Module 2
Planar Statics

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Objective of this Module

In this module we will introduce a unified representation for the forces and torques that act on a rigid body. Pure forces and torques will be viewed here as particular cases of a more general entity, called *wrench*. The introduction of wrenches simplifies and makes the static analysis of robotic mechanisms more compact. By static analysis we mean the determination of the relationship between the forces and torques at the actuated joints and the forces and torques that the end effector exerts on the surroundings, while in equilibrium. We will determine this relationship and observe how it behaves in regular and singular situations, in the case of a parallel 3RPR manipulator.

2.1 Plücker coordinates of a line in the XY plane

2.1.1 Line through two points

We are quite used to give the coordinates of points in the plane. For the points 1 and 2 of Fig. 2.1, for example, we say that they have coordinates (x_1, y_1) and (x_2, y_2), respectively. In what follows we will see that in a very similar fashion we are able to provide the coordinates of a line in a plane, in order to identify it unambiguously.

Let $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ be the position vectors of points 1 and 2, and let's define $\boldsymbol{S} = \boldsymbol{r}_2 - \boldsymbol{r}_1$, which is a direction vector of the line passing through 1 and 2. The X and Y components of $\boldsymbol{S}$ are:

$$\begin{aligned} L = &\quad x_2 - x_1 \\ M = &\quad y_2 - y_1 \end{aligned}$$

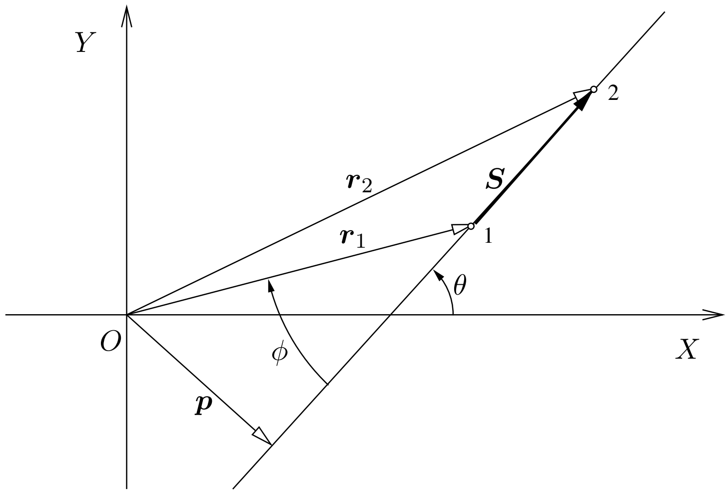


Figure 2.1: A line through points 1 and 2.

Let's compute the moment of $\boldsymbol{S}$ w.r.t. the origin O:

$$\begin{aligned}\boldsymbol{r}_1 \times \boldsymbol{S} &= |\boldsymbol{r}_1| \cdot |\boldsymbol{S}| \sin\phi\, \boldsymbol{k} = \\ &= |\boldsymbol{r}_1| \sin\phi\, |\boldsymbol{S}|\, \boldsymbol{k} = \\ &= p\, |\boldsymbol{S}|\, \boldsymbol{k} = R\boldsymbol{k}\end{aligned}$$

where $p = \mathrm{sign} \cdot |\boldsymbol{p}|$ is the signed distance of the line 1-2 to the origin O, that is, the standard point-line distance affected by a sign which is

- **positive**, if $\boldsymbol{S}$ "turns" in counterclockwise sense w.r.t. O,
- **negative**, if $\boldsymbol{S}$ "turns" in clockwise sense w.r.t. O (Fig. 2.2).

We should realize that even if we have computed the moment of $\boldsymbol{S}$ w.r.t. O assuming that the position of $\boldsymbol{S}$ is given by $\boldsymbol{r}_1$, we would have obtained exactly the same moment if we had considered the position vector $\boldsymbol{r}$ of any other point along the line 1-2 instead. In other words, displacing $\boldsymbol{S}$ along the line 1-2, R remains constant. In what follows we will see that the line 1-2 is the only one where R remains constant at this value. Thus we can state that the L and M components of $\boldsymbol{S}$, together with moment R constitute the components of the line 1-2.

Let's find the points (x, y) in the plane for which the moment of $\boldsymbol{S}$ w.r.t. O is equal to $\boldsymbol{r}_1 \times \boldsymbol{S}$. We have to solve the vector equation:

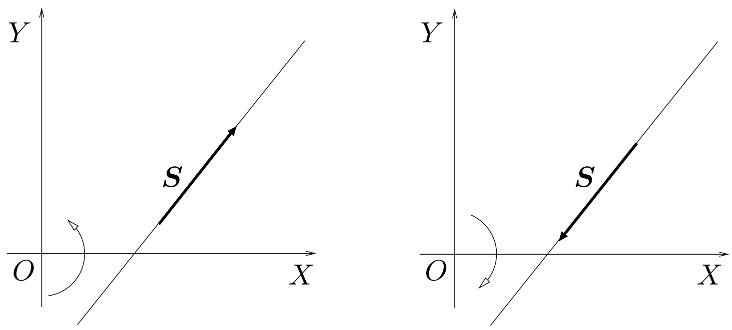


Figure 2.2: Counterclockwise and clockwise turns, positive and negative moments of $\boldsymbol{S}$ w.r.t. O.

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & 0 \\ L & M & 0 \end{vmatrix} \quad = \quad \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_1 & y_1 & 0 \\ L & M & 0 \end{vmatrix}$$

The i and j components in the equation are identically equal to zero. The k component is

$$\begin{vmatrix} x & y \\ L & M \end{vmatrix} = \begin{vmatrix} x_1 & y_1 \\ L & M \end{vmatrix} = R$$

We notice that R can also be rewritten (i.e., computed) in the following way:

$$R = \begin{vmatrix} x_1 & y_1 \\ x_2 - x_1 & y_2 - y_1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = R$$

We finally obtain the equation of the line:

$$Ly - Mx + R = 0,$$

which is the line that goes through 1 and 2, as this equation holds for (x_1, y_1) and (x_2, y_2).

Summarizing, it is clear that L, M and R define one and only one line and can be viewed as the **coordinates** that determine this line.

Observations:

Julius Plücker

1. These coordinates were first used by Julius Plücker (1801 - 1868) and therefore are usually called "Plücker coordinates"

2. They are called homogeneous coordinates, in the following sense: being λ a non-null scalar, λL, λM and λR define the same line.

3. We often represent the Plücker coordinates grouped in the following form:
$$\{L, M; R\}$$
L, M appear separated from R with a semicolon (;) because the first two have length units, whereas R has area units.

4. Note that the parameters $\{L, M; R\}$ identify uniquely the line $Ly - Mx + R = 0$, but they also determine uniquely a vector $\boldsymbol{S} = (L, M)$ aligned with this line, in an arbitrary position on this line. For this reason we say that $\{L, M; R\}$ defines a **line-bound vector** on the line $Ly - Mx + R = 0$.

5. The line-bound vector $\{L, M; R\}$ can be obtained in an elegant way from (x_1, y_1) and (x_2, y_2), using a rule introduced by Hermann Grassmann (1809 - 1877), which consists in arranging the points as follows
$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{bmatrix}$$
and computing the 2×2 determinants that result from deleting the 3rd, 2nd, and 1st columns. That is

$$L = \begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} \quad M = \begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix} \quad R = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

2.1.2 Points and lines at infinity

Hermann Grassmann

Up to now we have defined the coordinates of points and lines in the Euclidean plane. However, for a complete and accurate treatment of the different situations that may arise in Statics, we require a base geometry whose expressive power allows to deal with points and lines at infinity in the same manner as with points and lines located at finite distances from the origin (see Sec. 2.3.2). Projective Geometry meets this requirement: if we don't see the plane as the Euclidean plane $\mathbb{R}^2$ but as the projective $\mathbb{P}^2$, then all the points in the plane, the usual (or **proper**) points as well as the points at infinity (or **improper** points) are dealt with in a unified frame.

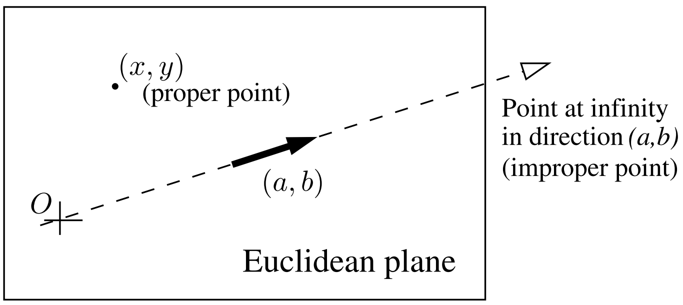


Figure 2.3: A proper point on the plane and an improper point at infinity given by direction (a, b).

