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Unknown Input Observer Design for Quadratically Inner Bounded - One Sided Lipschitz Noisy Nonlinear Systems

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In this paper, the problem of designing an observer for quadratically inner bounded (QIB) and one-sided Lipschitz (OSL) nonlinear systems considering noise and unknown inputs is addressed. The observer synthesis procedures are formulated as convex optimization problems. Sufficient conditions for observer gain synthesis are shown to be equivalent to solving finite sets of Linear Matrix Inequalities (LMIs) and Linear Matrix Equalities (LMEs). Three illustrative examples based on an isothermal CSTR reactor, a water tank-open channel system and FitzHugh-Nagumo system are used to illustrate how the proposed observer design approaches are used to design the QIB-UIO scheme. The obtained simulation results are presented to assess the proposed method.

KEYWORDS
One Sided Lipschitz, Quadratically Inner Bounded, Nonlinear systems, Unknown Input Observer, Observer design

1 | INTRODUCTION

The supervision, control and predictive maintenance of a system generally requires the knowledge of its states. However, economic and technical constraints require the reduction of the number of physical sensors [1]. Moreover, the evolution of the system can be influenced by unmeasured perturbations, usually modelled as unknown inputs [2].

There exist several works focusing on reconstructing the state of a system using input and output information in the presence of unknown inputs. Such a task involves the synthesis of an Unknown Input Observer (UIO). Several contributions towards the design of UIO for linear systems can be found in the literature, see as e.g. [3], [4]. However, the design of UIO for nonlinear systems (NLS) might become more challenging, since linear systems can only approximate the dynamic behavior of nonlinear systems in an interval around the operating point.

Nonlinear Lipschitz systems are a particular class of NLS, for which there are several contributions related to the design of UIO, see as e.g. [5], [6]. The UIO synthesis problem for nonlinear Lipschitz systems is formulated in the Linear Matrix Inequalitiy (LMI) framework, obtaining sufficient conditions to guarantee the convergence of the error dynamics. Another family of systems, which is less restrictive than the Lipschitz one, is the family of Quadratically Inner Bounded (QIB) One-sided Lipschitz Nonlinear Systems (OSL). This type of systems has attracted the attention of the scientific community due to the fact that the bounding condition of the derivative must only be satisfied in the growth rate, but not in the decay rate. This condition is less restrictive than in the case of Lipschitz systems, where both conditions must be satisfied. Additionally, since the QIB condition is equivalent to the Lipschitz condition [7], it can be adapted to the OSL condition by selecting sufficiently large QIB constants while also admitting negative constants, which is not possible when one wishes to describe a nonlinear system considering only the Lipschitz constant [8]. While the OSL-QIB condition is equivalent to the Lipschitz condition [7], the description of the non linearity is more detailed in the former case and should allow obtaining less conservative observer synthesis conditions. The possibility to find feasibility conditions for the formulated linear matrix inequality (LMI) using the OSL-QIB description, and which cannot be achieved considering the design of an observer with the classical Lipschitz constant [9], is highlighted in the present paper by means of an example.

Some examples of UIO design for QIB-OSL systems can be found in the literature. For instance, an UIO that is robust against time delays is designed in [10]. The design of UIO for OSL fractional NLS and Markovian jump NLS are presented in [11] and [12], respectively. The previously cited authors succeed in proving that the UIO design conditions are reduced to the satisfaction of certain LMI (convex optimization) constraints, which is also the case in the design of observers including conventional Lipschitz conditions. However, while UIOs are designed in previous works for OSL systems with different characteristics, e.g., time delays and Markovian jumps, the case in which noise is present in the measurement has not been considered yet. Including this design condition is of paramount importance, because measurement noise is always present in real applications. Therefore, it is important to minimize the effect of noise.

In this paper, an UIO is designed using an LMI approach for one-sided Lipschitz quadratically inner bounded nonlinear (OSL-QIB) systems. From a methodological point of view, the main contribution is the reformulation of the UIO synthesis problem for OSL-QIB nonlinear system as a convex optimization problem, while making the observer less sensitive to noise. This fact allows to solve the UIO synthesis problem for OSL-QIB and formulate the observer synthesis problem as a convex optimization problem by combining matrix inequalities and equalities. Using this approach, it becomes clear that sufficient conditions for UIO synthesis are identical to the one considered for classical observer synthesis using LMIs (as considered for instance in [13]) combined with decoupling conditions between the observer error and the unknown input.

The structure of the paper is the following: In Section 2, the problem statement is presented. Section 3 describes the UIO design procedures without and with noise. Section 4 shows the obtained results applying the proposed approaches using illustrative examples. Finally, Section 5 draws the main conclusions and points several future research paths.

Notations: $\mathbb{R}^n$ denotes the Euclidean space of dimension n, $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ matrices with dimension. Matrices $P \prec 0$ and $P \succ 0$ are squared and symmetric, and negative and positive definite, respectively. The operation

$\langle x, y \rangle = x^T y$ corresponds to the inner product between x, y and x^T corresponds to the transpose of vector x. The operator $\mathcal{H}(A) = A + A^T$ is the Hermitian of A. Finally, $\|x\|$ is the norm of vector x.

