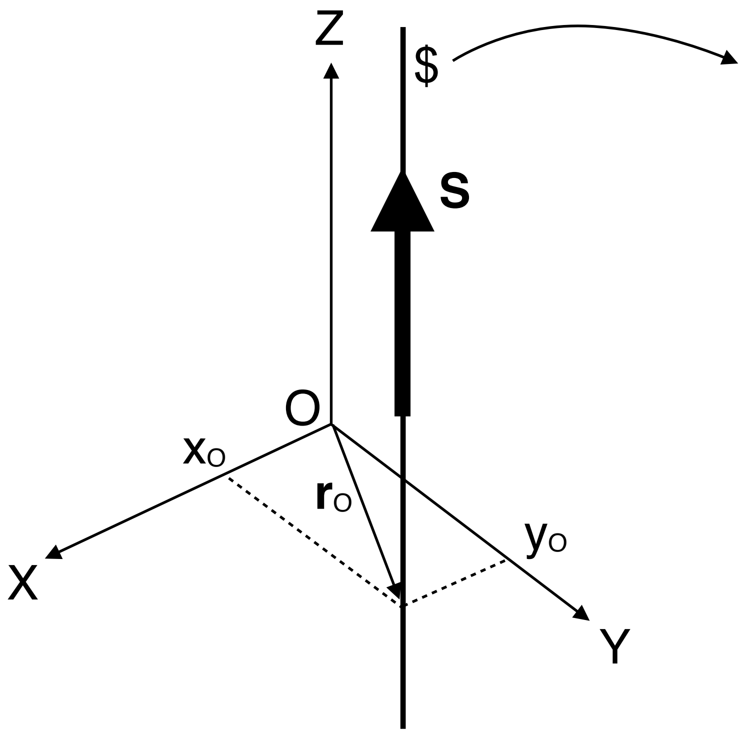
Module III

Instantaneous Kinematics

Analogy table

STATICS		KINEMATICS
• Line in XY:	$\hat{s} = \{L,M;R\}$	
• Unit coordinates:	$\hat{s} = \{c,s;p\}$	
• Wrench:	$\hat{w} = \{\mathbf{f};\mathbf{c}_O\}$	
• Pure couple:	$\hat{w} = \{\mathbf{0};\mathbf{c}_O\}$	
• Translation+rotation:	**[e]**	
• Statics of manipulators	**[j]**	

The coordinates of a line II to Z



$$\hat{s} = \{\mathbf{S};\mathbf{S}_o\}$$

$$\mathbf{S}_o = \mathbf{r}_o \times \mathbf{S}$$

$$\hat{s} = \{N;P,Q\}$$

$\mathbf{S}=N\mathbf{k}$ $\quad P = y_oN$

$Q = -x_oN$

LINE in XY plane: $\{L,M,0;0,0,R\}$

LINE II Z: $\{0,0,N;P,Q,0\}$ $\longrightarrow$ $\{L,M,N;P,Q,R\}$

$$\hat{s} = \{N;y_oN,-x_oN\} = N\underbrace{\{1;y_o,-x_o\}}_{\text{Normalized coords of the line}}$$