In order to make it possible we assume that the Euclidean plane is the plane $z = 1$ of $\mathbb{R}^3$, and we consider the bundle of lines in $\mathbb{R}^3$ passing through the origin. Each one of these lines represents a point of the Euclidean plane: the one resulting from the intersection of the line with the plane $z = 1$.

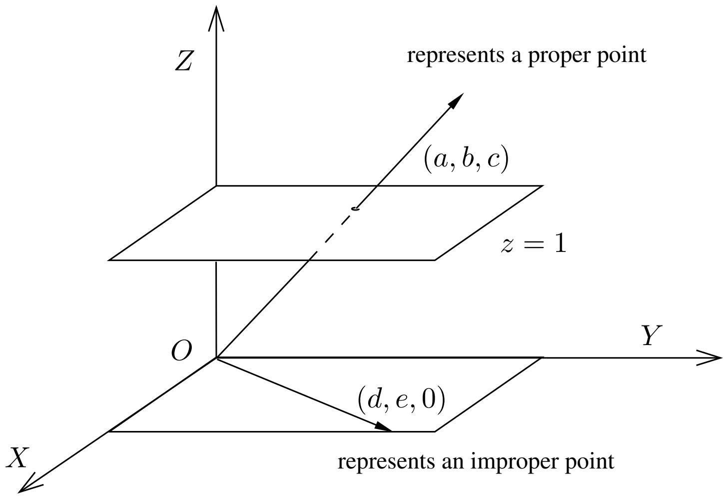


Figure 2.4: Correspondence between points in the Euclidean plane and lines in $\mathbb{R}^3$.

A line of the form $\lambda(a, b, c)$ with $c \neq 0$ represents a proper point P with coordinates $(\frac{a}{c}, \frac{b}{c}, 1)$. A line of the form $\lambda(d, e, 0)$ represents a point at infinity located in the direction (d, e) of plane $z = 0$.

In sum, the coordinates of a projective point have the form (x, y, z), with $z \neq 0$ if the point is proper and $z = 0$ if the point is improper. They are called **homogeneuos** coordinates because, if multiplied by a scalar $\lambda \neq 0$, they are always representing the same point.

Note that the Grassmann rule can be used to compute the coordinates of a line that goes through two projective points, be they proper or not. For example, the line in Fig. 2.1, which we have assumed to be defined by the proper projective points

$$\begin{array}{c}(x_1, y_1, 1)\\(x_2, y_2, 1),\end{array}$$

can also be defined by the points

$$\begin{array}{ll}(x_1, y_1, 1) & \text{(proper, the same as above)}\\(L, M, 0) & \text{(improper).}\end{array}$$

In the second case, applying Grassmann's rule, we should extract the 2×2 determinants of

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 0 & x_2 - x_1 & y_2 - y_1 \end{bmatrix}$$

which will be identical to the ones extracted from

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{bmatrix}$$

as if a row is added to another row, the value of the determinant does not change.

Projective Geometry does also allow to deal with the line at infinity as any other line. The line at infinity is the one containing all the improper points in the plane, which can be considered aligned.

This leads us to wonder which would be the coordinates $\{L, M; R\}$ of the line at infinity. To this end, we should mention first that in the same way that the points of the plane are respresented by lines through the origin in $\mathbb{R}^3$, in Projective Geometry the lines in the plane are represented by planes passing through (or containing) the origin (Fig. 2.5).

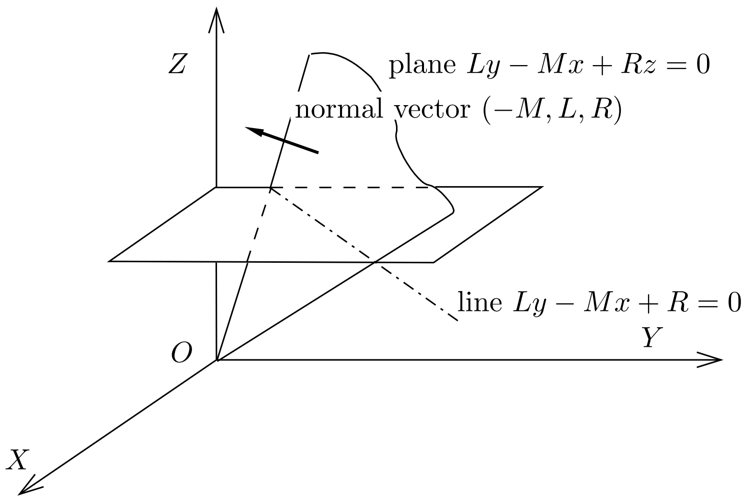


Figure 2.5: Correspondence between lines in the Euclidean plane and planes in $\mathbb{R}^3$.

The plane $Ly - Mx + Rz = 0$ represents the line resulting from the intersection of this plane with the plane $z = 1$, i.e., the line $Ly - Mx + R = 0$. Thus, we can see now another intepretation of the parameters L, M, R: they provide the components of the normal vector to the plane of $\mathbb{R}^3$ that represents the line $Ly - Mx + R = 0$.

The line through infinity is represented by the plane $z = 0$, whose direction vector takes the form $(0, 0, R)$. Thus, the line at infinity will have the coordinates

$$\{0, 0; R\}\,.$$

In Sec. 2.3.2, as well as in Module 3, we will see that the points and the lines at infinity have very useful physical meanings in statics and kinematics.

2.1.3 Interpretation of R when the line is proper

The area of a triangle defined by three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

In fact, this is an oriented area, which is positive if points 1, 2 and 3 are traversed in counter-clockwise order, or negative otherwise. Thus, the area of triangle $O12$ of Figure 2.1 is

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = \frac{1}{2}R,$$

i.e., R is twice the area of triangle $O12$. On the other hand, from Figure 2.1,

$$\Delta = \frac{1}{2}\,|\boldsymbol{S}|\,p = \frac{1}{2}p\sqrt{L^2 + M^2},$$

and comparing these two equations we obtain

$$p = \frac{R}{\sqrt{L^2 + M^2}} \tag{2.1}$$

2.1.4 Normalized coordinates of a line

When $|\boldsymbol{S}| = 1$, we have, from Figure 2.1

$$L = \cos\theta$$
$$M = \sin\theta$$

and from Eq. (2.1)

$$R = p$$

With the abbreviations $c = \cos\theta$ and $s = \sin\theta$ we have the following Plücker coordinates

$$\{c, s; p\}$$

when $|\boldsymbol{S}| = 1$. These are the **normalized** or **unit** coordinates of the line, and the equation of the line takes the form

$$cy - sx + p = 0 \tag{2.2}$$

It is easy to obtain the normalized Plücker coordinates of a line passing through point (x_Q, y_Q) forming an angle θ w.r.t. the abscissae (Fig. 2.6), since

$$c = \cos\theta$$
$$s = \sin\theta$$

and p can be computed by substituting (x_Q, y_Q) in Eq. (2.2).:

$$p = sx_Q - cy_Q$$

2.2 Point of the intersection of two lines

Let $\$_1$ and $\$_2$ be any two lines in the plane, with coordinates $\{L_1, M_1; R_1\}$ and $\{L_2, M_2; R_2\}$. Depending on the relative position of these lines, they can intersect at a proper or at an improper point P (Fig. 2.7):

In either case it is possible, using Projective Geometry, to find the coordinates of P. We just consider the planes representing $\$_1$ and $\$_2$

$$L_1y - M_1x + R_1z = 0 \quad (\text{ plane } \alpha_1)$$
$$L_2y - M_2x + R_2z = 0 \quad (\text{ plane } \alpha_2)$$

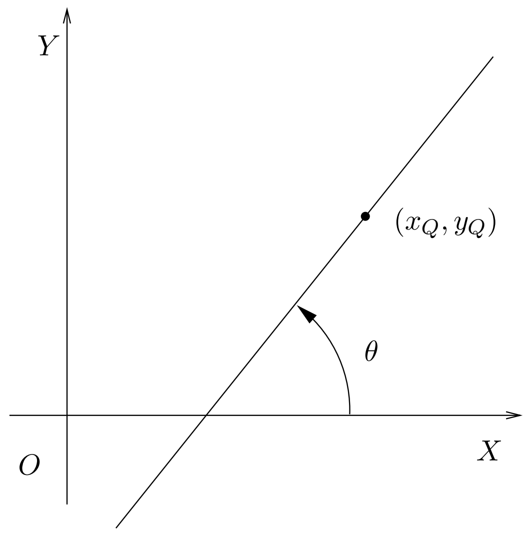


Figure 2.6: Computing the normalized coordinates of a line through (x_Q, y_Q).