2 | PROBLEM SET-UP

Consider the following nonlinear dynamical system:

$$\begin{aligned} \dot{x} &= Ax(t) + Bu(t) + D_f f(Hx, u) + Dv(t), \\ y &= Cx(t) + Rh(t), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ is the output, $v \in \mathbb{R}^p$ is an unknown input, $h \in \mathbb{R}^j$ is the measurement noise, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D_f \in \mathbb{R}^{n\times q}$, $H \in \mathbb{R}^{s\times n}$ and $R \in \mathbb{R}^{p\times j}$ are the system matrices. The non-linearity $f(Hx, u)$ is assumed to fulfill the following conditions [8].

Condition 1: The non linearity $f(Hx, u)$ satisfies the OSL condition if the following inequalities are fulfilled:

$$< f(H\hat{x}, u) - f(Hx, u), H(\hat{x} - x) > \leq \rho \|H(\hat{x} - x)\|^2, \tag{2}$$

such that writing $\Delta f = f(H\hat{x}, u) - f(Hx, u), e = \hat{x} - x$ is equivalent to

$$\Delta f^T He \leq \rho e^T H^T He, \tag{3}$$

where $\rho \in \mathbb{R}$ is the OSL constant.

Condition 2: As in Condition 1, the following inequalities are formulated to state the QIB property:

$$\|f(H\hat{x}, u) - f(Hx, u)\|^2 \leq \beta \|H(\hat{x} - x)\|^2 + \gamma \langle f(H\hat{x}, u) - f(Hx, u), H(\hat{x} - x) \rangle, \tag{4}$$

such that writing $\Delta f = f(H\hat{x}, u) - f(Hx, u), e = \hat{x} - x$ is equivalent to

$$\Delta f^T \Delta f \leq \beta e^T H^T He + \gamma e^T H \Delta f, \tag{5}$$

and the scalars γ and β in (5) are the QIB constants.

It should be noted that every vector function that is locally Lipschitz satisfies the one-sided Lipschitz property, but the inverse is not true [8]. For example, the scalar function $f(x) = -x^3$ is locally Lipschitz but not globally Lipschitz [14]. However, by applying (3) it is possible to conclude that the condition is fulfilled with $\rho = 0$, and thus the system is OSL. From the above, it can be inferred that the OSL condition encompasses a much larger family of dynamical systems than the classical Lipschitz condition, so that the design of an observer with this property could estimate the hidden state of a wide variety of systems (more than those including classical Lipschitz property). To guarantee an implementable design of an observer, it is of vital importance to guarantee robustness against noise in the measurement since it can negatively alter the performance of the observer. Thus, the goal of this work is to propose a formulation of a state estimator that is robust to noise and which also includes the OSL-QIB property. With these characteristics, the implementation of the observer becomes interesting for real-time state estimation and filtering applications.

The following preliminary definitions and lemmas are introduced to present the results obtained in this paper:

Definition (Exponential Observer) [15]: An observer $\hat{x}$ is an exponential observer of x with convergence rate α and overshoot Ω if the following inequality holds for all $t \geq 0$:

$$\|x(t) - \hat{x}(t)\| \leq \Omega \|x(0) - \hat{x}(0)\| e^{-\alpha t}. \tag{6}$$

Definition (Continuous time robust observer)[15]: An observer $\hat{x}$ of x with $\mathcal{L}_2$ gain μ is said to be robust with respect to perturbation v if

$$\int_0^\infty \left(e(s)^T e(s) - \mu^2 v(s)^T v(s) \right) ds < \infty, \tag{7}$$

where $e(s)$ corresponds to the estimation error

3 | UNKNOWN INPUT OBSERVER DESIGN FOR QIB-OSL NONLINEAR SYSTEMS

3.1 | UIO design, noiseless case (R=0)

This section will address the design of an UIO when the measurement is noise-free, that is $R = 0$. Initially, let us consider the following UIO structure [8]:

$$\begin{aligned} \dot{z} &= Nz + TBu + Gy + TD_f f(H\hat{x}, u), \\ \hat{x} &= z - Ey, \end{aligned} \tag{8}$$

where z is an auxiliary variable, $\hat{x}$ is the state estimate, and matrices N, J, L, E and $T = I_n + EC$ are the observer gains.

