ESTIMATOR STABILITY ANALYSIS IN SLAM

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Abstract: This work presents an analysis of the state estimation error dynamics for a linear system within the Kalman filter based approach to Simultaneous Localization and Map Building. Our objective is to demonstrate that such dynamics is marginally stable. The paper also presents the necessary modifications required in the observation model, in order to guarantee zero mean stable error dynamics. Simulations for a one-dimensional robot and a planar vehicle are presented.

Keywords: SLAM, Kalman Filter, Observability

1. INTRODUCTION

The study of stochastic models for Simultaneous Localization and Map Building (SLAM) in mobile robotics has been an active research topic for over fifteen years. Within the Kalman filter (KF) approach to SLAM, seminal work (Smith and Cheeseman, 1986) suggested that as successive landmark observations take place, the correlation between the estimates of the location of such landmarks in a map grows continuously. This observation was later ratified (Dissanayake *et al.*, 2001) with a proof showing that the estimated map converges monotonically to a relative map with zero uncertainty. They also showed how the absolute accuracy of the map reaches a lower bound defined only by the initial vehicle uncertainty.

In a dynamic system, such as the one typically used in SLAM, where both the state and measurement dynamics are corrupted by noise, it is important to know whether it is possible or not to reconstruct the entire state space from output measurements. In this paper, we present an analysis of the state estimator that shows how, with the typical measurement model, it is not possible to obtain zero mean error state estimation.

Zero mean state estimation does not pose a problem in SLAM if a perfectly accurate relative map is sought, such as the one in (Newman, 1999). However, when building an absolute map, and at the same time estimating the absolute vehicle location purely from sensor measurements and odometry, the results of the Kalman filter-based approach to SLAM will be subject to the error produced at the very first iteration.

Unfortunately, the state space constructed by appending the robot pose and the landmark locations is partially observable (Andrade-Cetto and Sanfeliu, 2004), and the consequence is precisely, that the absolute map accuracy is bounded by the initial vehicle uncertainty. In this paper, we develop the equations for the simplest case in SLAM, a one-dimensional vehicle with one landmark, in order to show the behavior of the observer. We are able to see in the end, that the entire closed loop system, including the estimator, has a pole

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in one, making the filter marginally stable. One way to make sure that the entire system is fully observable, is by slightly modifying the observation model, either anchoring the state estimate to one landmark, or by adding an external sensor.

The paper is structured as follows. The system and the estimation procedures are presented in Section 2. In Section 3 we show the marginally stable estimation error dynamics for the one-landmark monobot vehicle, and the way to improve the behavior of the filter. In the same section simulations results are presented. Section 4 shows results for a planar vehicle model. Conclusions are presented in the last section.

2. STATE ESTIMATION IN SLAM

2.1 The Kalman Filter

Considering the problem of SLAM as stochastic estimation of a discrete-time linear system, and expressed as a vector difference equation with additive white Gaussian noise that models unpredictable disturbances; the dynamic plant equation is simply

$$\mathbf{x}_{k+1} = \mathbf{F}\mathbf{x}_k + \mathbf{G}\mathbf{u}_k + \mathbf{v}_k \tag{1}$$

where $\mathbf{x}_k$ is the augmented state vector formed by appending the vehicle state estimate and the landmark location estimates, $\mathbf{u}_k$ is a known input vector, and $\mathbf{v}_k$, is the k-th term of a sequence of zero-mean Gaussian process noise with covariance $\mathbf{V}_k = E\left[\mathbf{v}_k\mathbf{v}_k^\top\right]$.

The measurement equation is

$$\mathbf{z}_k = \mathbf{H}\mathbf{x}_k + \mathbf{w}_k \tag{2}$$

with $\mathbf{w}_k$ the k-th term of a sequence of zero-mean white Gaussian measurement noise with covariance $\mathbf{W}_k = E\left[\mathbf{w}_k\mathbf{w}_k^\top\right]$.

