The Kalman Filter

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The Kalman Filter developed in the early sixties by R.E. Kalman [7, 8] is a recursive state estimator for partially observed non-stationary stochastic processes. It gives an optimal estimate in the least squares sense of the actual value of a state vector from noisy observations.

1 Recursive State Estimation

Consider a discrete-time stochastic process

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k, \mathbf{v}_k) \tag{1}$$

with system input $\mathbf{u}$ and unmodeled process dynamics plus noise $\mathbf{v}$. The task at hand is to find an estimate of the state vector $\mathbf{x}$. However, $\mathbf{x}$ is only accessible from noise distorted sensor measurements

$$\mathbf{z}_k = \mathbf{h}(\mathbf{x}_k, \mathbf{w}_k) \tag{2}$$

in which as with the process model, $\mathbf{w}$ represents observation model inaccuracies and sensor noise.

Recursive state estimation consists on iteratively reconstructing the state vector from our knowledge of the process dynamics, the measurement model, and the sensed data.

Let $\mathbf{x}_{i|j}$, $i \geq j$, be the estimate of the state $\mathbf{x}_i$ using the observation information up to and including time j, $\mathbf{Z}^j = \{\mathbf{z}_0, \ldots, \mathbf{z}_j\}$. Given an estimate $\mathbf{x}_{k|k}$, and the input to the system $\mathbf{u}_k$, the predicted state $\mathbf{x}_{k+1|k}$ is ideally given by the expectation

$$\mathbf{x}_{k+1|k} = E[\mathbf{f}(\mathbf{x}_k, \mathbf{u}_k, \mathbf{v}_k)|\mathbf{Z}^k]. \tag{3}$$

We call $\mathbf{x}_{k+1|k}$ the *a priori* estimate of $\mathbf{x}_{k+1}$, and compute it from a noise-free version of Eq. 1, the estimate $\mathbf{x}_{k|k}$, and the input that hypothetically would drive the process from $\mathbf{x}_k$ to $\mathbf{x}_{k+1}$

$$\mathbf{x}_{k+1|k} = \mathbf{f}(\mathbf{x}_{k|k}, \mathbf{u}_k, \mathbf{0}). \tag{4}$$

Combining this result with the discrete-time measurement model from Eq. 2, we can also predict a noise-free a priori estimate of the sensor measurements

$$\mathbf{z}_{k+1|k} = \mathbf{h}(\mathbf{x}_{k+1|k}, \mathbf{0}). \tag{5}$$

By comparing the actual measurement vector $\mathbf{z}_{k+1}$ with the predicted data $\mathbf{z}_{k+1|k}$, we obtain an observation prediction error which in turn is added in a correction term to the a priori state estimate to produce an *a posteriori* state estimate.

$$\mathbf{x}_{k+1|k+1} = \mathbf{x}_{k+1|k} + \mathbf{K}_{k+1}(\mathbf{z}_{k+1} - \mathbf{z}_{k+1|k}). \tag{6}$$

The choice of the gain matrix $\mathbf{K}$ usually meets some optimality criteria. In the case of the Kalman Filter, the stochastic nature of the process and measurement dynamics is taken into account in the derivation of $\mathbf{K}$, producing an optimal linear estimator that minimizes the squared error on the expected value of the state estimate $\mathbf{x}_{k+1|k+1}$.

2 Linear Kalman Filter

Consider the case in which the process and measurement models correspond to a possibly non-stationary[1] discrete-time linear system, and that both the process and sensor noises are zero-mean white[2] and Gaussian with covariance matrices $\mathbf{Q}_k$ and $\mathbf{R}_k$ respectively, then Eqs. 1 and 2 become

$$\mathbf{x}_{k+1} = \mathbf{F}_k\mathbf{x}_k + \mathbf{u}_k + \mathbf{v}_k \tag{7}$$
$$\mathbf{z}_k = \mathbf{H}_k\mathbf{x}_k + \mathbf{w}_k \tag{8}$$

where

$$E[\mathbf{v}_k] = \mathbf{0}, \quad E[\mathbf{v}_k{\mathbf{v}_k}^\top] = \mathbf{Q}_k, \quad \text{and} \quad E[\mathbf{v}_i{\mathbf{v}_j}^\top] = \mathbf{0}, \quad \forall i \neq j \tag{9}$$
$$E[\mathbf{w}_k] = \mathbf{0}, \quad E[\mathbf{w}_k{\mathbf{w}_k}^\top] = \mathbf{R}_k, \quad \text{and} \quad E[\mathbf{w}_i{\mathbf{w}_j}^\top] = \mathbf{0}, \quad \forall i \neq j \tag{10}$$