The subtraction between the estimated state and the true state allows to define the estimation error

$$\begin{aligned} e &= \hat{x} - x = z - ECx - x = z - Tx, \\ T &= (I_n + EC). \end{aligned} \tag{9}$$

Taking the time derivative of (9), the error dynamics can be deduced:

$$\begin{aligned} \dot{e} &= \dot{z} - T\dot{x}, \\ \dot{e} &= Ne + TD_f \Delta f + (NT + GC - TA)x - TDv. \end{aligned} \tag{10}$$

Once the error dynamics are obtained, it is possible to proceed with the synthesis of the UIO [9]. In the following equations, the observer gainsare stated as functions of other matrices (that will be used as matrix variables in LME

and LMI expressions):

$$\begin{aligned} E &= P^{-1}S, \\ K &= P^{-1}Q, \\ N &= TA - KC, \\ G &= K + KCE - TAE. \end{aligned} \tag{11}$$

Theorem 1 *If there exist matrices $P \succ 0$, S , Q and positive scalars τ_1, τ_2 such that the following relations holds:*

$$\Lambda = D_f^T P + D_f^T C^T S^T - \tau_1 H + \tau_2 \gamma H \tag{12}$$

$$\begin{bmatrix} \mathcal{H}(PA + SCA - QC) + 2\tau_2 \beta H^T H + 2\tau_1 \rho H^T H & \Lambda \\ \Lambda^T & -2\tau_2 I_n \end{bmatrix} \prec 0 \tag{13}$$

$$PD + SCD = 0 \tag{14}$$

$$P \succ 0 \tag{15}$$

then observer (8) is an exponential observer of system (1)*, and its gains are given by (11).*

Proof: By assuming $PD+SCD = 0$, and since P is invertible, it follows that $D+P^{-1}SCD = 0$. Moreover, since $P^{-1}S = E$ and $(I + EC)D = 0$, it follows that $TD = 0$. Furthermore:

$$\begin{aligned} NT + GC - TA &= N + NEC + GC - TA \\ &= -KC + NEC + GC \qquad (16) \\ &= NEC + (KCE - TAE)C \\ &= (N + KC - TA)EC \qquad (17) \\ &= 0. \end{aligned}$$

With the above considerations, the simplified error dynamics are obtained:

$$\dot{e} = Ne + TD_f \Delta f. \tag{18}$$

Considering the error dynamics (18), the quadratic Lyapunov function (LF) $V = e^T Pe$ is proposed. Taking the LF derivative, the following expression is found:

$$\dot{V} = e^T \left(N^T P + PN \right) e + \mathcal{H}\left(\Delta f \left(D_f^T P + D_f^T C^T S^T \right) e \right) \tag{19}$$

Pre-multiplying conditions (3) and (5) with the constants $2\tau_1$ and $2\tau_2 \in \mathbb{R}$, respectively, it is possible to find an upper bound for the LF derivative:

$$\Psi = \rho e^T H^T He - \Delta f He > 0, \tag{20}$$

$$\Gamma = \Delta f^T \Delta f + \gamma e^T H^T \Delta f + \beta e^T H^T He > 0, \tag{21}$$

$$\dot{V} \leq \dot{V} + \tau_1 \mathcal{H}(\Psi) + \tau_2 \mathcal{H}(\Gamma). \tag{22}$$

Rewriting inequality (22) leads to the following LMI:

$$\Lambda = D_f^T P + D_f^T C^T S^T - \tau_1 H + \tau_2 \gamma H,$$

$$\eta = [e \quad \Delta f],$$

$$\eta^T \begin{bmatrix} \mathcal{H}(PA + SCA - QC) + 2\tau_2 \beta H^T H + 2\tau_1 \rho H^T H & \Lambda^T \\ \Lambda & -2\tau_2 I_n \end{bmatrix} \eta < 0. \tag{23}$$

With the determination of LMI (23), the proof of Theorem 1 is finished.

3.2 | UIO design, including noise ($R \neq 0$)

Considering that $R \neq 0$, it is possible to obtain the following modified error dynamics:

$$\dot{e} = Ne + (GR + NER)h + TD_f \Delta f - ER\dot{h}. \tag{24}$$

From (24), it can be observed that that the estimation error dynamics are affected by the noise and its derivative. Theorem 2 states some conditions order to reduce the impact of the noise.

Theorem 2 *If there exist matrices $P \succ 0$, S , Q and positive scalars τ_1, τ_2, τ_3 and μ such that the following relations holds:*

$$\begin{bmatrix} \mathcal{H}(PA + SCA - QC) + a_1 & \Lambda^T & QR \\ \Lambda & -2\tau_2 I_n & 0_n \\ R^T Q^T & 0_n & -\tau_3 \mu^2 I_n \end{bmatrix} \prec 0 \tag{25}$$

$$PD + SCD = 0 \tag{26}$$

$$SR = 0 \tag{27}$$

$$P \succ 0 \tag{28}$$

$$a_1 = 2(\tau_1 \rho + \tau_2 \beta) H^T H + \tau_3 I_n \tag{29}$$

then observer (8) is a robust observer of system (1)*, and its gains are given by (11).*

Proof: With the new error dynamics (24), the quadratic LF $V = e^T Pe$ is proposed. Taking the LF derivative with respect

to time, and adding the LME constraint $SR = 0$ (with the aim to reduce the impact of the noise, since $GR = QR$, $ER = P^{-1}SR = 0$), expression (30) is obtained:

$$\dot{V} = e^T(N^TP + PN)e + \mathcal{H}(h^TQRe) + \mathcal{H}\left(\Delta f D_f^T T^T Pe\right). \tag{30}$$