Using the aforementioned linear system, the algorithm for the optimal state estimator (the Kalman filter) is as follows:

First, compute the a priori state prediction

$$\mathbf{x}_{k+1|k} = \mathbf{F}\mathbf{x}_{k|k} + \mathbf{G}\mathbf{u}_k \tag{3}$$

Followed by an a priori measurement prediction

$$\mathbf{z}_{k+1|k} = \mathbf{H}\mathbf{x}_{k+1|k} \tag{4}$$

Next, compute the a posteriori state estimate, known also as the update of the state estimate

$$\mathbf{x}_{k+1|k+1} = \mathbf{x}_{k+1|k} + \mathbf{K}\left(\mathbf{z}_{k+1} - \mathbf{z}_{k+1|k}\right) \tag{5}$$

where $\mathbf{K}$ is the Kalman gain for optimal estimation in the mean square error sense.

Finally, replacing the system from equations 3 and 4 in equation 5, the closed-loop system becomes

$$\begin{aligned}\mathbf{x}_{k+1|k+1} = &\left(\mathbf{F} - \mathbf{KHF}\right)\mathbf{x}_{k|k} + \\ &\left(\mathbf{G} - \mathbf{KHG}\right)\mathbf{u}_k + \mathbf{K}\mathbf{z}_{k+1}\end{aligned} \tag{6}$$

In order to compute the optimal filter gain for the linear system one needs:

The state prediction covariance

$$\mathbf{P}_{k+1|k} = \mathbf{F}\mathbf{P}_{k|k}\mathbf{F}^\top + \mathbf{V}_k, \tag{7}$$

the innovation covariance

$$\mathbf{S}_{k+1} = \mathbf{W}_{k+1} + \mathbf{H}\mathbf{P}_{k+1|k}\mathbf{H}^\top \tag{8}$$

and finally, the filter gain

$$\mathbf{K} = \mathbf{P}_{k+1|k}\mathbf{H}_{k+1}^\top\mathbf{S}_{k+1}^{-1} \tag{9}$$

The update of the state covariance is computed with

$$\mathbf{P}_{k+1|k+1} = \mathbf{P}_{k+1|k} - \mathbf{K}\mathbf{S}_{k+1}\mathbf{K}^\top \tag{10}$$

For constant plant and sensor covariances, the steady state value for the covariance matrix is given by the solution of the Riccati equation

$$\mathbf{P} = \mathbf{F}(\mathbf{P} - \mathbf{P}\mathbf{H}^\top(\mathbf{H}\mathbf{P}\mathbf{H}^\top + \mathbf{W})^{-1}\mathbf{H}\mathbf{P})\mathbf{F}^\top + \mathbf{V} \tag{11}$$

Such solution to the Riccati equation will converge, in the linear case, to a positive semi definite steady state covariance in terms of $\mathbf{P}_{r,0|0}$, $\mathbf{V}$, $\mathbf{W}$, and the total number of landmarks n (Gibbens *et al.*, 2000). Notice however, that for the nonlinear case, the computation of the Jacobians $\mathbf{F}$ and $\mathbf{H}$ will in general also depend on the steady state value of $\mathbf{x}$.

Now, consider the case of suboptimal filtering, in which covariance inflation is used for decorrelation (Julier, 2003). In that case, partial observability of the pair $\{\mathbf{F}, \mathbf{H}\}$ might produce in some cases an unbounded value for $\mathbf{P}$, unless, full observability is guaranteed (Bar-Shalom *et al.*, 2001).

For the optimal filter, under detectability, there is at least one positive semi definite solution of equation 11 such that the filter is marginally stable (Kailath *et al.*, 2000). We show next how in the linear case of SLAM, any solution of equation 11 produces a marginally stable solution for the estimation error.