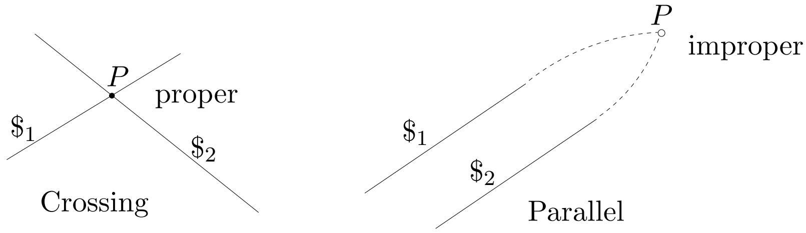


Figure 2.7: Skew and parallel lines intersect at a proper and improper point respectively.

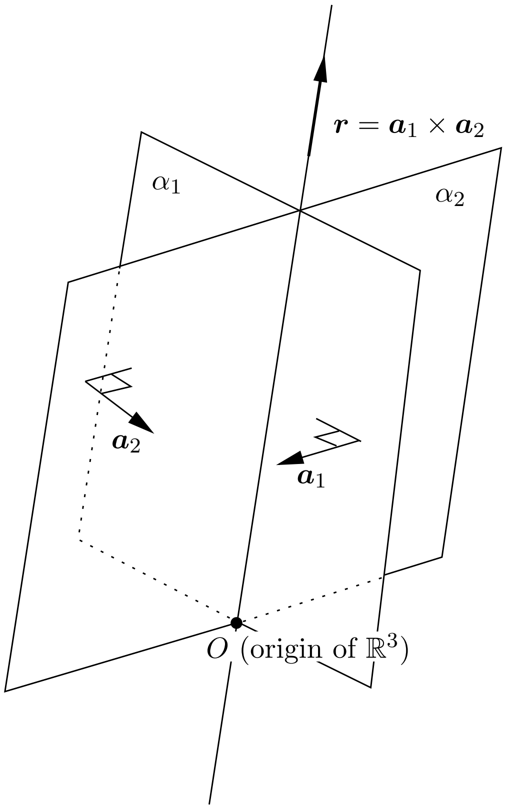


Figure 2.8: Line that represents the intersection point P of $\$_1$ and $\$_2$.

and intersect them. Their intersection line represents P (Fig. 2.8). This line goes through the origin, and its direction vector is $\boldsymbol{r} = \boldsymbol{a}_1 \times \boldsymbol{a}_2$, where $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are the normal vectors to the two planes:

$$\begin{aligned} \boldsymbol{a}_1 &= [-M_1, L_1, R_1]^T \\ \boldsymbol{a}_2 &= [-M_2, L_2, R_2]^T \end{aligned}$$

Therefore,

$$\boldsymbol{r} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -M_1 & L_1 & R_1 \\ -M_2 & L_2 & R_2 \end{vmatrix},$$

and expanding the determinant as follows

$$\boldsymbol{r} = \underbrace{\begin{vmatrix} L_1 & R_1 \\ L_2 & R_2 \end{vmatrix}}_{r_x} \boldsymbol{i} + \underbrace{\begin{vmatrix} M_1 & R_1 \\ M_2 & R_2 \end{vmatrix}}_{r_y} \boldsymbol{j} + \underbrace{\begin{vmatrix} L_1 & M_1 \\ L_2 & M_2 \end{vmatrix}}_{r_z} \boldsymbol{k},$$

we obtain

$$\boldsymbol{r} = r_x \boldsymbol{i} + r_y \boldsymbol{j} + r_z \boldsymbol{k},$$

so that P has the projective coordinates

$$(r_x, r_y, r_z)$$

If $r_z \neq 0$, then P is a proper point, with Euclidean coordinates

$$(\frac{r_x}{r_z}, \frac{r_y}{r_z})$$

otherwise it is the improper point of the Euclidean plane located in direction (r_x, r_y).

The coordinates r_y, r_x, and r_z (note the order in the subindices) can be obtained as the 2×2 determinants that result from eliminating the 1st, 2nd and 3rd columns, respectively, of the matrix

$$\begin{bmatrix} L_1 & M_1 & R_1 \\ L_2 & M_2 & R_2 \end{bmatrix} \tag{2.3}$$

This rule was also given by Grassmann.

2.3 Statics of rigid planar systems

2.3.1 The wrench of a force

As stated earlier, the coordinates $\{L, M; R\}$ define a line-bound vector (L, M) on the line $Ly - Mx + R = 0$. Therefore, we can use these coordinates to represent a force vector (L, M) together with its line of application, which acts on a lamina (planar rigid body).

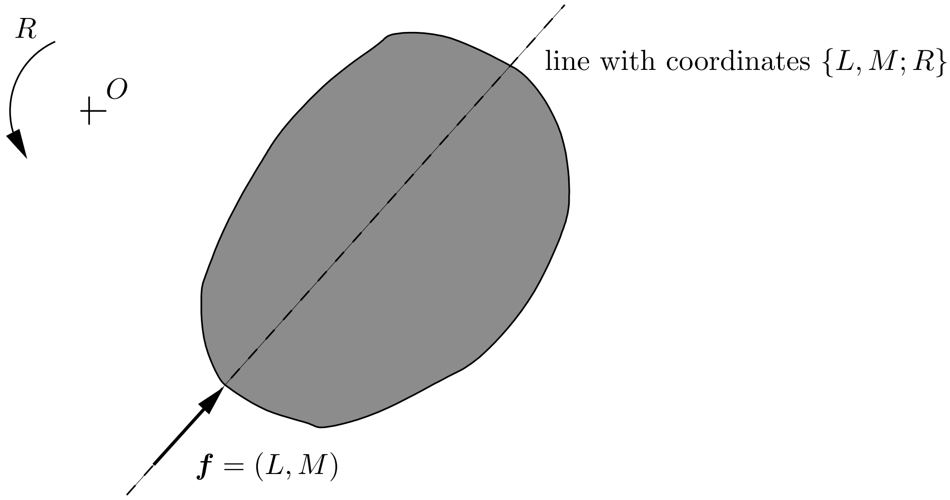


Figure 2.9: Force acting on a lamina.

Geometric entity	**Physical interpretation**
(L, M)	Force vector
$\{L, M; R\}$	Line of application
$\sqrt{L^2 + M^2}$	Force intensity
R	Force moment w.r.t. origin O

Thus, a force $\boldsymbol{f}$ as the one in Fig. 2.9 will be represented by a vector

$$\hat{w} = \begin{bmatrix} L \\ M \\ R \end{bmatrix}$$

called **wrench**, which encodes both the force and its action line.

From the laws of Statics we know that if we have two forces $\boldsymbol{f}_1 = (L_1, M_1)$ and $\boldsymbol{f}_2 = (L_2, M_2)$ acting on a lamina, with moments R_1 and R_2 w.r.t. the origin, then they can be replaced by one force called **resultant**, whose vector is $\boldsymbol{f} = \boldsymbol{f}_1 + \boldsymbol{f}_2$, acting on a line such that the moment of $\boldsymbol{f}$ w.r.t. the origin is $R_1 + R_2$. Clearly, if $\hat{w}_1 = [L_1, M_1, R_1]^T$ and $\hat{w}_2 = [L_2, M_2, R_2]^T$ are the wrenches of $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$, the wrench of $\boldsymbol{f}$ will be $\hat{w} = \hat{w}_1 + \hat{w}_2$. In general, if we have n forces of wrenches $\hat{w}_i = [L_i, M_i, R_i]^T$, with $i = 1, \ldots, n$ acting on a lamina, the resultant force will have a wrench

$$\hat{w} = \begin{bmatrix} L \\ M \\ R \end{bmatrix} = \begin{bmatrix} L_1+ & \ldots & +L_n \\ M_1+ & \ldots & +M_n \\ R_1+ & \ldots & +R_n \end{bmatrix},$$

and thus we will quickly obtain the action line of the resultant. It will be $Ly - Mx + R = 0$ (if $R \neq 0$).