Considering (20) and (21) together with Definition (7), inequality (31) is defined for the LF derivative:

$$\dot{V} \leq \dot{V} + \tau_1\mathcal{H}(\Psi) + \tau_2\mathcal{H}(\Gamma) + \tau_3 e^T e - \tau_3\mu^2 h^T h. \tag{31}$$

By introducing the variable $\zeta = [e \quad \Delta f \quad h]$, expression (31) can be rewritten as an LMI as shown in (32):

$$\zeta^T \begin{bmatrix} \mathcal{H}(PA + SCA - QC) + a_1 & \Lambda^T & QR \\ \Lambda & -2\tau_2 I_n & 0_n \\ R^TQ^T & 0_n & -\tau_3\mu^2 I_n \end{bmatrix} \zeta < 0, \tag{32}$$

with

$$a_1 = 2(\tau_1\rho + \tau_2\beta)H^TH + \tau_3 I_n. \tag{33}$$

According to (31), it is possible to minimize the impact of the noise on the system by replacing $\tau_3\mu^2$ with a new LMI variable ϵ such that $\frac{\epsilon}{\tau_3}$ is minimized. This minimizes the $\mathcal{L}_2$ gain μ. While $\frac{\epsilon}{\tau_3}$ is not a convex function of the optimization variable, an approximate solution can be found by replacing the cost function by $\epsilon - \tau_3$ such that the resulting optimization problem is convex. Once (32) has been obtained, the proof of Theorem 2 is concluded.

3.3 | UIO design including noise, alternative approach

Analyzing the requirements of Theorem 2, it can be observed that the condition $SR = 0$ is too restrictive and renders the optimization problem infeasible. In order to overcome this problem, Theorem 3 is proposed.

Theorem 3 *If there exist matrices $P \succ 0$, S, Q and positive scalars $\tau_1, \tau_2, \tau_3, \tau_4$ and μ, μ_2 such that the following relations hold:*

$$\begin{bmatrix} \mathcal{H}(PA + SCA - QC) + a_1 & \Lambda & QR & -SR \\ \Lambda^T & -2\tau_2 I_n & 0_n & 0_n \\ R^TQ^T & 0_n & -\tau_3\mu^2 I_n & 0_n \\ R^TS^T & 0_n & 0_n & -\tau_4\mu_2^2 I_n \end{bmatrix} \prec 0, \tag{34}$$

$$a_1 = 2(\tau_1\rho + \tau_2\beta)H^TH + \tau_3 I_n, \tag{35}$$

$$PD + SCD = 0, \tag{36}$$

$$P \succ 0, \tag{37}$$

then observer (8) is a robust observer of system (1)*, and its gains are given by (11).*

Proof: Considering the noisy error dynamics (24), the quadratic LF $V = e^T Pe$ is proposed. Taking LF derivative, and adding the LME constraint $PD + SCD = 0$, the following expression is obtained:

$$\dot{V} = \mathcal{H}(e^T PNe) + \mathcal{H}(e^T QRh) - \mathcal{H}(e^T SR\dot{h}) + \mathcal{H}(e^T PTD_f \Delta f). \tag{38}$$

Taking (20) and (21) together with (7) in Definition 2 (proposing $\mathcal{L}_2$ gains for the noise and its derivative), inequality (39) is defined for the LF derivative:

$$\dot{V} \leq \dot{V} + \tau_1 \mathcal{H}(\Psi) + \tau_2 \mathcal{H}(\Gamma) + (\tau_3 + \tau_4) e^T e - \tau_3 \mu^2 h^T h - \tau_4 \mu_2^2 \dot{h}^T \dot{h} \tag{39}$$

Introducing the variable $\Xi = \begin{bmatrix} e & \Delta f & h & \dot{h} \end{bmatrix}$, inequality (39) can be rewritten as

$$\Xi \begin{bmatrix} \mathcal{H}(PA + SCA - QC) + a_1 & \Lambda & QR & -SR \\ \Lambda^T & -2\tau_2 I_n & 0_n & 0_n \\ R^T Q^T & 0_n & -\tau_3 \mu^2 I_n & 0_n \\ -R^T S^T & 0_n & 0_n & -\tau_4 \mu_2^2 I_n \end{bmatrix} \Xi \prec 0, \tag{40}$$

where

$$a_1 = 2(\tau_1 \rho + \tau_2 \beta) H^T H + \tau_3 I_n. \tag{41}$$

The effect of noise is minimized in the same way as in Theorem 2. The variable change $\epsilon_1 = \tau_3 \mu^2, \epsilon_2 = \tau_4 \mu_2^2$ is initially proposed, and then the semidefinite programming (SDP) $\epsilon_1 - \tau_3 + \epsilon_2 - \tau_4$ is solved. This concludes the proof of Theorem 3.