3. STATE ESTIMATE ERROR DYNAMICS

Defining the estimation error $\tilde{\mathbf{x}}_k$ as

$$\tilde{\mathbf{x}}_k = \mathbf{x}_k - \mathbf{x}_{k|k} \tag{12}$$

Then, with the appropriate substitutions we obtain

$$\tilde{\mathbf{x}}_{k+1} = (\mathbf{F} - \mathbf{KHF})\tilde{\mathbf{x}}_k + (\mathbf{I} - \mathbf{KH})\mathbf{v}_k - \mathbf{K}\mathbf{w}_{k+1} \tag{13}$$

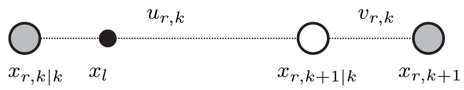

Fig. 1. Monobot, one-dimensional mobile robot.

Unfortunately, the state vector formed by appending the vehicle position estimate with the landmark location estimates is not fully observable, and only partially controllable (one-controllable in the case of the monobot); making one eigenvalue of the matrix $(\mathbf{F}-\mathbf{KHF})$ lay on the unitary circle.

That means the matrix $(\mathbf{F}-\mathbf{KHF})$ is marginally stable as a consequence of the partial observability. The pole equal to one corresponds to a state estimate signal corrupted with constant amplitude.

3.1 One Landmark Monobot

Consider the one-dimensional robot (monobot) from Figure 1. The robot location is $x_{r,k}$ and the motion command is u_k. The robot error dynamics is modeled with the additive term v_k, and the entire system is simply

$$x_{r,k+1} = x_{r,k} + u_k + v_k \quad (14)$$

$$x_{l,k+1} = x_{l,k} \quad (15)$$

The map for this simple model is just the static landmark $\mathbf{x}_l$. The observation model for such landmark is

$$z_{k+1} = x_{l,k+1} - x_{r,k+1} + w_k \quad (16)$$

with w_k the landmark observation error.

For any value of the Kalman gain in this simplest SLAM configuration, $\mathbf{K} = [k_1, k_2]^\top$, the eigenvalues of the matrix

$$\mathbf{F} - \mathbf{KHF} = \begin{bmatrix} k_1+1 & -k_1 \\ k_2 & -k_2+1 \end{bmatrix} \quad (17)$$

are $\{1, k_1 - k_2 + 1\}$. Appendix A shows the actual form of the eigenvalues of $(\mathbf{F}-\mathbf{KHF})$ as a function of the variances $\mathbf{V}$ and $\mathbf{W}$, in the very first iteration of the algorithm. The result is that regardless of the value of these variances, the error dynamics will be marginally stable, and the steady state error will tend to a constant value dependant on the initial conditions, instead of an error with zero mean.

3.2 Full observability in SLAM

Two different forms to make the system observable are presented in (Andrade-Cetto and Sanfeliu, 2004). That is, two different ways to make $(\mathbf{F}-\mathbf{KHF})$ asymptotically stable. One is by fixing

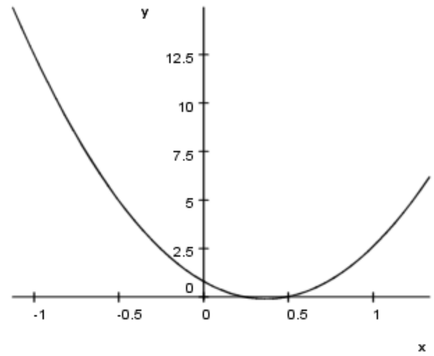

Fig. 2. One landmark monobot characteristic polynomial for the first iteration of $(\mathbf{F}-\mathbf{KHF})$.

a global reference at the origin. In that case, the measurement model for the one landmark linear system (monobot) with anchor becomes

$$\begin{bmatrix} z_k^{(0)} \\ z_k \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} w_k^{(0)} \\ w_k \end{bmatrix} \quad (18)$$

The anchor is taken as a global reference at the world origin. No map state is needed for it. The zero-th superscript in the measurement vector is used for the consistent indexing of landmarks and observations with respect to the original model. It can be easily shown that the observability matrix for this augmented measurement model is full rank.

