ell = \deg\left(\Phi_\ell\right), \quad 0 \leq \ell \leq r+1, \tag{8}$$

and the following property is satisfied: for any index $0 \leq \ell \leq r$ and any monic polynomial $g \in K[x]$ with $0 < \deg(g) < m_{\ell+1}$, there exists $\phi \in \Phi_\ell$ such that

$$\operatorname{wt}(g) \leq \operatorname{wt}(\phi). \tag{9}$$

Definition. An *Okutsu frame* of F is a list of monic irreducible polynomials as in (7), having degrees as in (8), and satisfying the fundamental property (9).

Clearly, we may replace each Φ_ℓ with a suitable subset so that the following additional properties are satisfied, for all $0 \leq \ell \leq r$:

(OF1) $\#\Phi_\ell = 1$ whenever $\max\left(W_{m_{\ell+1}}\right)$ exists.

(OF2) If $\max\left(W_{m_{\ell+1}}\right)$ does not exist, we may consider a total ordering on Φ_ℓ determined by the action of v_F:

$$\phi < \phi' \iff v_F(\phi) < v_F(\phi').$$

(OF3) For all $\phi \in \Phi_\ell$, $\varphi \in \Phi_{\ell+1}$, we have $\operatorname{wt}(\phi) < \operatorname{wt}(\varphi)$.

From now on, we shall assume that our Okutsu frames satisfy these additional properties. Note that $w_{m_1}(F) < \cdots < w_{m_{r+1}}(F) = w_n(F)$ and

$$w_{m_{\ell+1}}(F) = \begin{cases} \operatorname{wt}(\phi), & \text{if } \Phi_\ell = \{\phi\}, \\ \sup\{\operatorname{wt}(\chi_i) \mid i \in A\}, & \text{if } \Phi_\ell = \{\chi_i \mid i \in A\}. \end{cases}$$

4.2. **Okutsu frames and Mac Lane–Vaquié chains.** In this section, we show that any MLV chain of v_F determines an Okutsu frame of F.

Lemma 4.3. *Let $[\Phi_0, \Phi_1, \dots, \Phi_r, \{F\}]$ be an Okutsu frame of F. For $0 \le \ell \le r$, let*

$$V_{m_\ell} = \{v_F(f) \mid f \in K[x] \text{ monic of degree } m_\ell\}.$$

Then, there exists $\max\left(W_{m_{\ell+1}}\right)$ if and only if there exists $\max\left(V_{m_\ell}\right)$.

Proof. If $\max\left(W_{m_{\ell+1}}\right)$ exists, then $\Phi_\ell = \{\phi\}$ with $\max\left(W_{m_{\ell+1}}\right) = \operatorname{wt}(\phi)$, by the property (9). Obviously, $\max\left(V_{m_\ell}\right) = v_F(\phi)$.

Suppose now that $\max\left(V_{m_\ell}\right) = v_F(\varphi)$ for some monic $\varphi \in K[x]$ of degree m_ℓ. By (9), for all monic $f \in K[x]$ with $\deg(f) < m_{\ell+1}$ we have $\operatorname{wt}(f) \le \operatorname{wt}(\phi)$, for some $\phi \in \Phi_\ell$. Since $v_F(\phi) \le v_F(\varphi)$, we deduce that $\operatorname{wt}(\varphi) = \max\left(W_{m_{\ell+1}}\right)$. □

Theorem 4.4. *Let $F \in \operatorname{Irr}(K)$ and consider a MLV chain of v_F:*

$$\mu_0 \xrightarrow{\phi_1,\gamma_1} \mu_1 \longrightarrow \cdots \xrightarrow{\phi_{r-1},\gamma_{r-1}} \mu_{r-1} \xrightarrow{\phi_r,\gamma_r} \mu_r \xrightarrow{F,\infty} \mu_{r+1} = v_F.$$