3.4 | OSL-QIB constants calculation

As a preliminary step to implement the UIO, it is important to determine the OSL and QIB constants. The calculation of these constants can be a complex step for systems with numerous state variables. However, a simple methodology is proposed in [7] to obtain the values of these constants. To find the OSL constant ρ, matrix $\Psi = \frac{1}{2}(J_f + J_f^T)$ must be computed, where J_f is the Jacobian of the system. Finding the mathematical expression for Ψ, the optimization problem (42) is solved to find $\rho \in \mathbb{R}$ as

$$\rho = \max_{i \in n} \left(\max_{x,u \in \Omega} \left(\Psi_{i,i} + \sum_{j \in n | i} |\Psi_{i,j}| \right) \right), \tag{42}$$

where Ω is the set of feasible inputs and states. According to [16], QIB constants are parameterized in terms of the non-negative variables ϵ_1 and ϵ_2. It is important to note that these two variables provide a degree of freedom in the calculation of the QIB constants, which can be very useful when designing observers and controllers. From (5) and

[16], the first QIB constant is equal to $\beta = \epsilon_2 - \epsilon_1$, while γ is the solution of the following optimization problems:

$$\gamma = \epsilon_1 \underline{\gamma} - \epsilon_2 \bar{\gamma} + \max_{x,u \in \Omega} \sum_{i \in n} \|\nabla \xi_i\|^2 \geq 0, \tag{43}$$

$$\underline{\gamma} = \max_{x,u \in \Omega} \lambda_{max}(\Psi), \tag{44}$$

$$\bar{\gamma} = \min_{x,u \in \Omega} \lambda_{min}(\Psi), \tag{45}$$

where λ are the eigenvalues of matrix Ψ, and $\|\nabla \xi_i\|^2$ is the norm of each row of J_f.

4 | APPLICATION EXAMPLES

Three different real case studies are considered in this section to test the effectiveness of the approach: a continuous stirred-tank reactor (CSTR), a water tank-open channel system, and the FHN system. The consideration of these three physical systems is interesting since they are characterized by different properties. While the CSTR and the FHN are locally Lipschitz, the water tank-open channel is OSL in the region of study, hence demonstrating the adaptability of the UIO to the different types of nonlinearities that can occur when designing the proposed state estimation scheme.

The integral square error (ISE) is employed to measure the performance of the UIO, and is defined as

$$ISE = \int_0^t \| e(t) \|^2 \, dt. \tag{46}$$

4.1 | Isothermal CSTR

CSTR are key equipment within the stages of a process, since they are in charge of generating higher value-added products [17]. Operation of CSTR involves the addition of different reagents to maximize conversion and generation of added value products. Thus, it it is important to monitor the concentrations of the reagents and products in the reaction. To this end, the UIO will be used. A schematic representation of the case study is shown in Figure 1.

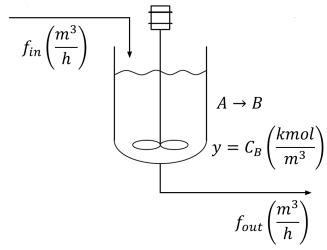

FIGURE 1 Isothermal CSTR reactor with single reaction [17]

Assuming constant level and first order isothermal reaction, the following molar balances are proposed for the

Parameter	Units	Value
Reactor volume (V)	m^3	6.6
Preexponential constant (k_0)	h^{-1}	9703.3600
Activation energy (E_a)	$\frac{kcal}{kmol}$	11843
Gas constant (R)	$\frac{kcal}{kmolK}$	1.987
Reactor temperature (T)	$^\circ C$	60
Density (ρ)	$\frac{kg}{m^3}$	1000
Feed concentration of A $\left(\frac{N_{A,in}}{V}\right)$	$\frac{kmol}{m^3}$	0.5
Input flow (F_1)	$\frac{m^3}{h}$	5

TABLE 1 Parameters and inputs of the CSTR reactor

chemical species involved in the process [17]:

$$\dot{N}_A = \frac{F_1}{V}(N_{A,in} - N_A) - r_A V, \tag{47}$$

$$\dot{N}_B = -\frac{F_1}{V}N_B + r_A V, \tag{48}$$

$$r_A = k_0 e^{-\frac{Ea}{RT}}\frac{N_A}{V}, \tag{49}$$

$$x = \begin{bmatrix} N_A & N_B \end{bmatrix}, \quad y = N_B, \quad u = F_1, \tag{50}$$

where N_A and N_B are the reagent and product moles, respectively, F_1 is the volumetric input flow, and r_A is the velocity of consumption for the reagent. The parameters in (47)–(49) are summarized in Table 1 [17].

From the knowledge of the process dynamics and parameters, the system can be rewritten in the form of (1) with the following matrices:

$$A = \begin{bmatrix} -k_0 e^{-\frac{Ea}{RT}} & 0 \\ k_0 e^{-\frac{Ea}{RT}} & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{51}$$

$$H = D_f = I_2, \tag{52}$$

$$R = C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D = R^T, \tag{53}$$

$$f = \begin{bmatrix} \frac{F_1}{V}(N_{A,in} - N_A) \\ -\frac{F_1}{V} N_B \end{bmatrix}. \tag{54}$$

For the considered input interval, the OSL-QIB constants $\rho = 0, \beta = -1$ and $\gamma = -0.2237$ can be computed using (42) and (43). Initially, the observer is tested with the measurement free of noise and the initial conditions $x(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}$, $z(0) = \begin{bmatrix} 0.6 & 0.6 \end{bmatrix}$. Additionally, after 20 hours of operation, an additional flow of product is added to the system with the objective of accelerating the production, such that $Dv = \begin{bmatrix} 0 & 2N_B/V \end{bmatrix}$. This flow can be considered as an unknown input. Figure 2 presents the observer performance in the absence of noise.