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STATICS

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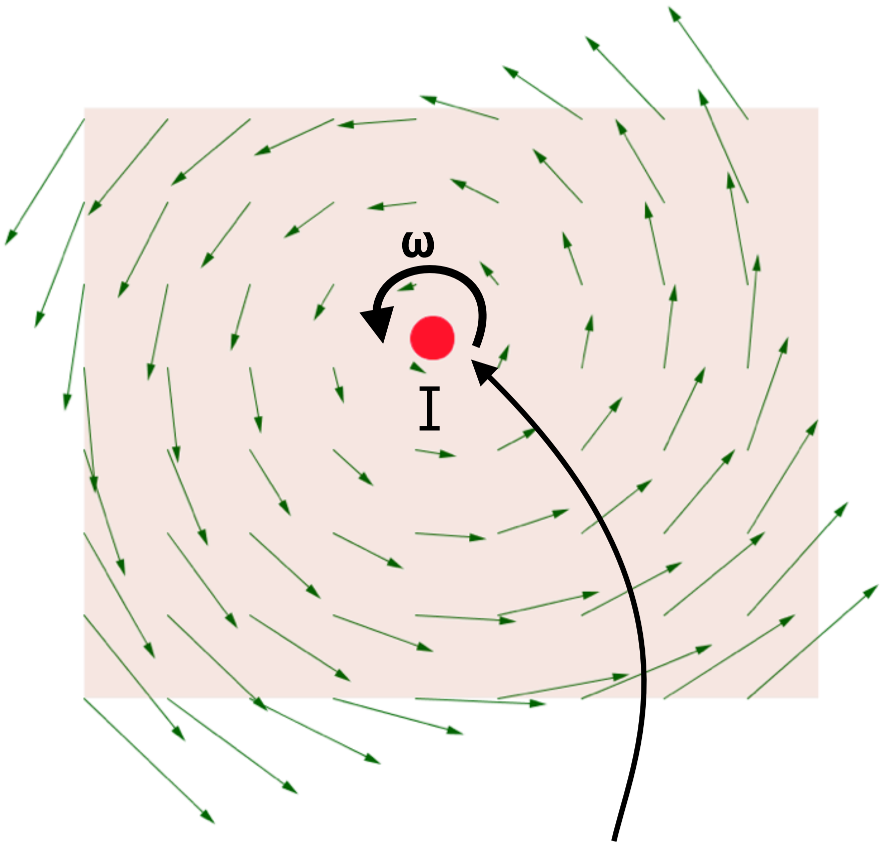
- Translation+rotation: **[e]**
- Statics of manipulators **[j]**

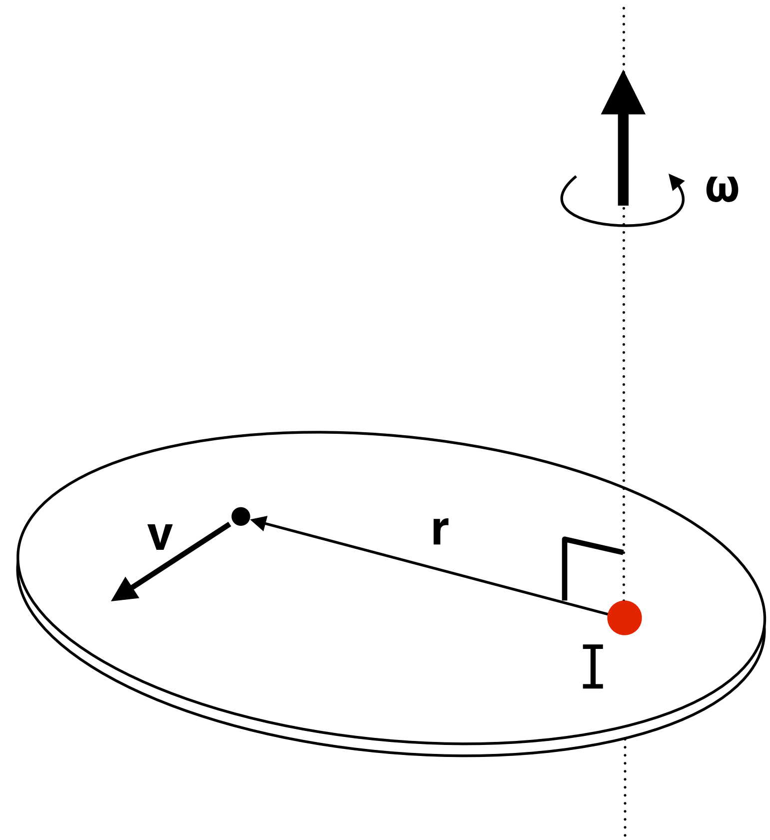
KINEMATICS

- Line II to Z axis: $\hat{s} = \{N;P,Q\}$
- Unit coordinates: $\hat{s} = \{1;y_O,-x_O\}$

Angular velocities & the instant center



Instant center of rotation (ICR)

