INFINITESIMAL DISPLACEMENTS OF A LAMINA

DEFINITION AND PROPERTIES

Suppose that a lamina is instantly rotating about a vertical axis $\$$, with the instant center in $Q=(x_Q, y_Q)$, at an angular velocity $\vec{\omega} = \omega \vec{k}$ (fig. A1). The twist of the lamina is

$$\hat{T} = \omega \begin{bmatrix} y_Q \\ -x_Q \\ 1 \end{bmatrix} = \begin{bmatrix} v_{ox} \\ v_{oy} \\ \omega \end{bmatrix}$$

where $\vec{v}_o = v_{ox}\vec{i} + v_{oy}\vec{j}$ is the velocity of the origin point of the lamina. Now, let us assume that the lamina rotates an infinitesimal angle $\delta\phi$ during a very small time increment δt. Then, by analogy to the twist, we can define the <u>infinitesimal displacement</u> of the lamina, $\delta\hat{D}$, as the vector

$$\delta\hat{D} = \delta\phi \begin{bmatrix} y_Q \\ -x_Q \\ 1 \end{bmatrix} = \begin{bmatrix} \delta r_{ox} \\ \delta r_{oy} \\ \delta\phi \end{bmatrix}$$

where

$$\delta r_{ox} = \delta\phi \cdot y_Q \quad , \quad \delta r_{oy} = -\delta\phi \, x_Q$$

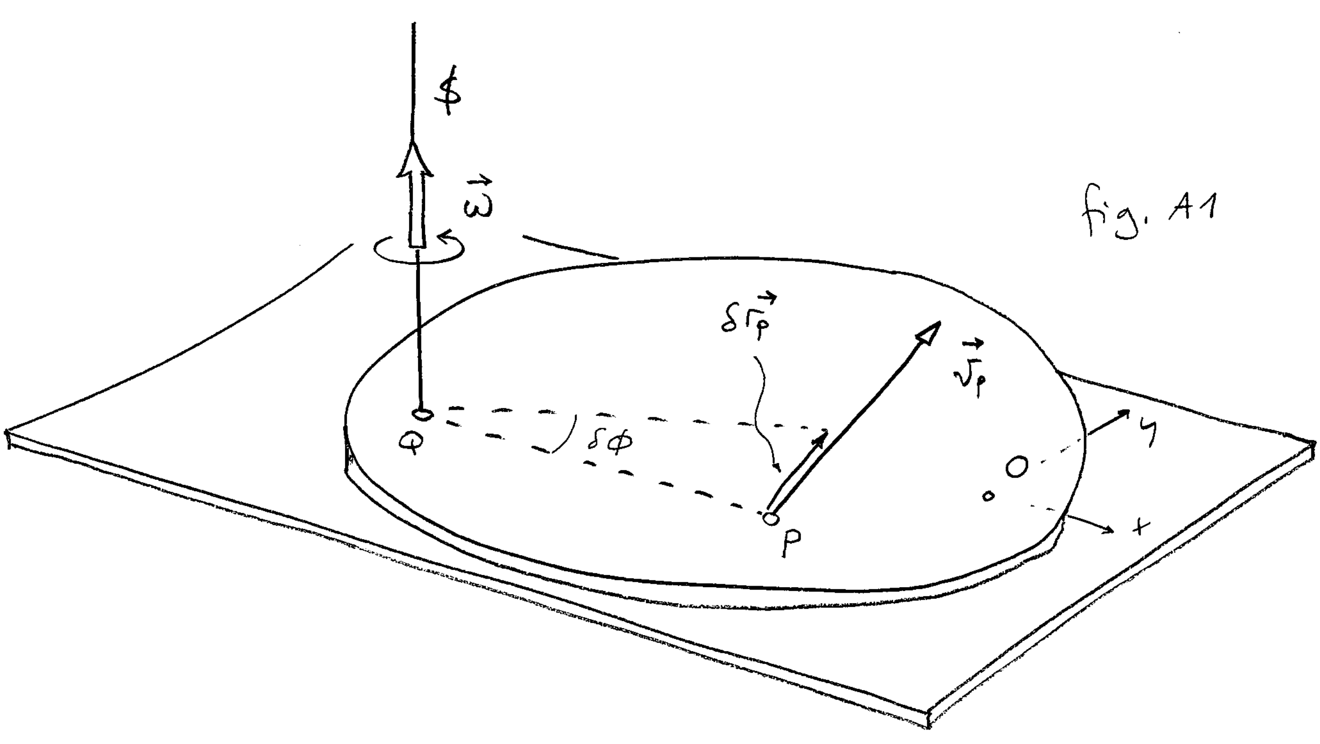

fig. A1

Note that, since $\omega = \frac{\delta\phi}{\delta t}$, then $\delta\hat{D}$ can be obtained by multiplying the twist $\hat{T}$ by δt. Recall that the twist $\hat{T}$ allows to compute the velocity of any point P of the lamina (fig. A1), since

$$\boxed{\vec{v}_P = \vec{\omega} \times \vec{r}_P + \vec{v}_O} \qquad (\text{where } \vec{r}_P = \vec{OP})$$