Take $\Phi_{r+1} = \{F\}$. For each $0 \le \ell \le r$, consider the following set of monic irreducible polynomials of constant degree:

$$\Phi_\ell = \begin{cases} \{\phi_\ell\}, & \text{if } \mu_{\ell+1} = [\mu_\ell;\, \phi_{\ell+1}, \gamma_{\ell+1}] \text{ is an ordinary augmentation,} \\ \{\chi_i \mid i \in A_\ell\}, & \text{if } \mu_{\ell+1} = [\mathcal{A}_\ell;\, \phi_{\ell+1}, \gamma_{\ell+1}] \text{ is a limit augmentation,} \end{cases}$$

where $\mathcal{A}_\ell = (\rho_i)_{i \in A_\ell}$ with $\rho_i = [\mu_\ell;\, \chi_i, v_F(\chi_i)]$ for all $i \in A_\ell$. Then, $[\Phi_0, \dots, \Phi_r, \Phi_{r+1}]$ is an Okutsu frame of F.

Proof. Let us first prove that the fundamental property (9) holds for all $0 \le \ell \le r$.

Suppose that $\mu_\ell \to \mu_{\ell+1}$ is an ordinary augmentation. By the definition of a MLV chain, we have $m_\ell = \deg(\phi_\ell) < m_{\ell+1} = \deg(\phi_{\ell+1})$. Hence, for all monic $g \in K[x]$ of degree $0 < \deg(g) < m_{\ell+1}$, we have simultaneously $\phi_{\ell+1} \nmid_{\mu_\ell} \phi_\ell$ and $\phi_{\ell+1} \nmid_{\mu_\ell} g$. Since $\mathbf{t}(\mu_\ell, v_F) = [\phi_{\ell+1}]_{\mu_\ell}$, Lemma 1.4 shows that $\mu_\ell(\phi_\ell) = v_F(\phi_\ell)$ and $\mu_\ell(g) = v_F(g)$.

Now, since ϕ_ℓ is a key polynomial for μ_ℓ, Theorem 1.2 implies

$$\operatorname{wt}(g) = \frac{\mu_\ell(g)}{\deg(g)} \le \frac{\mu_\ell(\phi_\ell)}{m_\ell} = \operatorname{wt}(\phi_\ell).$$

This ends the proof concerning all ordinary augmentation steps.

Suppose that $\mu_\ell \to \mu_{\ell+1}$ is a limit augmentation. We mimic the above arguments, just by replacing the pair μ_ℓ, ϕ_ℓ with the pair ρ_i, χ_i for a sufficiently large $i \in A_\ell$.

Let $\mu_{\mathcal{A}_\ell}=[\mathcal{A}_\ell;\,\phi_{\ell+1},\gamma_{\mathcal{A}_\ell}]$ be the minimal limit augmentation of the essential continuous family $\mathcal{A}_\ell$. By the definition of a MLV chain, we have

$$m_\ell=\deg(\chi_i)<m_{\ell+1}=\deg(\phi_{\ell+1})\quad\text{for all } i\in A_\ell.$$

Hence, for all monic $g\in K[x]$ of degree $0<\deg(g)<m_{\ell+1}$, we have simultaneously $\phi_{\ell+1}\nmid_{\mu_\ell}\chi_i$ and $\phi_{\ell+1}\nmid_{\mu_\ell} g$. Since $\mathbf{t}(\mu_{\mathcal{A}_\ell},v_F)=[\phi_{\ell+1}]_{\mu_{\mathcal{A}_\ell}}$, Lemma 1.4 shows that

$$\mu_{\mathcal{A}_\ell}(g)=v_F(g),\qquad \mu_{\mathcal{A}_\ell}(\chi_i)=v_F(\chi_i)\quad\text{for all } i\in A_\ell.$$