The stability of the matrix $(\mathbf{F}-\mathbf{KHF})$ is subject to the values on the entries in $\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$. Appendix B shows explicitly these terms for the first iteration of a one landmark monobot; and presents also an expression for the computation of the characteristic polynomial of $(\mathbf{F}-\mathbf{KHF})$, assuming equal anchor and landmark variances.

Figure 2 plots for example, such characteristic polynomial for initial variances $\mathbf{P}_{r,0|0} = \mathbf{V} = \mathbf{W} = 1$. The roots in the plot indicate the value for the eigenvalues of $(\mathbf{F}-\mathbf{KHF})$, $\lambda_1 = 0.1491, \lambda_2 = 0.4663$; and as we can see, for the fully observable SLAM case, these lay within the unitary circle, thus guaranteeing filter stability, and consequently, zero mean state estimation.

In general terms, given that in the fully observable case the observability matrix has full rank, for any values of $\mathbf{V}, \mathbf{W}$, and $\mathbf{P}_{r,0|0}$, the eigenvalues of $(\mathbf{F}-\mathbf{KHF})$ will always be inside the unit circle of the complex z plane.

3.3 Simulations

Next, we show the effects of marginal stability on a vehicle under Brownian motion; and how

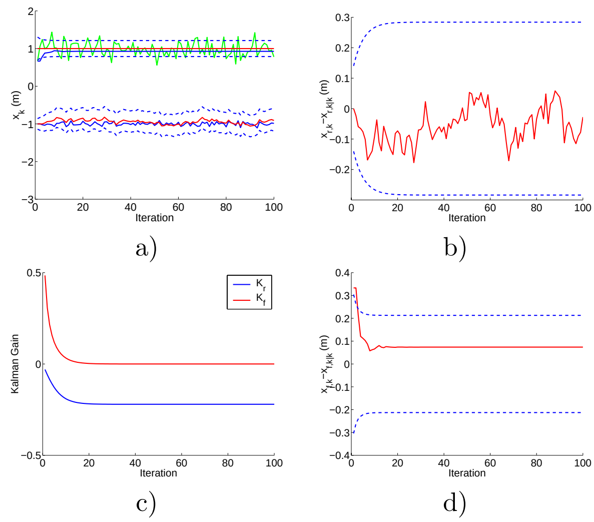

Fig. 3. KF SLAM for a one landmark monobot under Brownian motion.

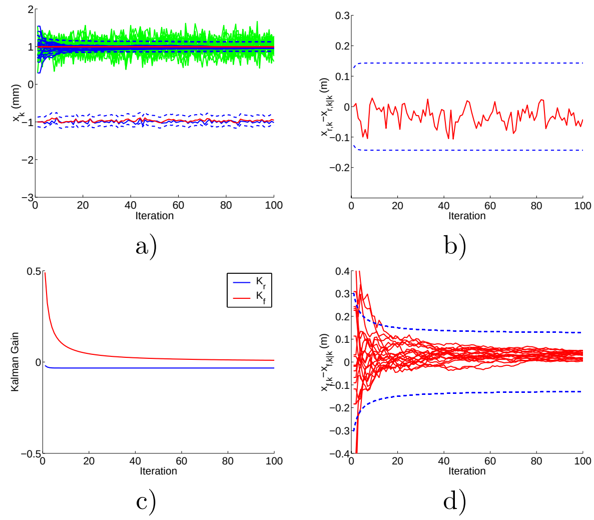

Fig. 4. KF SLAM for a twenty landmarks monobot under Brownian motion.

these effects are eliminated once full observability is guaranteed. Figures 3 and 4 plot the results of using the original Kalman filter approach to SLAM for a monobot that starts at location $x_{r,0} = -1m$, and is governed by Brownian motion only. All landmarks are located at $x_l^{(i)} = 1m$. The plots correspond as follows: a) full state estimate, b) vehicle estimation error with 2σ uncertainty bounds, c) vehicle and landmark Kalman gains, and d) landmark estimation error, also with 2σ uncertainty bounds.