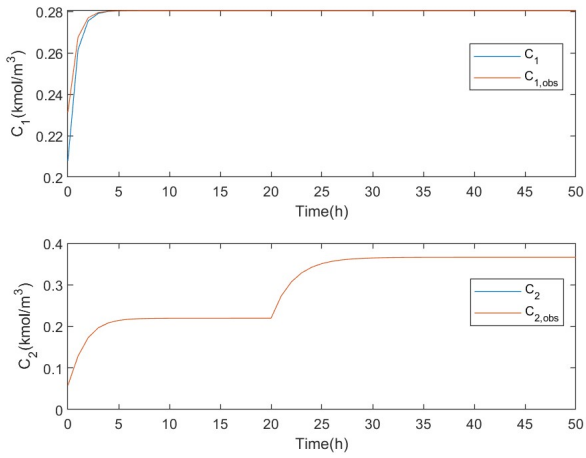

FIGURE 2 UIO considering Theorem 1, CSTR free of noise case ($ISE = 0.01$)

From the results presented in Figure 2, it can be inferred that the observer converges to the true state in finite time while adapting correctly to the unknown input applied to the reactor. From Figure 2, it can be seen that the unknown input only affects the product concentration (in order to fulfill the condition $rank(C) = rank(CD)$) so that the reagent concentration remains insensitive to this change. The difference between the observer convergence speed for each of the states is due to the fact that the effect of the non-measurable variable takes longer to replicate than the measurable one.

For the second simulation scenario, the previously established conditions will be retained. However, the measured variable will be contaminated with Gaussian noise (zero mean with a standard deviation equal to 1) to evaluate the properties of the observer in this new situation. The corresponding simulation results are depicted in Figure 3, from which it can be concluded that the observer presents a satisfactory performance since noise filtering is observed. Moreover, the observer is able to retain the properties established with Theorem 1, i.e., robustness to unknown inputs and convergence in finite time.

4.2 | Water tank-open channel system

Water is a very important element in most production systems [18], and an efficient way to transport and store it is through an open tank-channel system [19]. The tank is in charge of collecting a water flow f_0, which is then supplied

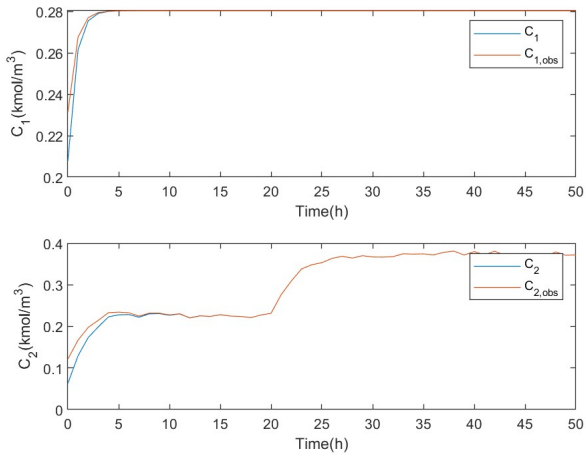

FIGURE 3 UIO considering Theorem 3, CSTR noisy case ($ISE = 1.41$)

through a valve f_{out} to the open channel with a downstream gate f_{out_2} for its subsequent transport and supply. Figure 4 shows the details of the considered case study.

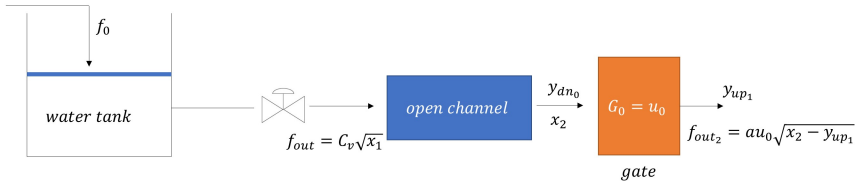

FIGURE 4 Water tank-open channel system scheme

The mathematical model representing the system dynamics is given by the following equations:

$$\dot{h} = f_0 - C_v\sqrt{h}, \tag{55}$$

$$\dot{y}_{dn_0} = C_v\sqrt{h} - au_0\sqrt{y_{dn_0} - y_{up_1}}, \tag{56}$$

$$y = y_{dn_0}, \quad u = f_0, \quad x = \begin{bmatrix} h & y_{dn_0} \end{bmatrix}. \tag{57}$$

Equations (55) and (56) represent the mass balances applied over the system in Figure 4. The term $C_v = 0.1$ is the valve coefficient, $a = 3.98$ is a factor which takes into account the gate properties (geometry and discharge coefficient) and the gravity, $u_0 = 0.4$ is the degree of gate opening, $h = x_1$ is the height of the tank, $x_2 = y_{dn_0}$ is the height before the gate discharge and $y_{up_1} = 0.1$ is the upstream height.