On the other hand, $m_{\ell+1}=\deg_\infty(\mathcal{A}_\ell)$ is the unstable degree of $\mathcal{A}_\ell$. Hence, g and χ_i are $\mathcal{A}_\ell$-stable. Take a sufficiently large i such that

$$\rho_i(g)=\rho_{\mathcal{A}_\ell}(g)=\mu_{\mathcal{A}_\ell}(g)=v_F(g).$$

By [2, Lem. 4.12], we may assume that $\chi_j\nmid_{\rho_i}\chi_i$ for all $j>i$. This implies

$$\rho_i(\chi_i)=\rho_j(\chi_i)=\rho_{\mathcal{A}_\ell}(\chi_i)=\mu_{\mathcal{A}_\ell}(\chi_i)=v_F(\chi_i),$$

as well. Since χ_i is a key polynomial for ρ_i, Theorem 1.2 implies

$$\operatorname{wt}(g)=\frac{\rho_i(g)}{\deg(g)}\le\frac{\rho_i(\chi_i)}{m_\ell}=\operatorname{wt}(\chi_i).$$

This ends the proof of (9).

Finally, by [17, Thm. 4.7], the augmentation step $\mu_\ell\to\mu_{\ell+1}$ is ordinary if and only if the set V_{m_ℓ} contains a maximal element. By Lemma 4.3, $\#\Phi_\ell=1$ if and only if $W_{m_{\ell+1}}$ contains a maximal element. ☐

In particular, the length $r+1$ of this Okutsu frame of F is equal to the Mac Lane–Vaquié depth of v_F.

Conversely, all Okutsu frames of F arise in this way.

Theorem 4.5. *Let $[\Phi_0,\dots,\Phi_r,\Phi_{r+1}=\{F\}]$ be an Okutsu frame of $F\in\operatorname{Irr}(K)$. For all $0\le\ell\le r+1$, choose an arbitrary $\phi_\ell\in\Phi_\ell$ and denote $\gamma_\ell=v_F(\phi_\ell)$. Then, the truncation of v_F by ϕ_ℓ is a valuation μ_ℓ fitting into a MLV chain of v_F:*

$$\mu_0\ \xrightarrow{\phi_1,\gamma_1}\ \mu_1\ \longrightarrow\ \cdots\ \xrightarrow{\phi_{r-1},\gamma_{r-1}}\ \mu_{r-1}\ \xrightarrow{\phi_r,\gamma_r}\ \mu_r\ \xrightarrow{F,\infty}\ \mu_{r+1}=v_F.$$

If $\Phi_\ell=\{\phi_\ell\}$, then $\mu_{\ell+1}=[\mu_\ell;\,\phi_{\ell+1},\gamma_{\ell+1}]$ is an ordinary augmentation.

If $\Phi_\ell=\{\chi_i\mid i\in I_\ell\}$, then $\mu_{\ell+1}=[\mathcal{A}_\ell;\,\phi_{\ell+1},\gamma_{\ell+1}]$ is a limit augmentation with respect to the essential continuous family $\mathcal{A}_\ell=\{\mu_\ell\}\cup(\rho_i)_{i\in A_\ell}$, where $\rho_i=[\mu_\ell;\,\chi_i,v_F(\chi_i)]$ and $A_\ell\subset I_\ell$ contains the indices i such that $v_F(\chi_i)>\gamma_\ell$.

Proof. Let $\phi_0=x-a$ for some $a\in K$. Since $\operatorname{supp}(v_F)=FK[x]$ and $\deg(F)>1$, we have $\gamma_0:=v_F(\phi_0)<\infty$. Thus, the truncation of v_F by ϕ_0 is the depth-zero valuation $\mu_0=\omega_{a,\gamma_0}\le v_F$. Since μ_0 has trivial support, we have $\mu_0<v_F$ and ϕ_0 is a key polynomial for μ_0 of minimal degree. Also, the equality $\mu_0(\phi_0)=\gamma_0=v_F(\phi_0)$ shows that $\phi_0\notin\mathbf{t}(\mu_0,v_F)$.

Now, suppose that for some $0\le\ell\le r$ we have constructed a MLV chain of the valuation μ_ℓ:

$$\mu_0\ \xrightarrow{\phi_1,\gamma_1}\ \mu_1\ \longrightarrow\ \cdots\ \xrightarrow{\phi_\ell,\gamma_\ell}\ \mu_\ell<v_F,$$

satisfying all conditions of the theorem, for all indices $0,\dots,\ell$. Since $\mu_\ell(\phi_\ell)=\gamma_\ell=v_F(\phi_\ell)$, we have $\phi_\ell\notin\mathbf{t}(\mu_\ell,v_F)$.