The a priori and a posteriori state estimation errors can be written as

$$\mathbf{e}_{k+1|k} = \mathbf{x}_{k+1} - \mathbf{x}_{k+1|k} \tag{11}$$
$$\mathbf{e}_{k+1|k+1} = \mathbf{x}_{k+1} - \mathbf{x}_{k+1|k+1} \tag{12}$$

[1]Hence Kalman filter's beauty, compared to its predecessor the Weiner filter that only works for stationary linear systems.

[2]Temporally uncorrelated and with equal power at all frequencies.

and from the linear model in Eq. 7, the noise-free a priori state estimate in Eq. 4 takes the form

$$\mathbf{x}_{k+1|k} = \mathbf{F}_k \mathbf{x}_{k|k} + \mathbf{u}_k. \tag{13}$$

It follows that the a priori state estimate error is given by

$$\mathbf{e}_{k+1|k} = \mathbf{F}_k \mathbf{e}_{k|k} + \mathbf{v}_k. \tag{14}$$

Substituting Eq 6 and the observation models

$$\mathbf{z}_{k+1|k} = \mathbf{H}_{k+1} \mathbf{x}_{k+1|k}$$

and

$$\mathbf{z}_{k+1} = \mathbf{H}_{k+1} \mathbf{x}_{k+1} + \mathbf{w}_{k+1}$$

in Eq. 12, we obtain a recursive expression for the a posteriori state estimation error

$$\mathbf{e}_{k+1|k+1} = \mathbf{e}_{k+1|k} - \mathbf{K}_{k+1}(\mathbf{H}_{k+1} \mathbf{e}_{k+1|k} + \mathbf{w}_{k+1}). \tag{15}$$

The state error covariances are given by the expectations of the square of the state errors.

$$\mathbf{P}_{k+1|k} = E[\mathbf{e}_{k+1|k} \mathbf{e}_{k+1|k}{}^\top] \tag{16}$$

$$\mathbf{P}_{k+1|k+1} = E[\mathbf{e}_{k+1|k+1} \mathbf{e}_{k+1|k+1}{}^\top]. \tag{17}$$

Substituting Eq. 14 in Eq. 16 and taking the expectations on $\mathbf{v}$, we get the following expression for the a priori state error covariance

$$\mathbf{P}_{k+1|k} = \mathbf{F}_k \mathbf{P}_{k|k} \mathbf{F}_k{}^\top + \mathbf{Q}_k. \tag{18}$$

For simplicity of notation, in the sequel we rewrite the dependencies $(k+1|k)$ and $(k+1|k+1)$ as $^\ominus$ and $^\oplus$ respectively, and when no step reference is provided, $(k+1)$ is assumed. Substituting Eq. 12 in Eq. 17 and taking the expectations on $\mathbf{w}$ and $\mathbf{e}^\ominus$, the a posteriori error covariance takes the form

$$\mathbf{P}^\oplus = \mathbf{P}^\ominus - \mathbf{P}^\ominus \mathbf{H}^\top \mathbf{K}^\top - \mathbf{K}\mathbf{H}\mathbf{P}^\ominus + \mathbf{K}(\mathbf{H}\mathbf{P}^\ominus \mathbf{H}^\top + \mathbf{R})\mathbf{K}^\top. \tag{19}$$

The gain matrix $\mathbf{K}$ is chosen to minimize the a posteriori error covariance. Making the derivative of the trace of $\mathbf{P}^\oplus$ with respect to $\mathbf{K}$ equal to $\mathbf{0}$, and solving for $\mathbf{K}$ we get the optimal gain for the computation of Eq. 6, i.e., the Kalman gain

$$\mathbf{K} = \mathbf{P}^\ominus \mathbf{H}^\top (\mathbf{H}\mathbf{P}^\ominus \mathbf{H}^\top + \mathbf{R})^{-1}. \tag{20}$$