Rewriting the system in the form of (1), the following matrices are obtained:

$$A = \begin{bmatrix} -0.0333 & 0 \\ 0.0333 & -12.6 \end{bmatrix}, \tag{58}$$

$$H = D_f = I_2, \tag{59}$$

$$R = C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \qquad D = R^T, \qquad B = \begin{bmatrix} 1 & 0 \end{bmatrix}^T \tag{60}$$

$$f = \begin{bmatrix} -C_v\sqrt{h} \\ C_v\sqrt{h} - au_0\sqrt{y_{dn_0} - y_{up_1}} \end{bmatrix} - Ax. \tag{61}$$

In order to implement the observer, the OSL-QIB constants were calculated, obtaining the following values: $\rho = 0$, $\beta = -0.1$ and $\gamma = -1$ (since γ is unbounded, the selected value easily adapts to the OSL condition). The performance of the observer is first evaluated when the measurement is noise-free and with initial conditions $x(0) = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}$, $z(0) = \begin{bmatrix} 0.6 & 0.6 \end{bmatrix}$. After 160 hours of operation, an unknown input, which can be interpreted as a leak, is applied to the system in order to test the performance of the observer. The results for the simulation of the UIO applied to the system under consideration can be found in Figure 5. The trajectories of the variables show that the observer is robust to unknown inputs and converges in finite time to the real values of the state variables. Note that the unknown input does not affect the unmeasured variable, since the first dynamic is decoupled from the tank height.

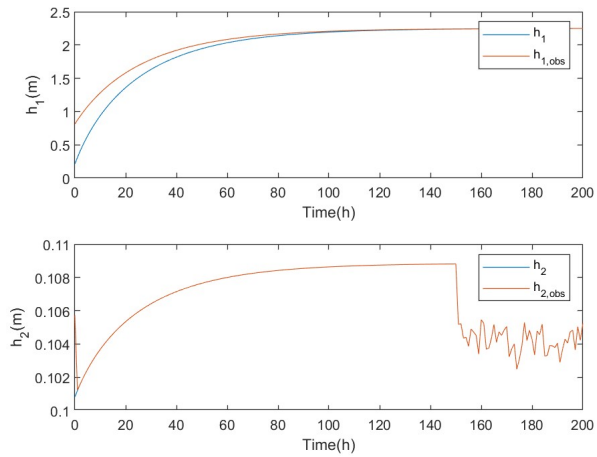

FIGURE 5 UIO water tank- open channel free of noise, Theorem 1 ($ISE = 3.51$)

A second simulation scenario is considered, but this time the measurement is contaminated with Gaussian noise (zero mean with standard deviation equal to 1), while the rest of conditions are maintained as in the previous case

(noise-free tank-open channel). Analysis of the results provided in Figure 6 allows to conclude that the observer retains the robustness against the unknown input while filtering the noise.

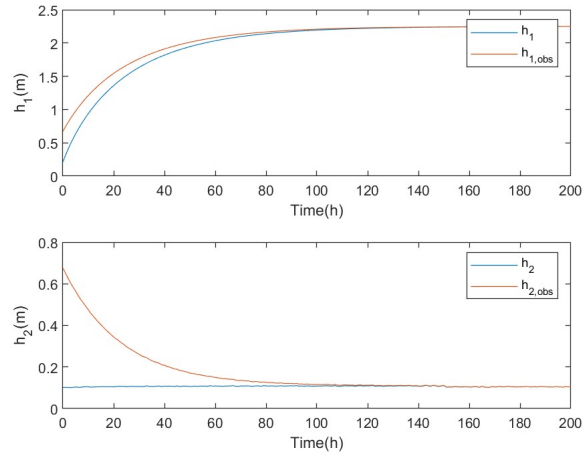

FIGURE 6 UIO water tank-open channel with noise, Theorem 3 ($ISE = 6.03$)

4.3 | FitzHugh-Nagumo (FHN) system

FHN systems are widely used to model neural behavior, chemical reaction kinetics, artificial neural networks and electronic oscillators, and also to study external stimulation, e.g., deep brain stimulation, therapies for effective treatment of brain disorders [20]. The following state-space model represents two coupled synchronous FHN neurons [20]:

$$A = \begin{bmatrix} -g_1 - 1 & -v_1 & g_1 & 0 & 0 \\ b_1 & 0 & 0 & 0 & 0 \\ g_2 & 0 & -g_2 - 1 & -v_2 & 0 \\ 0 & 0 & b_2 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T, \tag{62}$$

$$R = C, \quad D = C^T, \tag{63}$$

$$H = D_f = diag\left(\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \end{bmatrix}\right), \tag{64}$$

$$f = \begin{bmatrix} -rx_1^3 + (1+r)x_1^2 + \frac{a}{\omega}cos\omega t \\ 0 \\ -rx_3^3 + (1+r)x_1^3 + \frac{a}{\omega}cos\omega t \\ 0 \\ 0 \end{bmatrix}, \tag{65}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad y = x_1. \tag{66}$$