Let us construct a further step of the MLV chain, satisfying the conditions of the theorem for the index $\ell+1$ as well.

Denote $m_\ell=\deg(\phi_\ell)$ for all $0\le \ell\le r$. Suppose that $\Phi_\ell=\{\phi_\ell\}$. Since ϕ_ℓ is a key polynomial for μ_ℓ of minimal degree, Theorem 1.2 shows that

$$\frac{\mu_\ell(\phi_{\ell+1})}{m_{\ell+1}}\le\frac{\mu_\ell(\phi_\ell)}{m_\ell}=\operatorname{wt}(\phi_\ell)<\operatorname{wt}(\phi_{\ell+1}).$$

Hence, $\mu_\ell(\phi_{\ell+1})<v_F(\phi_{\ell+1})$. We claim that $m_{\ell+1}$ is the least degree of a monic polynomial in $K[x]$ satisfying this inequality. Indeed, suppose that $g\in K[x]$ is a monic polynomial of minimal degree such that $\mu_\ell(g)<v_F(g)$. By Lemma 1.4, g is a key polynomial for μ_ℓ; hence, it is μ_ℓ-minimal and Theorem 1.2 shows that

$$\frac{\mu_\ell(g)}{\deg(g)}=\frac{\mu_\ell(\phi_\ell)}{m_\ell}=\operatorname{wt}(\mu_\ell).$$

If $\deg(g)<m_{\ell+1}$, the fundamental property (9) would imply:

$$\operatorname{wt}(g)\le\operatorname{wt}(\phi_\ell)=\frac{\mu_\ell(\phi_\ell)}{m_\ell}=\frac{\mu_\ell(g)}{\deg(g)},$$

contradicting our initial assumption.

Thus, $\phi_{\ell+1}\in K[x]$ is a monic polynomial of minimal degree satisfying $\mu_\ell(\phi_{\ell+1})<v_F(\phi_{\ell+1})$. By Lemma 1.4, $\phi_{\ell+1}$ is a key polynomial for μ_ℓ. The ordinary augmentation

$$\mu_{\ell+1}:=[\mu_\ell;\phi_{\ell+1},\gamma_{\ell+1}],\qquad \gamma_{\ell+1}=v_F\left(\phi_{\ell+1}\right),$$

satisfies $\mu_\ell<\mu_{\ell+1}\le v_F$. If we add the augmentation step $\mu_\ell\to\mu_{\ell+1}$ to the previous MLV chain, we obtain a MLV chain of $\mu_{\ell+1}$, because

$$\deg(\mu_\ell)=m_\ell<m_{\ell+1}=\deg(\mu_{\ell+1}).$$

If $\ell=r$, then $\phi_{\ell+1}=F$ and $\gamma_{\ell+1}=\infty$, so that $\mu_{\ell+1}=v_F$ and the theorem is proven.

If $\ell<r$, then $m_{\ell+1}<\deg(F)$ and this implies $\gamma_{\ell+1}<\infty$. In this case, $\mu_{\ell+1}$ has trivial support and $\mu_{\ell+1}<v_F$. Also, $\phi_{\ell+1}$ is a key polynomial for $\mu_{\ell+1}$ of minimal degree. By Lemma 1.1, the truncation of v_F by $\phi_{\ell+1}$ is equal to:

$$\left(v_F\right)_{\phi_{\ell+1}}=\left(\mu_{\ell+1}\right)_{\phi_{\ell+1}}=\mu_{\ell+1}.$$

Finally, since $\mu_{\ell+1}\left(\phi_{\ell+1}\right)=\gamma_{\ell+1}=v_F\left(\phi_{\ell+1}\right)$, we have necessarily $\phi_{\ell+1}\not\in\mathbf{t}(\mu_{\ell+1},v_F)$. This ends the recurrence step in the case $\Phi_\ell=\{\phi_\ell\}$.