The vehicle and landmark estimation errors do not converge to a zero mean signal (plots b and d). However, thanks to marginal filter stability, the filter does converge to a constant value. This value is less sensitive to the initial conditions when a large number of landmarks is used. Figures 5 and 6, in which partial observability has been revised, show zero mean steady state estimation error, and

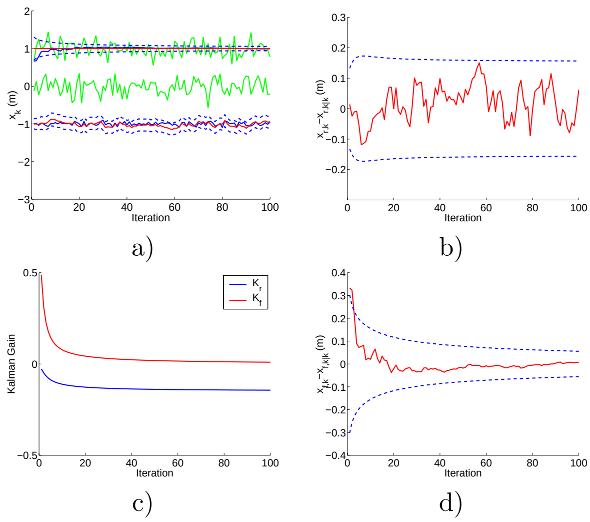

Fig. 5. KF SLAM for a one landmark plus anchor monobot under Brownian motion.

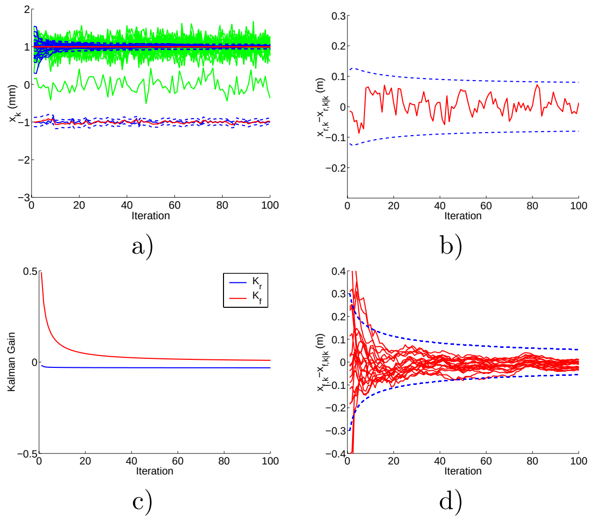

Fig. 6. KF SLAM for a twenty landmarks plus anchor monobot under Brownian motion.

smaller vehicle and landmark variances than those in Figures 3 and 4.

4. PLANAR VEHICLE

The dynamics of a more realistic mobile robot and sensor are governed by the following discrete-time nonlinear state transition model,

$$\mathbf{x}_{k+1} = \mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k, \mathbf{v}_k\right) \tag{19}$$

$$\mathbf{z}_k = \mathbf{h}\left(\mathbf{x}_k, \mathbf{w}_k\right) \tag{20}$$

The state vector $\mathbf{x}_k$ contains usually the pose (and sometimes the velocity) of the robot $\mathbf{x}_{r,k}$ at time step k, and a vector of stationary landmarks $\mathbf{x}_l^{(i)}$, i.e.,

$$\mathbf{x}_k = \begin{bmatrix} \mathbf{x}_{r,k} \\ \mathbf{x}_l^{(i)} \end{bmatrix} \tag{21}$$

where, in the case of our planar robot $\mathbf{x}_{r,k} = \left[x_k \; y_k \; \theta_k \right]^\top$ are the horizontal and vertical vehicle position and orientation, respectively. The vector $\mathbf{u}_k$ represents the velocity commands, and $\mathbf{v}_k$ and $\mathbf{w}_k$ are unmodeled motion and measurement noises, respectively.