Substituting Eq. 20 back in Eq. 19 reduces $\mathbf{P}^\oplus$ to the well known form

$$\mathbf{P}^\oplus = \mathbf{P}^\ominus - \mathbf{K}\mathbf{H}\mathbf{P}^\ominus. \tag{21}$$

By inspecting the Kalman filter equations the behavior of the filter agrees with our intuition. The Kalman gain is proportional to the uncertainty in the state estimate and inversely proportional to that in the measurements. If sensor readings are very uncertain, and the state estimate is relatively precise, then the Kalman gain has little impact on the update of the state estimate in Eq. 6, and the system relies heavily on the system model. If, on the other hand, the uncertainty in the measurement is small and that in the state estimate is large, then $\mathbf{K}$ is also large, thus trusting more in sensor measurements for the correction of the state estimate.

However, when sensor measurements are uncertain the second term in Eq. 21 is small and the state estimate error covariance sees little reduction. Conversely, accurate sensor measurements contribute considerably in reducing the state estimation error.

Given the initial conditions $\mathbf{x}_{0|0}$ and $\mathbf{P}_{0|0}$, the complete recursion in the Kalman filter is computed iteratively with the following steps:

- Predict the a priori state, error covariance, and observation estimates

$$\begin{aligned} \mathbf{x}^{\ominus} &= \mathbf{F}_k \mathbf{x}_{k|k} + \mathbf{u}_k \\ \mathbf{P}^{\ominus} &= \mathbf{F}_k \mathbf{P}_{k|k} \mathbf{F}_k{}^{\top} + \mathbf{Q}_k \\ \mathbf{z}^{\ominus} &= \mathbf{H}\mathbf{x}^{\ominus} \end{aligned}$$

- Compute the Kalman gain and correct the state and state error covariance estimates

$$\begin{aligned} \mathbf{K} &= \mathbf{P}^{\ominus}\mathbf{H}^{\top}(\mathbf{H}\mathbf{P}^{\ominus}\mathbf{H}^{\top} + \mathbf{R})^{-1} \\ \mathbf{x}^{\oplus} &= \mathbf{x}^{\ominus} + \mathbf{K}(\mathbf{z} - \mathbf{z}^{\ominus}) \\ \mathbf{P}^{\oplus} &= \mathbf{P}^{\ominus} - \mathbf{K}\mathbf{H}\mathbf{P}^{\ominus} \end{aligned}$$

3 Extended Kalman Filter

Consider now the case when the process and observation models in Eqs. 1 and 2 are non-linear. The Extended Kalman Filter (EKF) provides a solution by linearizing the process about the current state, and linearizing the measurement model about the predicted observation.

The linearization of $\mathbf{f}$ about the current estimate $\mathbf{x}_{k|k}$ can be formulated as a Taylor series with the higher order terms dropped, that is:

$$\mathbf{x} \approx \mathbf{x}^{\ominus} + \nabla \mathbf{f}_{\mathbf{x}}(\mathbf{x}_k - \mathbf{x}_{k|k}) + \nabla \mathbf{f}_{\mathbf{v}} \mathbf{v}_k.$$

Similarly, the linearization of the observation model takes the form

$$\mathbf{z} \approx \mathbf{z}^{\ominus} + \nabla \mathbf{h}_{\mathbf{x}}(\mathbf{x} - \mathbf{x}^{\ominus}) + \nabla \mathbf{h}_{\mathbf{w}} \mathbf{w}.$$

The noise-free estimates $\mathbf{x}^{\ominus}$ and $\mathbf{z}^{\ominus}$ are given in Eqs. 4 and 5, and the various Jacobian matrices contain the partial derivatives of $\mathbf{f}$ and $\mathbf{h}$ with respect to $\mathbf{x}$

and the noises $\mathbf{v}$ and $\mathbf{w}$

$$\begin{aligned}
\nabla \mathbf{f_x} &= \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{(\mathbf{x}_{k|k},\mathbf{u}_k,\mathbf{0})} \\
\nabla \mathbf{f_v} &= \left.\frac{\partial \mathbf{f}}{\partial \mathbf{v}}\right|_{(\mathbf{x}_{k|k},\mathbf{u}_k,\mathbf{0})} \\
\nabla \mathbf{h_x} &= \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{(\mathbf{x}^{\ominus},\mathbf{0})} \\
\nabla \mathbf{h_w} &= \left.\frac{\partial \mathbf{h}}{\partial \mathbf{w}}\right|_{(\mathbf{x}^{\ominus},\mathbf{0})}
\end{aligned}$$