Now, suppose that $\Phi_\ell=\{\chi_i\mid i\in I\}$ and let $i_0\in I$ be the index for which $\phi_\ell=\chi_{i_0}$.

By our recurrence assumption, μ_ℓ is a valuation and ϕ_ℓ is a key polynomial for μ_ℓ of minimal degree. Since $\deg\,\mathbf{t}(\mu_\ell,v_F)\ge\deg(\mu_\ell)=m_\ell$, Lemma 1.4 shows that

$$\mu_\ell(a)=v_F(a)\quad\text{ for all } a\in K[x],\ \text{ with }\ \deg(a)<m_\ell,$$

because $\phi\nmid_{\mu_\ell}a$ for any $\phi\in\mathbf{t}(\mu_\ell,v_F)$.

Denote $\beta_i=v_F(\chi_i)$ for all $i\in I$. By the additional property (i) of the Okutsu frame, we have $\beta_{i_0}<\beta_i$ for all $i_0<i$ in I. Since all $\chi_i\in K[x]$ are monic polynomials of degree m_ℓ, we deduce that

$$\mu_\ell\left(\chi_{i_0}-\chi_i\right)=v_F\left(\chi_{i_0}-\chi_i\right)=\beta_{i_0}=\operatorname{sv}(\mu_\ell)\quad\text{ for all } i_0<i.$$

By [16, Prop. 6.3], all these $\chi_i \in \Phi_\ell$ are key polynomials for μ_ℓ of degree m_ℓ. Hence, we may consider the family of ordinary augmentations

$$\rho_i = [\mu_\ell;\, \chi_i, \beta_i] \quad \text{for all } i > i_0.$$

By comparing their action of χ_j-expansions, we clearly have

$$\mu_\ell < \rho_i < \rho_j < v_F \quad \text{for all } \ i_0 < i < j \ \text{ in } I.$$

Denote $\rho_{i_0} := \mu_\ell$. For the totally ordered set of indices $A = I_{\geq i_0}$, we may consider a continuous family of valuations $\mathcal{A} = (\rho_i)_{i\in A}$ of stable degree m_ℓ.

Let us show that $\phi_{\ell+1}$ is $\mathcal{A}$-unstable. For all $i \in A$, Theorem 1.2 shows that

$$\frac{\rho_i\,(\phi_{\ell+1})}{m_{\ell+1}} \leq \frac{\mathrm{sv}(\rho_i)}{m_\ell} = \frac{\beta_i}{m_\ell} < \mathrm{wt}(\phi_{\ell+1}),$$

the last inequality by the additional property (OF3) of the Okutsu frame. Hence, $\rho_i\,(\phi_{\ell+1}) < v_F(\phi_{\ell+1})$. By [17, Cor. 2.5], this implies

$$\rho_i\,(\phi_{\ell+1}) < \rho_j\,(\phi_{\ell+1}) \quad \text{for all } \ i < j,$$

so that $\phi_{\ell+1}$ is $\mathcal{A}$-unstable.

Let us now show that $m_{\ell+1}$ is the minimal degree of $\mathcal{A}$-unstability; that is, all monic $g \in K[x]$ with $\deg(g) < m_{\ell+1}$ are $\mathcal{A}$-stable. Since the product of $\mathcal{A}$-stable polynomials is $\mathcal{A}$-stable, we may assume that g is irreducible. By the fundamental property (9), there exists $\chi_i \in \Phi_\ell$ such that

$$\mathrm{wt}(g) \leq \mathrm{wt}(\chi_i) = \frac{\beta_i}{m_\ell} = \mathrm{wt}(\rho_i). \tag{10}$$

We claim that $\rho_i(g) = \rho_j(g)$ for all $j > i$. Indeed, suppose that $\rho_i(g) < \rho_j(g) \leq v_F(g)$ for some $j > i$. On one hand, from (10) we deduce

$$\frac{\rho_i(g)}{\deg(g)} < \mathrm{wt}(g) \leq \mathrm{wt}(\rho_i),$$

so that g is not ρ_i-minimal, by Theorem 1.2. This contradicts Corollary 3.2.