2.3.2 The wrench of a couple

A couple is a system of two forces with equal norm but opposite direction, which act on parallel application lines on a lamina, tending to make the lamina to rotate. An intuitive example is the couple of forces applied to a steering wheel when we drive (Fig. 2.10).

Figure 2.10: Couple of forces acting on a steering wheel.

The wrenches of these forces are $\hat{w}_1 = [L_1, M_1, R_1]^T$ and $\hat{w}_2 = [-L_1, -M_1, R_2]^T$, with $R_1 \neq -R_2$. The resultant wrench will be

$$[0, 0, R_1 + R_2]^T$$

which corresponds to the vector of the line at infinity. Therefore, a couple of forces can also be viewed as a particular case of force. We say that it is a force of infinitesimal magnitude acting on the line at infinity.

It may be difficult to understand that a force whose magnitude tends to zero, infinitely far away from the origin, can provide a finite couple $R_1 + R_2$. But it is certainly so, as we can see in the following example.

EXAMPLE.- On a lamina two antiparallel forces, of magnitude 2 N and m N respectively, act as shown in Fig. 2.11.

Be aware of the units of the wrench components:

$$\hat{w}_1 = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix} \quad \hat{w}_2 = \begin{bmatrix} 0 \\ m \\ 2m \end{bmatrix} \quad \begin{array}{l} \longleftarrow \text{ Newton} \\ \longleftarrow \text{ Newton} \\ \longleftarrow \text{ Newton} \cdot \text{meter} \end{array}$$

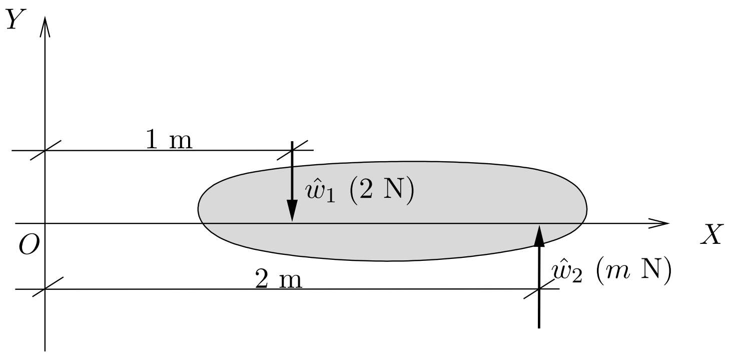


Figure 2.11: Antiparallel forces acting on a lamina.

The resulting wrench, as a function of m, is

$$\hat{w} = \hat{w}_1 + \hat{w}_2 = \begin{bmatrix} 0 \\ m-2 \\ 2m-2 \end{bmatrix} \begin{array}{l} \longleftarrow L(m) \\ \longleftarrow M(m) \\ \longleftarrow R(m) \end{array}$$

which comes out to be a vertical force of magnitude

$$M(m) = m - 2$$

applied on a line which is at a distance of (applying Eq. 2.1)

$$p(m) = \frac{2m-2}{m-2}$$

from the origin.

Assume that at the beginning $m = 3$. Then, the resultant

$$\hat{w} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$$

is a vertical force of 1 N, applied at $x = 4$ m. If we now make m tend to 2, we have:

$$\lim_{m \to 2^+} M(m) = \lim_{m \to 2^+} m - 2 = 0$$

(i.e., the magnitude of $\hat{w}$ tends to zero)

$$\lim_{m \to 2^+} p(m) = \lim_{m \to 2^+} \frac{2m-2}{m-2} = +\infty$$

(i.e., the distance tends to $+\infty$), but the product $M(m) \cdot p(m)$, i.e., the moment, tends to a finite value:

$$\lim_{m \to 2^+} M(m) \cdot p(m) = \lim_{m \to 2^+} 2m - 2 = 2\text{Nm}$$

End of the example.

2.4 Translation and rotation of coordinate systems

In what follows we will see how to determine L, M, and R in a coordinate system, when they are given in a different system. The process is valid independently of whether L, M, and R are the coordinates of a line or a wrench.

2.4.1 Pure translation

Consider the line through points 1 and 2 in Fig. 2.12, with coordinates

$$\left.\begin{matrix} (x_1, y_1) \\ (x_2, y_2) \end{matrix}\right\} \qquad \text{in the coordinate system } OXY$$

$$\left.\begin{matrix} (x'_1, y'_1) \\ (x'_2, y'_2) \end{matrix}\right\} \qquad \text{in the coordinate system } O'X'Y'$$

The coordinates of line 1-2 in the systems OXY and $O'X'Y'$ are, respectively,

$$L = x_2 - x_1 \quad M = y_2 - y_1 \quad R = x_1 y_2 - x_2 y_1 \tag{2.4}$$

$$L' = x'_2 - x'_1 \quad M' = y'_2 - y'_1 \quad R' = x'_1 y'_2 - x'_2 y'_1 \tag{2.5}$$

To find the transformation that expresses the coordinates L, M, and R in terms of the L', M', and R', we write

$$\begin{matrix} x_1 = x'_1 + a & y_1 = y'_1 + b \\ x_2 = x'_2 + a & y_2 = y'_2 + b \end{matrix}$$

Replacing these relations in Eq. (2.5) and considering also Eq. (2.4) we obtain

$$L = L' \quad M = M' \quad R = R' - L'b + M'a$$

or, in matrix form,

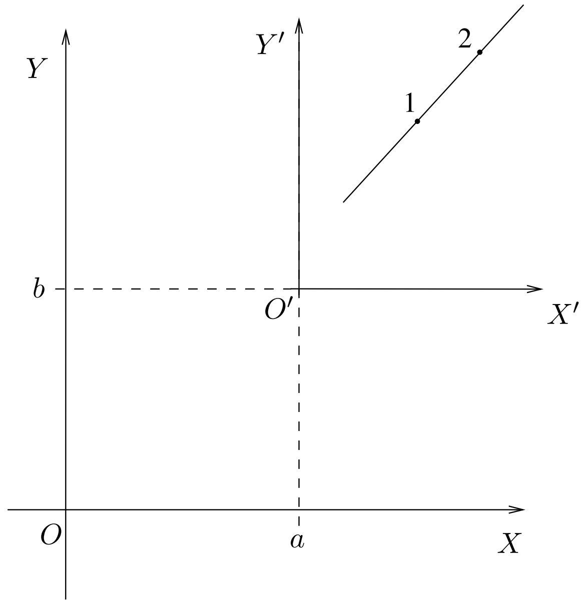


Figure 2.12: Translation of the coordinate system.

$$\begin{bmatrix} L \\ M \\ R \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & a & 1 \end{bmatrix} \begin{bmatrix} L' \\ M' \\ R' \end{bmatrix}$$

which expresses the transformation we were looking for.

2.4.2 Pure rotation

Consider the situation shown in Fig. 2.13, where the line 1-2 can be expressed in the $O'X'Y'$ or $O''X''Y''$ coordinate systems. The two systems have a common origin, that is, $O' = O''$, and $O''X''Y''$ has been rotated an angle ϕ w.r.t. $O'X'Y'$, as shown.

Let the coordinates of point i be

(x'_i, y'_i) in the $O'X'Y'$coordinate system
(x''_i, y''_i) in the $O''X''Y''$coordinate system

It is well-known that the relation between (x'_i, y'_i) and (x''_i, y''_i) is given by

$$\begin{bmatrix} x'_i \\ y'_i \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x''_i \\ y''_i \end{bmatrix} \tag{2.6}$$

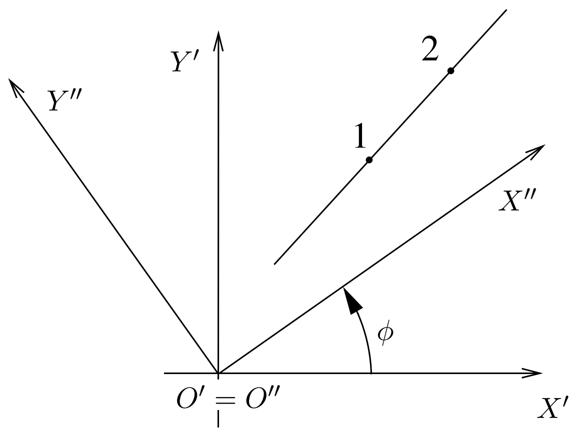


Figure 2.13: Rotation of the coordinate system.