Following the same discussion as in the previous section but with this new linear model, it is easy to show how the complete recursion for the Extended Kalman Filter involves the following steps:

- Predict the a priori state and observation estimates as well as the a priori state error covariance estimate

$$\begin{aligned}
\mathbf{x}^{\ominus} &= \mathbf{f}(\mathbf{x}_{k|k},\mathbf{u}_k,\mathbf{0}) \\
\mathbf{P}^{\ominus} &= \nabla \mathbf{f_x} \mathbf{P}_{k|k} \nabla \mathbf{f_x}^{\top} + \nabla \mathbf{f_v} \mathbf{Q}_k \nabla \mathbf{f_v}^{\top} \\
\mathbf{z}^{\ominus} &= \mathbf{h}(\mathbf{x}^{\ominus},\mathbf{0})
\end{aligned}$$

- Compute the Kalman gain and correct the state and state error covariance estimates

$$\begin{aligned}
\mathbf{K} &= \mathbf{P}^{\ominus} \nabla \mathbf{h_x}^{\top} (\nabla \mathbf{h_x} \mathbf{P}^{\ominus} \nabla \mathbf{h_x}^{\top} + \nabla \mathbf{h_w} \mathbf{R} \nabla \mathbf{h_w}^{\top})^{-1} \\
\mathbf{x}^{\oplus} &= \mathbf{x}^{\ominus} + \mathbf{K}(\mathbf{z} - \mathbf{z}^{\ominus}) \\
\mathbf{P}^{\oplus} &= \mathbf{P}^{\ominus} - \mathbf{K} \nabla \mathbf{h_x} \mathbf{P}^{\ominus}
\end{aligned}$$

It is important to note however, that the linearization of the nonlinear process and measurement models in the EKF does not preserve the distributions of the state and measurement random variables as normal. This may lead to difficulties in the implementation and tuning of the EKF, making it only reliable for systems that are almost linear on the time scale interval $(k, k+1)$.

4 Conditioning

It turns out that the recursion in Eq. 21 is ill-conditioned. As the filter converges, the cancelling of significant digits on $\mathbf{P}^{\oplus}$ may lead to asymmetries or to a non *positive semi definite* (psd) matrix, which cannot be true from the definition in Eq. 17 of the a posteriori error covariance matrix.

An algebraic manipulation that guarantees $\mathbf{P}^{\oplus}$ psd is obtained by multiplying Eq. 20 by $(\mathbf{H}\mathbf{P}^{\ominus}\mathbf{H}^{\top} + \mathbf{R})\mathbf{K}^{\top}$, rearranging terms

$$\mathbf{K}\mathbf{H}\mathbf{P}^{\ominus}\mathbf{H}^{\top}\mathbf{K}^{\top} - \mathbf{P}^{\ominus}\mathbf{H}^{\top}\mathbf{K}^{\top} + \mathbf{K}\mathbf{R}\mathbf{K}^{\top} = \mathbf{0} \tag{22}$$

and adding Eq. 22 into Eq. 21

$$\mathbf{P}^{\oplus} = (\mathbf{I} - \mathbf{KH})\mathbf{P}^{\ominus}(\mathbf{I} - \mathbf{KH})^{\top} + \mathbf{KRK}^{\top} \tag{23}$$

The recursivity in Eq. 23 is known as the Joseph form of the a posteriori error covariance matrix, and given its quadratic nature it is obviously psd.

5 Sequential Innovation

When combining information from multiple sensors or from multiple data sources, the observation vector $\mathbf{z}$ can be seen as a collection of n independent measurements $\mathbf{z}^{(i)}$ coming from the same number of independent sources at any particular time instance $(k+1)$.

It is possible to process each of these observations independently provided $\mathbf{R}$ is block diagonal. This is, when the set of measurements taken at the same time interval are uncorrelated. Even when the measurements are correlated, they may always be transformed into uncorrelated data which then may be treated sequentially. The process is called *whitening* (see [3]).