In a similar way, we next see that the infinitesimal displacement $\delta\hat{D}$ determines the small displacement $\delta\vec{r}_P$ undergone by P due to the small rotation of angle $\delta\phi$, because it will turn out that

$$\boxed{\delta\vec{r}_P = \delta\vec{\phi} \times \vec{r}_P + \delta\vec{r}_O}$$

where $\delta\vec{\phi} = \delta\phi \cdot \vec{k}$ and $\delta\vec{r}_O = \delta r_{Ox}\vec{i} + \delta r_{Oy}\vec{j}$

PROOF

First of all, we see that $\delta\vec{r}_O$ is the small displacement suffered by the origin point O of the lamina (this is a slight abuse of notation because we are giving the same name O to the origin point of the lamina, and to the origin of the coordinate system, but you understand it). Certainly, note that due to the small rotation $\delta\phi$ about $, O follows a circular path centered in Q, of radius $|\vec{QO}|$, and since $\delta\phi$ is very small, we can approximate $\delta\vec{r}_O$

by the length of the arc of circumference described (Fig. A2)

Thus, we can say that $\delta\vec{r}_0$ is a vector of the same direction than $\delta\phi \cdot \vec{k} \times \vec{QO}$, with norm $\delta\phi \cdot |\vec{QO}|$.

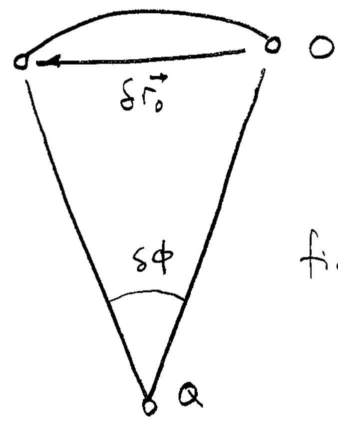

fig. A2

Thus it will be:

$$\delta\vec{r}_0 = \delta\vec{\phi} \times \vec{QO}$$

i.e.,

$$\delta\vec{r}_0 = \begin{bmatrix} 0 \\ 0 \\ \delta\phi \end{bmatrix} \times \begin{bmatrix} -x_Q \\ -y_Q \\ 0 \end{bmatrix} = \begin{bmatrix} \delta\phi \cdot y_Q \\ -\delta\phi \cdot x_Q \\ 0 \end{bmatrix} = \begin{bmatrix} \delta r_{0x} \\ \delta r_{0y} \\ 0 \end{bmatrix}$$

In the second place, we see that the displacement $\delta\vec{r}_P$ of any point P of the lamina is

$$\delta\vec{r}_P = \delta\vec{\phi} \times \vec{QP} = \delta\vec{\phi} \times (\vec{QO} + \vec{OP}) =$$

$$= \delta\vec{\phi} \times \underbrace{\vec{OP}}_{=\vec{r}_P} + \underbrace{\delta\vec{\phi} \times \vec{QO}}_{=\delta\vec{r}_0}$$

$$= \delta\phi \times \vec{r}_P + \delta\vec{r}_0,$$

as we wished to prove.

IMPORTANT OBSERVATION

Since the infinitesimal displacement $\delta\hat{D}$ is a multiple of the twist $\hat{T}$, the vector space of twists of freedom coincides with the vector space of feasible infinitesimal displacements of the lamina.

KINEMATIC EQUATIONS WITH INFINITESIMAL DISPLACEMENTS

The kinematic behaviour of a serial robot is captured by the equation

$$\hat{T} = J \cdot \vec{\gamma}$$

end-effector twist

actuated joint velocities

Multiplying this equation by δt, we obtain the analogous expression

$$\delta \hat{D} = J \cdot \delta \vec{\theta}$$

infinitesimal displacement of the end-effector

$$\delta \vec{\theta} = \begin{bmatrix} \delta\theta_1 \\ \delta\theta_2 \\ \delta\theta_3 \end{bmatrix}$$

small angles rotated by the joints

This is the typical equation embedded in a position control loop. If $\delta \hat{D}$ is the infinitesimal displacement that corrects a position error of the end-effector, we use the equation to compute the small angles $\delta\theta_i$ that produce such a $\delta \hat{D}$.

This, and other equations written in terms of infinitesimal displacements will be used in Module 5 to implement hybrid control strategies.