We do also know that the coordinates of the line 1-2 in the coordinate systems $O'X'Y'$ and $O''X''Y''$ are, respectively,

$$L' = x'_2 - x'_1 \quad M' = y'_2 - y'_1 \quad R' = x'_1 y'_2 - x'_2 y'_1 \tag{2.7}$$

$$L'' = x''_2 - x''_1 \quad M'' = y''_2 - y''_1 \quad R'' = x''_1 y''_2 - x''_2 y''_1 \tag{2.8}$$

Replacing Eq. (2.6) in Eq. (2.7) and taking Eq. (2.8) into account we obtain

$$\begin{bmatrix} L' \\ M' \\ R' \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} L'' \\ M'' \\ R'' \end{bmatrix},$$

which provides the coordinate transformation between the two systems.

2.4.3 Translation and rotation

Consider now two arbitrary coordinate systems, which we will call OXY and $O''X''Y''$. The origin O'' of the second system lies in the position (a, b) of OXY, and the axes $X''Y''$ have been rotated an angle ϕ w.r.t. the axes XY (Fig. 2.14). Consider also a third coordinate system $O'X'Y'$ parallel to OXY, with $O' = O''$.

Now assume that we have the coordinates $\{L'', M''; R''\}$ of line 1-2 in the system $O''X''Y''$ and we want to express them in the system OXY. We make the conversion in two steps: first obtaining the coordinates in system $O'X'Y'$ and then converting these to OXY. Clearly, the sequence of the two steps corresponds to the following matrix composition, where $c = \cos\phi$ and $s = \sin\phi$:

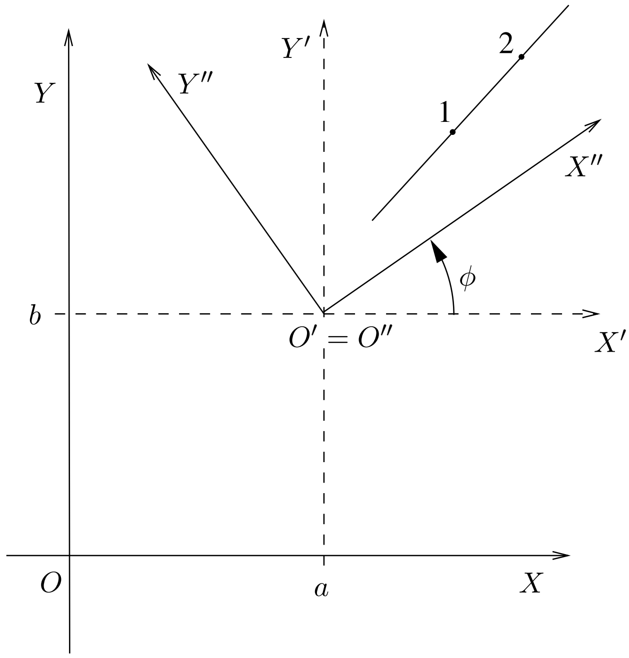


Figure 2.14: Translation and rotation of the coordinate systems.

$$\begin{bmatrix} L \\ M \\ R \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & a & 1 \end{bmatrix} \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} L'' \\ M'' \\ R'' \end{bmatrix}$$

Or, in condensed form,

$$\begin{bmatrix} L \\ M \\ R \end{bmatrix} = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ as - bc & ac + bs & 1 \end{bmatrix} \begin{bmatrix} L'' \\ M'' \\ R'' \end{bmatrix} = [\boldsymbol{e}] \begin{bmatrix} L'' \\ M'' \\ R'' \end{bmatrix}$$

which is the relationship we were seeking.

2.5 Symbolic representation of a wrench

We have seen that a wrench takes the form

$$\hat{w} = \begin{bmatrix} L \\ M \\ R \end{bmatrix} \tag{2.9}$$

where (L, M) are the components of the associated force, and R is the moment of this force w.r.t. the origin. The notation in Eq. 2.9 uses brackets to display the vector $\hat{w}$ and is preferred when these vectors are to be operated with matrices. However, it is convenient to introduce another notation for $\hat{w}$, which will be useful in more abstract expressions and reflects better the nature of a wrench. We will write the wrench as follows

$$\hat{w} = \{\boldsymbol{f}; \boldsymbol{c}_o\}$$

where

- $\boldsymbol{f} = L\boldsymbol{i} + M\boldsymbol{j}$ is the force vector,
- $\boldsymbol{c}_o = R\boldsymbol{k}$ is the moment of this force w.r.t. the origin of the coordinate system.

This notation reflects the fact that a wrench is formed by two vectors, $\boldsymbol{f}$ and $\boldsymbol{c}_o$, whose nature and units are different (N and Nm, respectively, in the International System).

Let $\boldsymbol{S}$ be a vector in $\mathbb{R}^2$ with the same direction as $\boldsymbol{f}$, but with $|\boldsymbol{S}| = 1$. Consider also $\boldsymbol{S}_o = \boldsymbol{r} \times \boldsymbol{S}$, the moment of $\boldsymbol{S}$ w.r.t. the origin, where $\boldsymbol{r}$ is the position vector of any point of the application line \$ of $\hat{w}$ (Fig. 2.15).

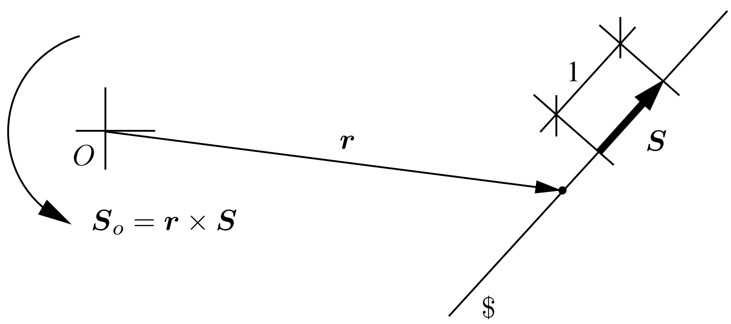


Figure 2.15: Meaning of the symbolic components of $\hat{w}$.

If $f = |\boldsymbol{f}|$ (magnitude of $\boldsymbol{f}$), then we can write

$$\begin{aligned} &\boldsymbol{f} = f\boldsymbol{S} \\ &\boldsymbol{c}_o = f(\boldsymbol{r} \times \boldsymbol{S}) = f\boldsymbol{S}_o \end{aligned}$$

and also

$$\hat{w} = \{\boldsymbol{f}; \boldsymbol{c}_o\} = \{f\boldsymbol{S}; f\boldsymbol{S}_o\} = f\underbrace{\{\boldsymbol{S}; \boldsymbol{S}_o\}}_{\hat{s}} = f\hat{s}.$$

Note that $\hat{s}$ contains all the normalized coordinates of line \$. Therefore, we have expressed the wrench $\hat{w}$ as a multiple of the normalized coordinates of the action line, where the multiplying factor is the magnitude f of force $\boldsymbol{f}$. This perspective corresponds to the fact that the wrench is a geometric entity (the line-bound vector $\boldsymbol{S}$ on the line \$) with a physical meaning (by multiplying it by the magnitude f of the force).

2.6 Statics of parallel manipulators

We next use the preceding concepts to perform the static analysis of parallel 3RPR manipulators, i.e., the computation of the relationship between the joint forces and the wrench that the platform applies to the environment, in situations of static equilibrium.

Consider the manipulator of Fig. 2.16, formed by a base that is fixed to the ground, and a mobile platform, joined by three RPR legs, aka connectors. We assume that the configuration of the manipulator is known (for example, it could have been computed resolving the direct kinematics of the manipulator, with a positional analysis method, as explained in Module 1).

The action lines of the connectors are displayed as $\$_1$, $\$_2$, and $\$_3$ in the figure, being at distances p_i w.r.t. the origin, and forming angles θ_i w.r.t. axis X. The prismatic joints P of each leg are actuated.