The values of the parameters are chosen according to [20]: $r = 10$, $a = 1$, $\omega = 0.26\pi$, $g_1 = 0.7$, $g_2 = 1$, $b_1 = 0.8$, $b_2 = 1.2$, $v_1 = b_1^{-1}$ and $v_2 = b_2^{-1}$. Moreover, the OSL-QIB constants take the following values: $\rho = \gamma = 0.1$ and $\beta = 0.2$. In the noise-free case, considering the unknown input $v = 0.15sin(10t)$ leads to the results depicted in Figure 7. Unlike the previous examples, in this situation there is a small bias in the estimation of the states. However, it is small in terms of magnitude (less than 16.16% in average), and it can be concluded that the observer presents a good performance. It should be noted that this system has stronger nonlinearities than the previous case studies: while the previous examples were just bilinear systems, the FHN system presents square and cubic terms with cosine functions, a fact that could affect the computation of the OSL-QIB constants.

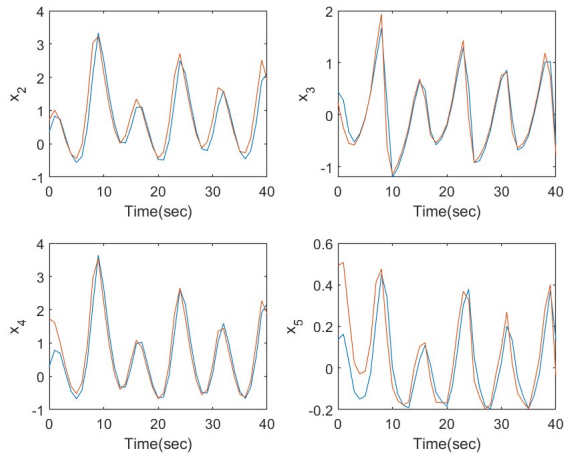

FIGURE 7 FHN system simulation without noise, Theorem 1 $ISE = 12.4$ Blue line real variable, orange line estimated variable

A second simulation scenario is proposed, considering noise in the measurement. Figure 8 shows that the error dynamics for each of the state variables converges to 0. This is due to the fact that, although the noise and its derivative are minimized during process operation, eliminating its negative effect is very demanding (since the condition $SR = 0$

cannot always be fulfilled). However, this proves the fact that the observer could be implemented in complex situations and that its performance is satisfactory when testing various application cases.

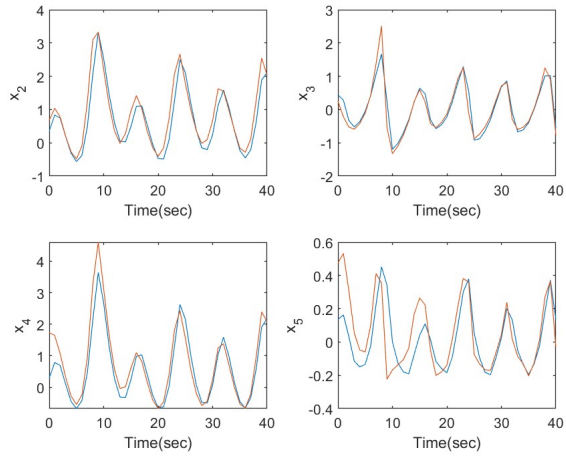

FIGURE 8 FHN system simulation with noise, Theorem 3 $ISE = 16.8$. Blue line real variable, orange line estimated variable

Remark: The observer proposed by [9] for Lipschitz systems was tested in the FHN example for comparison purposes. However, the feasibility conditions could not be satisfied just considering the Lipschitz constant. This case exemplifies the advantage of describing a nonlinear system under its OSL-QIB constants, since the Lipschitz constant does not provide sufficient information in certain cases to obtain satisfactory design conditions for nonlinear observers.

5 | CONCLUSIONS AND FUTURE RESEARCH

This paper presented LMI- and LME-based unknown input observer design strategies for nonlinear OSL-QIB noisy systems. Nonlinearities in the state and noise in the measurement were incorporated into the analysis. Based on Lyapunov stability theory, convergence conditions for the observer were developed and tested considering three different examples, namely a CTSR, a water-tank open channel and an FHS system, obtaining good performance in the scenarios considered. From the proposed methodology it was possible to develop an estimation algorithm that was capable of filtering the noise and obtaining finite-time convergence for the error dynamics. The results suggest that the observer could be useful in real applications where unknown or unmodeled inputs and instrumentation uncertainty (noise) are present.

As future research directions, it can be interesting to apply the observer strategy to real case studies such as water systems and chemical processes, and improve its performance in other situations that can arise in practice, e.g., parameter uncertainty and time delays.

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