Starting from $\mathbf{x}^{\oplus,0} = \mathbf{x}^{\ominus}$, and $\mathbf{P}^{\oplus,0} = \mathbf{P}^{\ominus}$, the a posteriori state estimate is iteratively given by

$$\mathbf{x}^{\oplus,i} = \mathbf{x}^{\oplus,i-1} + \mathbf{K}_{(i)}\left(\mathbf{z}^{(i)} - \mathbf{H}^{(i)}\mathbf{x}^{\oplus,i-1}\right).$$

The key advantage of the *sequential innovation* method is that the complexity in the computation of the Kalman gain is reduced considerably. From Eq. 20

$$\mathbf{K}_{(i)} = \mathbf{P}^{\oplus,i-1}\mathbf{H}^{(i)\top}\left(\mathbf{H}^{(i)}\mathbf{P}^{\oplus,i-1}\mathbf{H}^{(i)\top} + \mathbf{R}^{(i)}\right)^{-1}. \tag{24}$$

The required inverse in Eq. 24 has the dimension of each of the observed variables, and is considerably much smaller than the dimension of the entire measurement vector $\mathbf{z}$ as required in Eq. 20. When a sensor returns scalar values for each independent measurement, then the inverse in Eq. 24 becomes just a scalar division.

Given the initial conditions $\mathbf{x}_{0|0}$ and $\mathbf{P}_{0|0}$, the complete sequential innovation Kalman filter recursion is computed with the following steps:

- Predict the a priori state and state error covariance

$$\begin{aligned}\mathbf{x}^{\ominus} &= \mathbf{F}_k\mathbf{x}_{k|k} + \mathbf{u}_k\\ \mathbf{P}^{\ominus} &= \mathbf{F}_k\mathbf{P}_{k|k}\mathbf{F}_k^{\top} + \mathbf{Q}_k\end{aligned}$$

- For each measurement, iteratively compute the corresponding innovation and Kalman gain column and correct the state and state error covariance estimates

$$
\begin{array}{llcl}
\text{initialize} & \mathbf{x}^{\oplus,0} & = & \mathbf{x}^{\ominus} \\
 & \mathbf{P}^{\oplus,0} & = & \mathbf{P}^{\ominus} \\
\forall i & \mathbf{K}^{(i)} & = & \mathbf{P}^{\oplus,i-1}\mathbf{H}^{(i)\top}\left(\mathbf{H}^{(i)}\mathbf{P}^{\oplus,i-1}\mathbf{H}^{(i)\top}+\mathbf{R}^{(i)}\right)^{-1} \\
 & \mathbf{x}^{\oplus,i} & = & \mathbf{x}^{\oplus,i-1}+\mathbf{K}_{(i)}\left(\mathbf{z}^{(i)}-\mathbf{H}^{(i)}\mathbf{x}^{\oplus,i-1}\right) \\
 & \mathbf{P}^{\oplus,i} & = & \mathbf{P}^{\oplus,i-1}-\mathbf{K}_{(i)}\mathbf{H}^{(i)}\mathbf{P}^{\oplus,i-1} \\
\text{restore} & \mathbf{x}^{\oplus} & = & \mathbf{x}^{\oplus,n} \\
 & \mathbf{P}^{\oplus} & = & \mathbf{P}^{\oplus,n}
\end{array}
$$

6 Bibliographical Notes

The reader can find thorough discussions on the Kalman Filter in [1, 2, 7, 8, 9, 10, 11, 13], and on its predecessor the Weiner Filter in [4]. One approach to reduce the effect of nonlinearities is to apply iteratively the filter (IEKF) as in [15]. Another solution is to use the Unscented Kalman Filter (UKF), an extension to the EKF that takes into account the nonlinear transformation of means and covariances [5, 6]. Numerical instability may occur even with the Joseph form of the error covariance matrix. An alternative is the use of the square-root Kalman filter (SKF), in which recursive computations for $\mathbf{P}^{\oplus}$ are substituted by equations for a recursion in ${\mathbf{P}^{\oplus}}^{1/2}$ [2]. Sequential innovation in Kalman filtering is discussed in detail in [1, 2, 12, 14].

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