2.6.1 Forward static problem

Assume that the actuators apply forces $\boldsymbol{f}_1$, $\boldsymbol{f}_2$, and $\boldsymbol{f}_3$ at the legs. The platform will experience a resultant force $\boldsymbol{f}$ applied on a line \$. If $\hat{w}_i$ stands for the wrench of force $\boldsymbol{f}_i$ and $\hat{w}$ for the wrench of force $\boldsymbol{f}$, that is,

$$\hat{w}_i = \begin{bmatrix} \boldsymbol{f}_i \\ \boldsymbol{c}_i \end{bmatrix} \qquad\qquad \hat{w} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{c}_o \end{bmatrix}$$

then

$$\hat{w} = \hat{w}_1 + \hat{w}_2 + \hat{w}_3$$

which we can write as

$$\hat{w} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{c}_o \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{c}_1 \end{bmatrix} + \begin{bmatrix} \boldsymbol{f}_2 \\ \boldsymbol{c}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{f}_3 \\ \boldsymbol{c}_3 \end{bmatrix}$$

If we express each wrench $\hat{w}_i$ as

$$\hat{w}_i = f_i \hat{s}_i$$

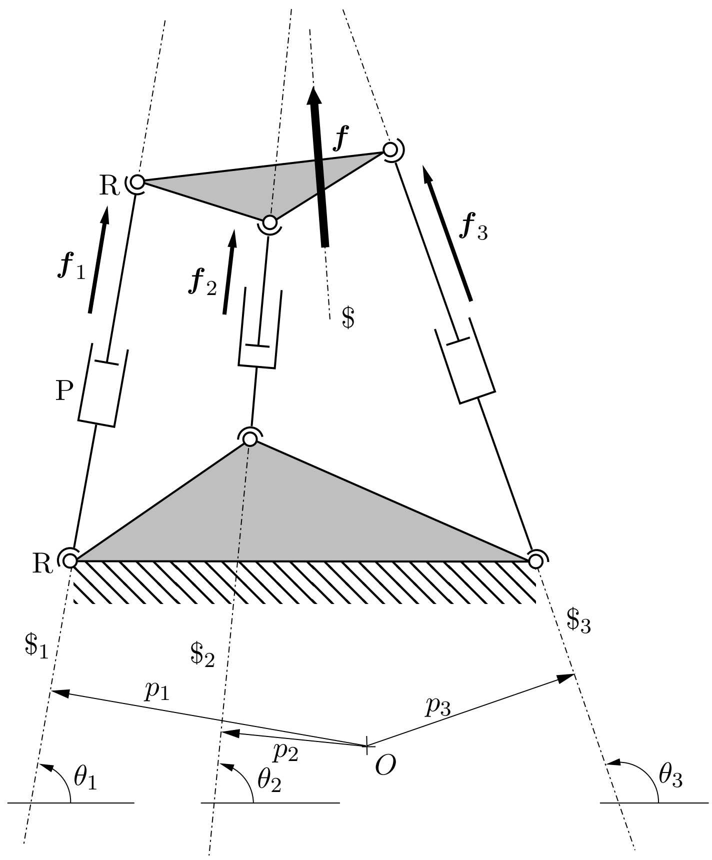


Figure 2.16: Static analysis in a parallel manipulator.

where f_i is the signed magnitude of the force at leg i, and $\hat{s}_i$ are the normalized coordinates of $\$_i$, then

$$\hat{w} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{c}_o \end{bmatrix} = f_1\hat{s}_1 + f_2\hat{s}_2 + f_3\hat{s}_3$$

If we take into account that the normalized coordinates $\hat{s}_i$ are

$$\hat{s}_i = \begin{bmatrix} c_i \\ s_i \\ p_i \end{bmatrix}$$

with

$$c_i = \cos\theta_i$$
$$s_i = \sin\theta_i$$

then we can write

$$\hat{w} = \underbrace{\begin{bmatrix} c_1 & c_2 & c_3 \\ s_1 & s_2 & s_3 \\ p_1 & p_2 & p_3 \end{bmatrix}}_{\boldsymbol{j}} \underbrace{\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}}_{\boldsymbol{\lambda}} = \boldsymbol{j\lambda},$$

where $\boldsymbol{j}$ is called the *force Jacobian* of the manipulator.

The expression $\hat{w} = \boldsymbol{j\lambda}$ provides the solution to the **forward static problem**: from the signed magnitudes of the joint forces applied at the joints (vector $\boldsymbol{\lambda}$) we obtain the **resultant wrench** $\hat{w}$ applied by the legs to the end effector (i.e., the platform). If the platform is in equilibrium with the environment, the environment will apply a force equal to $\boldsymbol{f}$ in norm but in the opposite direction on the same line \$. This force, called **equilibrant**, will have a wrench (Fig. 2.17):

$$\hat{w}_{eq} = -\hat{w}$$

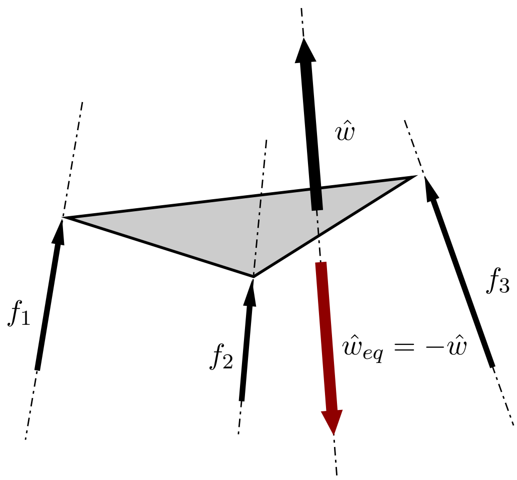


Figure 2.17: The wrench $\hat{w}$ is the resultant of the forces of magnitude f_1, f_2, and f_3 applied by the legs to the platform.

Application.- A possible application of $\hat{w} = \boldsymbol{j\lambda}$ appears when we want to use the 3RPR manipulator as a force sensor. If we install load cells at the legs, these cells provide the f_1, f_2, f_3 values of $\boldsymbol{\lambda}$, and we are able to determine the wrench $\hat{w}$, and thus the equilibrant wrench $\hat{w}_{eq}$ that the environment exerts on the platform. Notice that the three components L, M, and R of $\hat{w}_{eq}$ will give the equation of line \$, and thus we will know where this force is applied. Miniaturized 3D versions of this type of sensor exist nowadays, to be integrated in robotic fingers for example.

2.6.2 Inverse static problem

Inversely, when we have a wrench $\hat{w}_{ap}$ on the end effector (i.e., applied by the environment on the end effector), we often want to know which are the resultant forces

$$\boldsymbol{\lambda}_{res} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

that this wrench will cause on the legs, in order to know the forces that the actuators will have to exert to equilibrate them. This corresponds to solving the **inverse static problem**. Since we can write

$$\hat{w}_{ap} = \boldsymbol{j}\boldsymbol{\lambda}_{res},$$

this problem corresponds to solving the previous linear system of equations, with known $\hat{w}_{ap}$ and $\boldsymbol{j}$.

Often the nature of $\hat{w}_{ap}$ and $\boldsymbol{\lambda}_{res}$ is understood from their context, and the subindices "ap" and "res" can be omitted

$$\hat{w} = \boldsymbol{j}\boldsymbol{\lambda} \tag{2.10}$$

implying that $\hat{w}$ is the wrench applied by the environment to the end effector, and $\boldsymbol{\lambda} = [f_1, f_2, f_3]^T$ the vector of resultant joint forces (applied by the end effector on the legs). If $\det \boldsymbol{j} \neq 0$, then $\boldsymbol{j}$ is invertible, and the solution to the inverse static problem is

$$\boldsymbol{\lambda} = \boldsymbol{j}^{-1}\hat{w} \tag{2.11}$$

The vector of equilibrant forces that the actuators will have to perform in order to counteract f_1, f_2, and f_3 will be (Fig. 2.18)

$$\boldsymbol{\lambda}_{eq} = -\boldsymbol{\lambda}$$

If $\boldsymbol{j}$ is invertible we have, thus, a one-to-one relationship between the wrench $\hat{w}$ and vector $\boldsymbol{\lambda}$. For every $\hat{w}$ there is one, and only one, $\boldsymbol{\lambda}$ satisfying Eq. (2.10). If $\boldsymbol{j}$ is not invertible, the inverse static problem cannot be solved by applying Eq. (2.11), and, in fact, the one-to-oneness gets lost (as explained more thoroughly in Sec. 2.8).

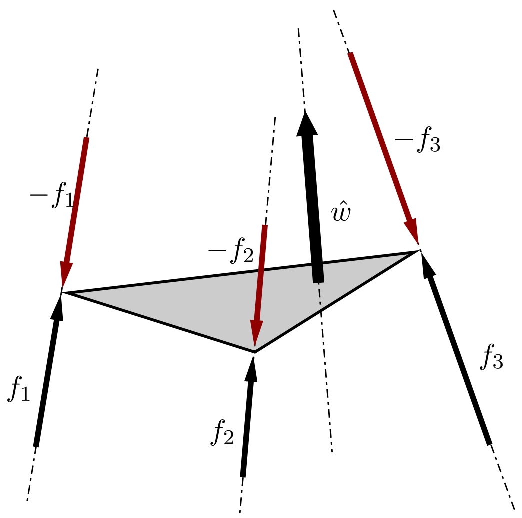


Figure 2.18: Forces on the platform: external force, resultant and equilibrant joint forces.