Therefore, $\phi_{\ell+1}$ is a monic $\mathcal{A}$-unstable polynomial of minimal degree. In other words, $\phi_{\ell+1} \in \mathrm{KP}_\infty(\mathcal{A})$. The limit augmentation

$$\mu_{\ell+1} := [\mathcal{A}; \phi_{\ell+1}, \gamma_{\ell+1}], \quad \gamma_{\ell+1} = v_F\,(\phi_{\ell+1})\,,$$

satisfies $\mu_{\ell+1} \leq v_F$. Since $\phi_\ell \notin \mathbf{t}(\mu_\ell, \mu_{\ell+1}) = \mathbf{t}(\mu_\ell, v_F)$, if we add the augmentation step $\mu_\ell \to \mu_{\ell+1}$ to the previous MLV chain, we obtain a MLV chain of $\mu_{\ell+1}$.

If $\ell = r$, we have $\phi_{\ell+1} = F$ and $\gamma_{\ell+1} = \infty$, so that $\mu_{\ell+1} = v_F$ and the theorem would be proven.

If $\ell < r$, then $m_{\ell+1} < \deg(F)$ and this implies $\gamma_{\ell+1} < \infty$. In this case, $\mu_{\ell+1}$ has trivial support and $\mu_{\ell+1} < v_F$. Also, $\phi_{\ell+1}$ is a key polynomial for $\mu_{\ell+1}$ of minimal degree. By Lemma 1.1, the truncation of v_F by $\phi_{\ell+1}$ is equal to:

$$(v_F)_{\phi_{\ell+1}} = (\mu_{\ell+1})_{\phi_{\ell+1}} = \mu_{\ell+1}.$$

Finally, since $\mu_{\ell+1}\,(\phi_{\ell+1}) = \gamma_{\ell+1} = v_F\,(\phi_{\ell+1})$, we have necessarily $\phi_{\ell+1} \notin \mathbf{t}(\mu_{\ell+1}, v_F)$. This ends the recurrence step in the case $\Phi_\ell = \{\chi_i \mid i \in I\}$. □

Corollary 4.6. *Let* $[\Phi_0, \dots, \Phi_r, \Phi_{r+1} = \{F\}]$ *be an Okutsu frame of some* $F \in \mathrm{Irr}(K)$*. Then,* $1 = m_0 \mid m_1 \mid \cdots \mid m_r \mid \deg(F)$*.*

Proof. By Theorem 4.5, $1 = m_0, \dots, m_{r+1} = \deg(F)$ are the degrees of a MLV chain of v_F. Since (K, v) is henselian, all jumps $m_{\ell+1}/m_\ell$ take integer values. Indeed, this follows from the results of Vaquié [25], described in [17, Sec. 6] as well. □

4.3. **Computation of ramification indices, residual degrees and defect.** Since (K, v) is henselian, the valuation $\overline{v}_F$ is the only extension of v to the field extension $K_F = K[x]/(F)$. By Ostrowski's lemma, we have

$$\deg(F) = e(F)f(F)d(F),$$

where $e(F) = e(\overline{v}_F/v)$ is the ramification index, $f(F) = f(\overline{v}_F/v)$ the residual degree and $d(F) = d(\overline{v}_F/v)$ the defect of $\overline{v}_F/v$. Also, Ostrowski showed that $d(F)$ is a power of the exponent characeristic, defined as

$$p = \begin{cases} \operatorname{char}(k), & \text{if } \operatorname{char}(k) > 0, \\ 1, & \text{if } \operatorname{char}(k) = 0. \end{cases}$$

In [17, Sec.6], it is shown how to compute these invariants $e(F)$, $f(F)$, $d(F)$ in terms of a MLV chain of v_F. Hence, as a consequence of Theorem 4.5, we can compute them in terms of an Okutsu frame.