The Extended Kalman Filter (EFK) is used in this non-linear case; and the linearization of $\mathbf{f}$ and $\mathbf{h}$ is evaluated at the a priori predictions $\mathbf{x}_{k+1|k}$ and $\mathbf{z}_{k+1|k}$ as a Taylor series with the higher order terms dropped.

(Andrade-Cetto and Sanfeliu, 2004) show that when using such linearized model for a planar vehicle, the resulting Jacobians still produce a partially observable SLAM, and proposes a way to make the system fully observable, similar to the linear case. For example, with the aid of a measurement model for a fixed global reference fixed at the origin

$$\mathbf{h}^{(0)} = -\mathbf{R}^\top \mathbf{t} + \mathbf{w}^{(0)} \tag{22}$$

with $\mathbf{w}^{(0)}$ the anchor observation error, and

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \mathbf{t} = [x_k, y_k]^\top \tag{23}$$

Visibility of a two dimensional anchor to aid as global reference guarantees fuul observability in this particular planar vehicle case.

4.1 Experiments

Figure 7 shows a run of the EKF SLAM algorithm over a $300m^2$ area in the second floor of the USC SAL building with data acquired with a laser range finder [1]. The effects of nonlinearities, together with the marginal stability of the filter produce large localization errors at the end of a nearly $90m$ loop.

The map obtained with the fully observable SLAM algorithm for the same data set is shown in Figure 8. In this run, the first observed landmark is used as an anchor. Furthermore, and to guarantee full observability during the entire run of the algorithm, every landmark revised more than 50 times by the filter was removed from the state vector, but its observations were still used to revise the remaining state vector elements.

5. CONCLUSIONS

Partial observability in SLAM produces a marginally stable filter, resulting in a unit norm eigenvalue for the error dynamics matrix $(\mathbf{F} - \mathbf{KHF})$, regardless of the plant and sensor variance parameters

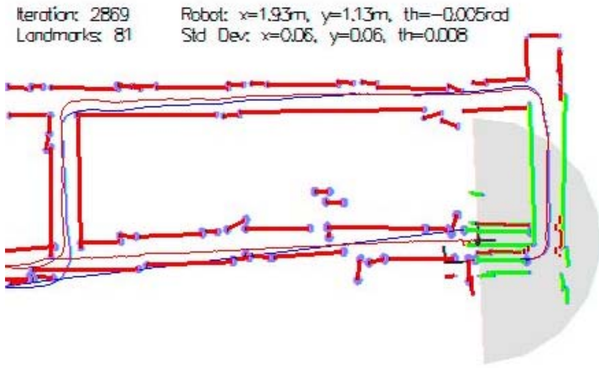

Fig. 7. EKF SLAM for a planar vehicle.

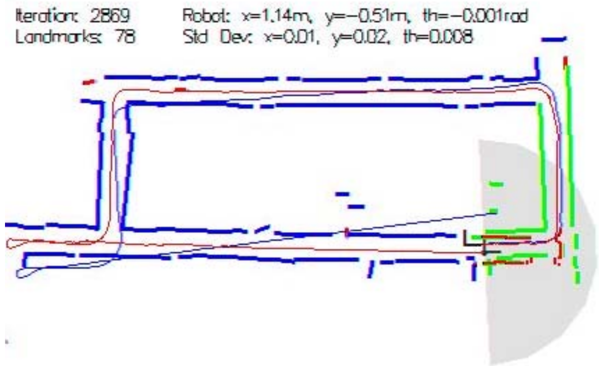

Fig. 8. EKF fully observable SLAM for a planar vehicle.

chosen. Marginal stability has the consequence of non-zero steady state vehicle and landmark estimation error. Even when a perfect relative map is computed, the absolute vehicle and landmark estimates will have non-zero steady state error; unless full observability is guaranteed.