2.7 Geometrical meaning of $\boldsymbol{j}^{-1}$

The inverse of

$$\boldsymbol{j} = \begin{bmatrix} c_1 & c_2 & c_3 \\ s_1 & s_2 & s_3 \\ p_1 & p_2 & p_3 \end{bmatrix}$$

can be expressed as

$$\boldsymbol{j}^{-1} = \frac{\boldsymbol{j'}^T}{\det \boldsymbol{j}}$$

where $\boldsymbol{j'}^T$ is the transposed adjoint matrix of $\boldsymbol{j}$ (i.e., $\boldsymbol{j'}$ is the matrix of cofactors of matrix $\boldsymbol{j}$) and $\det \boldsymbol{j}$ is the determinant of $\boldsymbol{j}$. We obtain the following expression:

$$\boldsymbol{j'}^T = \begin{bmatrix} (s_2 p_3 - s_3 p_2) & -(c_2 p_3 - c_3 p_2) & (c_2 s_3 - c_3 s_2) \\ (s_3 p_1 - s_1 p_3) & -(c_3 p_1 - c_1 p_3) & (c_3 s_1 - c_1 s_3) \\ (s_1 p_2 - s_2 p_1) & -(c_1 p_2 - c_2 p_1) & (c_1 s_2 - c_2 s_1) \end{bmatrix}$$

Now we will test that each row of $\boldsymbol{j'}^T$ provides the projective coordinates of the intersection point of two of the action lines $\$_1$, $\$_2$, $\$_3$ of the connectors (Fig. 2.19):

- Row 1 $\rightarrow \$_2 \cap \$_3 = \$_{23}$

- Row 2 $\rightarrow \$_3 \cap \$_1 = \$_{31}$
- Row 3 $\rightarrow \$_1 \cap \$_2 = \$_{12}$

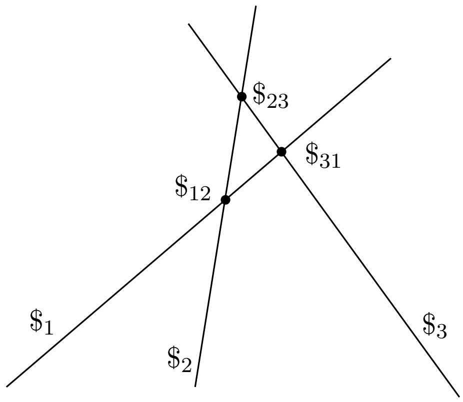


Figure 2.19: Intersections of the action lines of the connectors.

Indeed, applying Grassmann's rule for $\$_2$ and $\$_3$, the projective coordinates of the point of intersection of these two lines are (remember Eq. (2.3))

$$\begin{bmatrix} c_2 & s_2 & p_2 \\ c_3 & s_3 & p_3 \end{bmatrix} \rightarrow \quad r_y = \begin{bmatrix} s_2 & p_2 \\ s_3 & p_3 \end{bmatrix} \quad r_x = \begin{bmatrix} c_2 & p_2 \\ c_3 & p_3 \end{bmatrix} \quad r_z = \begin{bmatrix} c_2 & s_2 \\ c_3 & s_3 \end{bmatrix}$$

and it is straightforward to see that r_y, $-r_x$ and r_z are exactly the elements of the first row of $\boldsymbol{j'}^T$. The second and third rows can be tested for in a similar fashion.

2.8 Singular configurations

Note that Eq. (2.10) is in fact a system of three linear equations. If matrix $\boldsymbol{j}$ has rank 3 then it is invertible and we can express the solution to the inverse static problem as $\boldsymbol{\lambda} = \boldsymbol{j}^{-1}\hat{w}$, so that there is a one-to-one relationship between the wrenches on the end effector and joint forces. But, what happens if $\operatorname{rank} \boldsymbol{j} < 3$?

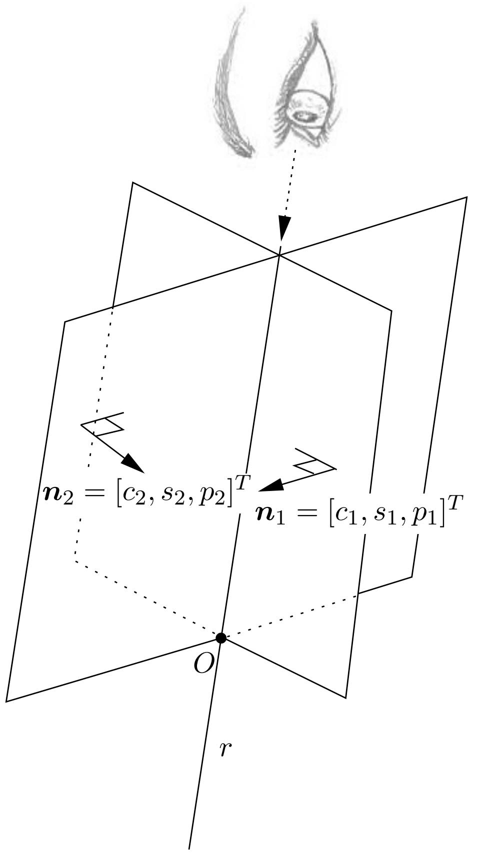


Figure 2.20: Intersection of the planes representing lines $\$_1$ and $\$_2$ in $\mathbb{R}^3$.

2.8.1 Concurrence condition

When the rank of $\boldsymbol{j}$ is less than 3 the manipulator is in a so-called **singular configuration**. In order to see which are these configurations, realize that

$$\det \begin{bmatrix} c_1 & c_2 & c_3 \\ s_1 & s_2 & s_3 \\ p_1 & p_2 & p_3 \end{bmatrix} = 0$$

if, and only if, the three lines $\$_1$, $\$_2$, $\$_3$ intersect in a common point (proper or improper). A visual way to verify so is to be aware that $\hat{w}_i = [c_i, s_i, p_i]^T$ is, besides the coordinate vector of line $\$_i$, the normal vector of the plane of $\mathbb{R}^3$ that represents this line.

Consider the planes representing $\$_1$ and $\$_2$ (Fig. 2.20). If we watch them "from above", in the direction of their intersection line r, we will have the

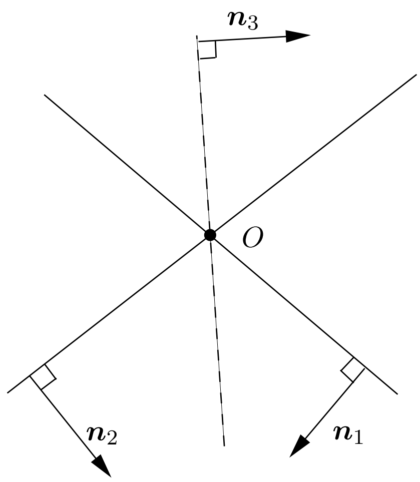


Figure 2.21: Linear dependence of $\boldsymbol{n}_3$ on $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$.

situation depicted in Fig. 2.21. As $\det \boldsymbol{j} = 0$, $\boldsymbol{n}_3$ has to be linearly dependent of $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$, and therefore the plane relative to $\boldsymbol{n}_3$, which necessarily has to pass through O, has to contain the line r. By cutting the three planes by $z = 1$ we obtain three lines intersecting at a point, which can be proper (as in the case depicted in the previous figures), or improper.

2.8.2 Concurrence at a proper point

As explained, the manipulator finds itself at a singular configuration when the three action lines of the connectors $\$_1$, $\$_2$, and $\$_3$ are concurrent (Fig. 2.22). In this situation there are end-effector wrenches that cannot be equilibrated by the joint forces. To visualize so, write Eq. (2.10), $\hat{w} = \boldsymbol{j}\boldsymbol{\lambda}$ assuming that the origin O of the coordinate system is in the intersection point Q of $\$_1$, $\$_2$, and $\$_3$:

$$\begin{bmatrix} L \\ M \\ R \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 \\ s_1 & s_2 & s_3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \tag{2.12}$$

Clearly, if the externally-applied wrench $\hat{w}_{ap} = [L, M, R]^T$ has $R \neq 0$, this system has no solution, implying that no combination of joint forces can equilibrate wrenches with $R \neq 0$. The system in Eq. (2.12) is in fact incompatible if, and only if, $R \neq 0$.