Let $[\Phi_0, \dots, \Phi_r, \Phi_{r+1} = \{F\}]$ be an Okutsu frame of $F \in \operatorname{Irr}(K)$. Consider the MLV chain of v_F described in Theorem 4.5:

$$\mu_0 \xrightarrow{\phi_1,\gamma_1} \mu_1 \longrightarrow \cdots \xrightarrow{\phi_{r-1},\gamma_{r-1}} \mu_{r-1} \xrightarrow{\phi_r,\gamma_r} \mu_r \xrightarrow{F,\infty} v_F.$$

Recall that $\gamma_\ell = v_F(\phi_\ell)$ for all $\ell \ge 0$.

By the MLV condition, this chain induces a chain of abelian groups:

$$\Gamma_{\mu_{-1}} := \Gamma \subset \Gamma_{\mu_0} \subset \Gamma_{\mu_1} \subset \cdots \Gamma_{\mu_r} = \Gamma_{v_F} = \Gamma_{\overline{v}_F},$$

with $\Gamma_{\mu_\ell} = \langle \Gamma, \gamma_0, \dots, \gamma_\ell \rangle$ for all $0 \le \ell \le r$ [17, Sec. 4.1]. Hence,

$$e(F) = (\Gamma_{\overline{v}_F} \colon \Gamma) = (\Gamma_{\mu_r} \colon \Gamma) = e_0 \cdots e_r,$$

where $e_\ell = \left(\Gamma_{\mu_\ell} \colon \Gamma_{\mu_{\ell-1}}\right)$ for all $0 \le \ell \le r$.

Thus, each index e_ℓ can be computed as the least positive integer e such that

$$e\, v_F(\phi_\ell) \in \langle \Gamma, v_F(\phi_0), \dots, v_F(\phi_{\ell-1}) \rangle.$$

On the other hand, Vaquié proved in [25] that $d(F) = d_1 \cdots d_r$, where

$$d_\ell = \begin{cases} 1, & \text{if } \mu_\ell \to \mu_{\ell+1} \text{ is an ordinary augmentation,} \\ m_{\ell+1}/m_\ell, & \text{if } \mu_\ell \to \mu_{\ell+1} \text{ is a limit augmentation,} \end{cases}$$

for all $0 \le \ell \le r$. Equivalently,

$$d_\ell = 1 \iff \#\Phi_\ell = 1 \iff \exists \max(W_{m_{\ell+1}}).$$

Thus, $d(F)$ may be computed solely in terms of the Okutsu frame too.

Finally, $f(F) = \deg(F)/e(F)d(F)$ is determined by $e(F)$ and $d(F)$.

4.4. **Okutsu frames and abstract key polynomials.** Abstract key polynomials were introduced by Herrera-Olalla-Spivakovsky as an alternative approach to the methods of Mac Lane and Vaquié, aiming at a thorough comprehension of the extensions to $K[x]$ of arbitrary (not necessarily henselian) valuations on a field K [7, 8].

These polynomials were further studied by several authors and the comparison with Mac Lane-Vaquié key polynomials is by now fully understood [12, 3, 19, 1, 20].

Although they were classically defined only for valuations with trivial support, the paper [1] develops their properties for arbitrary valuations on $K[x]$ too. Let us recall a concrete comparison result between the two sorts of "key polynomials".

Theorem 4.7. [1, Thm. 2.21] *Let ν be a valuation on $K[x]$ with nontrivial support. A monic polynomial $Q \in K[x] \setminus K$ is an abstract key polynomial for ν if and only if either* $\operatorname{supp}(\nu) = QK[x]$*, or the truncation ν_Q is a valuation and Q is a (MLV) key polynomial of minimal degree for ν_Q.*

A set Ψ of abstract key polynomials for ν is said to be *complete* if for all nonconstant $g \in K[x]$ there exists $Q \in \Psi$ such that

$$\deg(Q) \le \deg(g) \quad \text{and} \quad \nu_Q(g) = \nu(g). \tag{11}$$

As a consequence of Theorem 4.5, we derive another interpretation of abstract key polynomials.