6. APPENDICES

A. Eigenvalues of $(\mathbf{F}-\mathbf{KHF})$ *on the first iteration for a partially observable one landmark monobot.*

According to equations 1 and 2, the matrices of our reduced system are

$$\mathbf{F} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \; \mathbf{G} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \; \mathbf{H} = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

The KF gain for the one landmark monobot is computed as follows. First, the initial covariance is evaluated (Newman, n.d.)

$$\mathbf{P}_0 = \left[\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_r & 0 \\ 0 & W_0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^\top \right]^{-1}$$
$$= \begin{bmatrix} P_r & P_r \\ P_r & W_0 + P_r \end{bmatrix}$$

where $P_r = \mathbf{P}_{r,0|0}$ is the initial robot variance, and $W_0 = \mathbf{W}_0$ is the initial sensor error variance.

[1] Data from the Robotics Data Set Repository (Howard and Roy, 2003). Thanks to Andrew Howards.

We obtain the initial a priori estimate for the covariance matrix from equation 7,

$$\mathbf{P}_{1|0} = \begin{bmatrix} V_0 + P_r & P_r \\ P_r & W_0 + P_r \end{bmatrix}$$

with V_0 the initial variance for the plant error. The innovation term becomes

$$S_1 = V_0 + W_0 + W_1$$

Assuming the plant and measurement variances are constant, the last expression reduces to

$$S_1 = V + 2W$$

The KF gain for the first iteration in our system is thus,

$$\mathbf{K}_1 = \frac{1}{V + 2W} \begin{bmatrix} -V \\ W \end{bmatrix}$$

Substituting the above result in equation 17, the eigenvalues of the filter dynamics matrix at the first iteration for a one landmark monobot are $\{1, W/(V + 2W)\}$. A pole in one means marginal stability of $(\mathbf{F} - \mathbf{FKH})$.

B. Eigenvalues of $(\mathbf{F}-\mathbf{KHF})$ *on the first iteration for a fully observable one landmark monobot.*

Substituting for the fully observable measurement model we have that, for the first filter iteration

$$\begin{aligned} k_{11} &= (-2P_rVW + W^{(0)}(2W(V + P_r) + P_rV))/\gamma \\ k_{12} &= -VW^{(0)}(P_r + W^{(0)})/\gamma \\ k_{21} &= (-P_rVW - W^{(0)}(VW + 2P_rW + P_rV))/\gamma \\ k_{22} &= (P_rVW + \\ &\quad W^{(0)}(VW + 2P_rW + P_rv + WW^{(0)}))/\gamma \end{aligned}$$

where

$$\begin{aligned} \gamma = {} & 2VWP_r + W^{(0)}(2VW + 2VP_r + \\ & 4WP_r + VW^{(0)} + 2WW^{(0)}) \end{aligned}$$

The characteristic polynomial of $(\mathbf{F} - \mathbf{KHF})$, assuming equal anchor and landmark variances is

$$\begin{aligned} f(x) = x^2 - \frac{W\left(2VP_r + W(3W + 2V + 4P_r)\right)}{\gamma}x \\ + \frac{W^3(W + P_r)(4VP_r + 2W^2 + 3VW + 4WP_r)}{\gamma} \end{aligned}$$

The eigenvalues of the matrix $(\mathbf{F} - \mathbf{KHF})$, the roots of $f(x)$, are

$$\frac{1}{\gamma}\left(W(VP_r + \frac{3}{2}W^2 + VW + 2WP_r) \pm \frac{1}{2}\sqrt{\kappa}\right)$$

with $\kappa = W^2(W^4 + 8P_rWV^2 + 4V^2W^2 + 4V^2P_r^2)$.

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