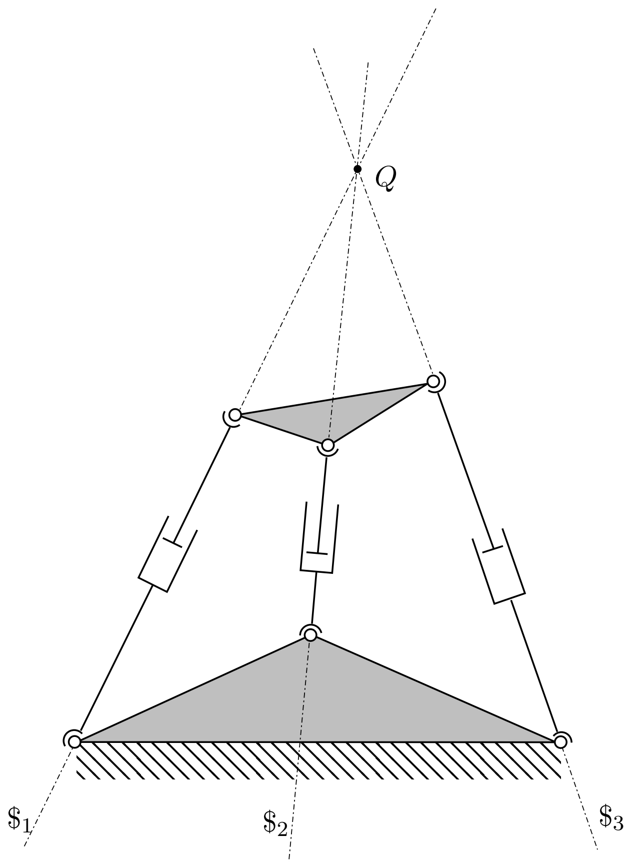


Figure 2.22: A singular configuration of a parallel 3RPR manipulator.

PROOF.- Compatible $\Leftrightarrow$ $\text{rank}(\boldsymbol{j}) = \text{rank}(\boldsymbol{j}, \hat{w}_{ap})$. Now, on the one hand:

$$\text{rank}\begin{bmatrix} c_1 & c_2 & c_3 \\ s_1 & s_2 & s_3 \\ 0 & 0 & 0 \end{bmatrix} = 2$$

as

$$\begin{vmatrix} c_1 & c_2 \\ s_1 & s_2 \end{vmatrix} = c_1 s_2 - s_1 c_2 = \sin(\theta_2 - \theta_1) \neq 0$$

Analogously, the other 2×2 minors are also different from 0.
On the other hand,

$$\text{rank}\begin{bmatrix} c_1 & c_2 & c_3 & L \\ s_1 & s_2 & s_3 & M \\ 0 & 0 & 0 & R \end{bmatrix} = 3 \Leftrightarrow R \neq 0$$

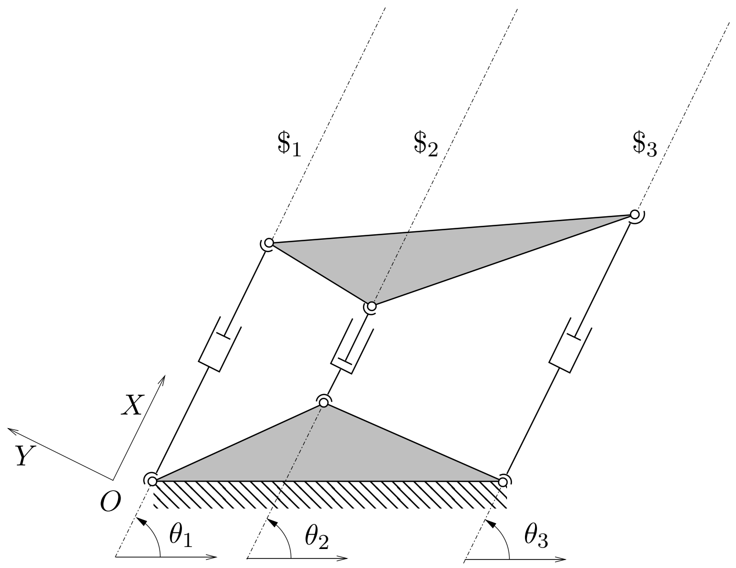


Figure 2.23: Another singular configuration of a parallel 3RPR manipulator.

as if $R = 0$ then this rank would be 2. □

Note that if the wrench applied on the end effector has the form $\hat{w}_{ap} = [L, M, 0]^T$, then the system of equations Eq. (2.12) is compatible but indeterminate, as it has now two equations with three unknowns. This means that there are infinite combinations of joint forces that can equilibrate the wrench $[L, M, 0]^T$. Thus, in a singularity it is sometimes possible to solve the inverse static problem, but when this happens there are infinite solutions.

2.8.3 Concurrence at an improper point

The manipulator of Fig. 2.23 is also in a singularity when the three lines $\$_1$, $\$_2$, $\$_3$ intersect at a point in infinity, or, in other words, when they are parallel. Since $\theta_1 = \theta_2 = \theta_3 = \theta$, we will have

$$\frac{s_1}{c_1} = \frac{s_2}{c_2} = \frac{s_3}{c_3} = \tan\theta$$

which means that the first and second rows of $\boldsymbol{j}$ are linearly dependent, and $\operatorname{rank} \boldsymbol{j} < 3$.

To see which end-effector wrenches won't be equilibrable, we write Eq. (2.10)

$\hat{w} = \boldsymbol{j}\boldsymbol{\lambda}$ in the coordinate system OXY shown in Fig. 2.23:

$$\begin{bmatrix} L \\ M \\ R \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ p_1 & p_2 & p_3 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \tag{2.13}$$

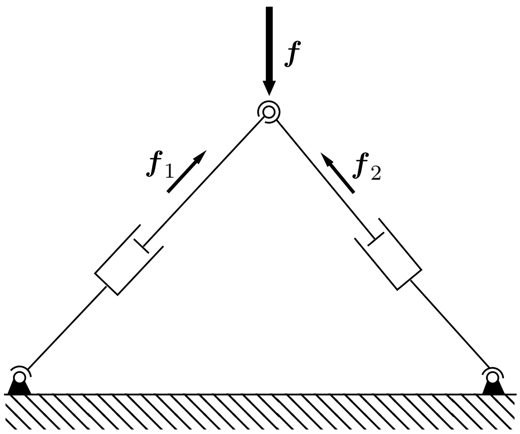
It is easy to see that this system has no solution if $M \neq 0$. Therefore, any wrench with non-null component M cannot be equilibrated by the manipulator in this configuration. One can check that only these wrenches cannot be equilibrated, since if $M = 0$, then the system in Eq. (2.13) has infinitely-many solutions.

2.8.4 Joint forces that tend to infinity

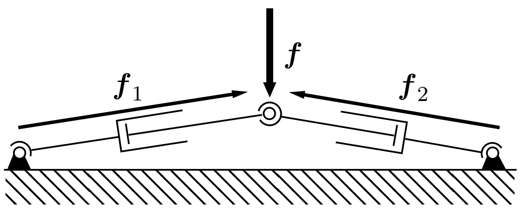
Accordingly on what has been stated in Sec. 2.7, equation $\boldsymbol{\lambda} = \boldsymbol{j}^{-1}\hat{w}$ can be expressed as

$$\boldsymbol{\lambda} = \frac{\boldsymbol{j}'^T}{\det \boldsymbol{j}}\hat{w}$$

Clearly, when we approach a singularity, $\det \boldsymbol{j}$ tends to zero, and therefore the values f_1, f_2 and f_3 of $\boldsymbol{\lambda}$ will tend to infinity. A similar phenomenon takes place in reality. The legs will support increasing forces and eventually may break. Therefore, from the point of view of material resistance, the approach of a parallel robot to a singularity has to be avoided. The phenomenon is analogous to the one shown in Table 1 below, for a 2RPR robot under the action of a vertical force.



Far from the singularity, the actuators have to exert small forces $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ to equilibrate $\boldsymbol{f}$.



Close to the singularity, the actuators have to exert very large forces $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ in order to generate enough vertical component of equilibrant force.

Table 1: 2RPR robot in a non-singular and a close-to-singular configuration.