Theorem 4.8. *Suppose that (K, v) is henselian and let $[\Phi_0, \dots, \Phi_r, \Phi_{r+1} = \{F\}]$ be an Okutsu frame of some $F \in \operatorname{Irr}(K)$. Then, the set $\Phi_0 \cup \cdots \cup \Phi_r \cup \{F\}$ is a complete set of abstract key polynomials for v_F.*

Proof. By Theorems 4.5 and 4.7, all polynomials in the set $\Phi_0 \cup \cdots \cup \Phi_r \cup \{F\}$ are abstract key polynomials for v_F.

Take a monic $g \in K[x] \setminus K$. If $\deg(g) \ge \deg(F)$, then (11) is satisfied for $Q = F$.

If $\deg(g) < \deg(F)$, then there exists $0 \le \ell \le r$ such that $m_\ell \le \deg(g) < m_{\ell+1}$.

With the notation in Theorem 4.5, consider the MLV chain of v_F:

$$\mu_0 \xrightarrow{\phi_1,\gamma_1} \mu_1 \longrightarrow \cdots \xrightarrow{\phi_{r-1},\gamma_{r-1}} \mu_{r-1} \xrightarrow{\phi_r,\gamma_r} \mu_r \xrightarrow{F,\infty} \mu_{r+1} = v_F.$$

If $\Phi_\ell = \{\phi_\ell\}$, then $\mu_{\ell+1} = [\mu_\ell; \phi_{\ell+1}, \gamma_{\ell+1}]$ is an ordinary augmentation and $\mathbf{t}(\mu_\ell, v_F) = \mathbf{t}(\mu_\ell, \mu_{\ell+1}) = [\phi_{\ell+1}]_{\mu_\ell}$. Since $\deg(g) < \deg(\phi_{\ell+1})$, necessarily $\phi_{\ell+1} \nmid_{\mu_\ell} g$ and $\mu_\ell(g) = v_F(g)$ by Lemma 1.4. Thus, (11) is satisfied for $Q = \phi_\ell$.

If $\Phi_\ell = \{\chi_i \mid i \in I_\ell\}$, then $\mu_{\ell+1} = [\mathcal{A}_\ell; \phi_{\ell+1}, \gamma_{\ell+1}]$ is a limit augmentation and $\phi_{\ell+1}$ is a limit key polynomial for $\mathcal{A}_\ell$. Hence, g is $\mathcal{A}_\ell$-stable and (11) is satisfied for $Q = \chi_i$, for a sufficiently large $i \in I_\ell$. □

The converse statement follows easily from the definitions. Consider a complete set of abstract key polynomials for $\nu := v_F$,

$$\Psi = \Psi_{m_0} \cup \cdots \Psi_{m_r} \cup \Psi_{m_{r+1}} = \{F\}, \quad m_0 = 1 < m_1 < \cdots < m_r < \deg(F),$$

where all polynomials in Ψ_{m_ℓ} have degree m_ℓ.

In order to show that the list $[\Psi_1, \dots, \Psi_r, \{F\}]$ is an Okutsu frame of F, we need only to check that the property (9) is satisfied.

Take a monic $g \in K[x]$ such that $0 < \deg(g) < m_{\ell+1}$ for some $0 \le \ell \le r$. By the completeness of Ψ, there exists $P \in \Psi$ such that $\deg(P) \le \deg(g)$ and $\nu_P(g) = \nu(g)$. Since $\deg(P) \le m_\ell$, there exists $Q \in \Psi_{m_\ell}$ such that $\nu_P \le \nu_Q \le \nu$. Thus, $\nu_Q(g) = \nu(g)$

as well. By Theorem 4.7, Q is a MLV key polynomial of minimal degree for ν_Q. Thus, Theorem 1.2 shows that

$$\mathrm{wt}(g) = \frac{\nu(g)}{\deg(g)} = \frac{\nu_Q(g)}{\deg(g)} \le \frac{\nu_Q(Q)}{m_\ell} = \frac{\nu(Q)}{m_\ell} = \mathrm{wt}(Q).$$

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