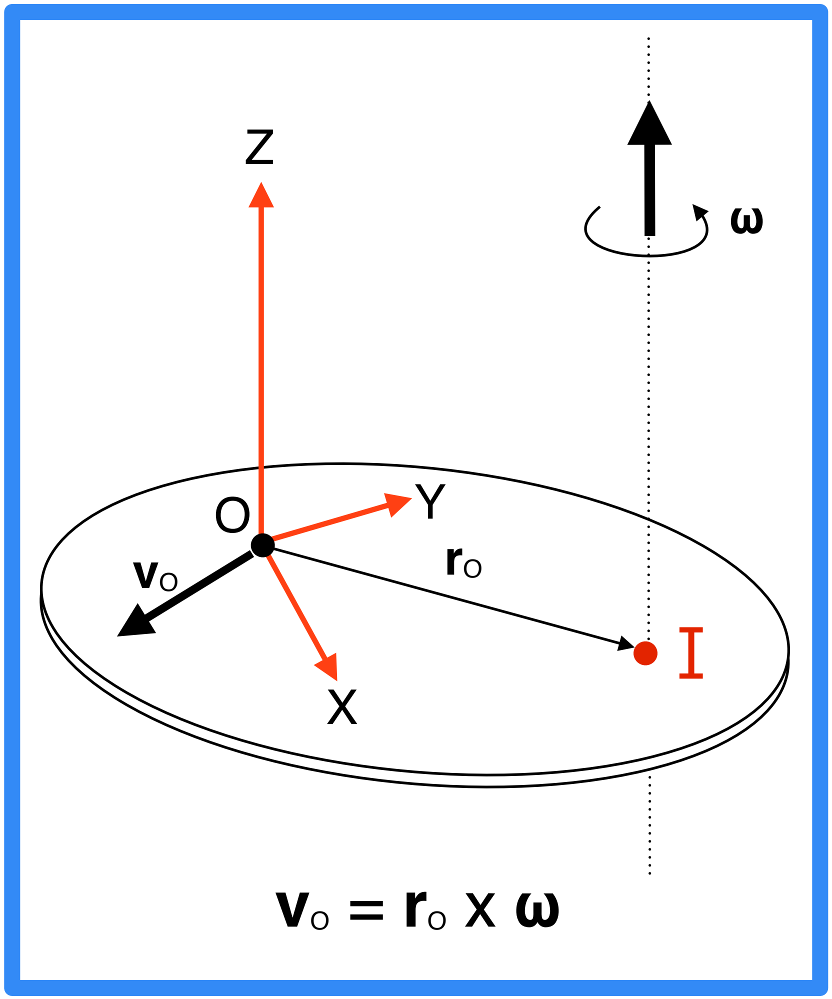


$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

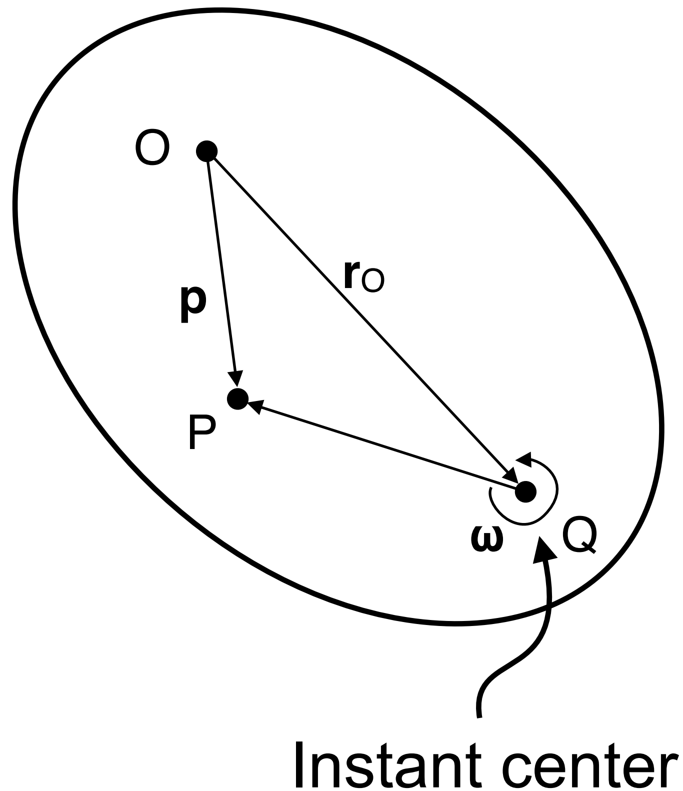
Find a compact way to describe the velocity field

Twist

$\hat{t} = \{\omega; v_o\}$

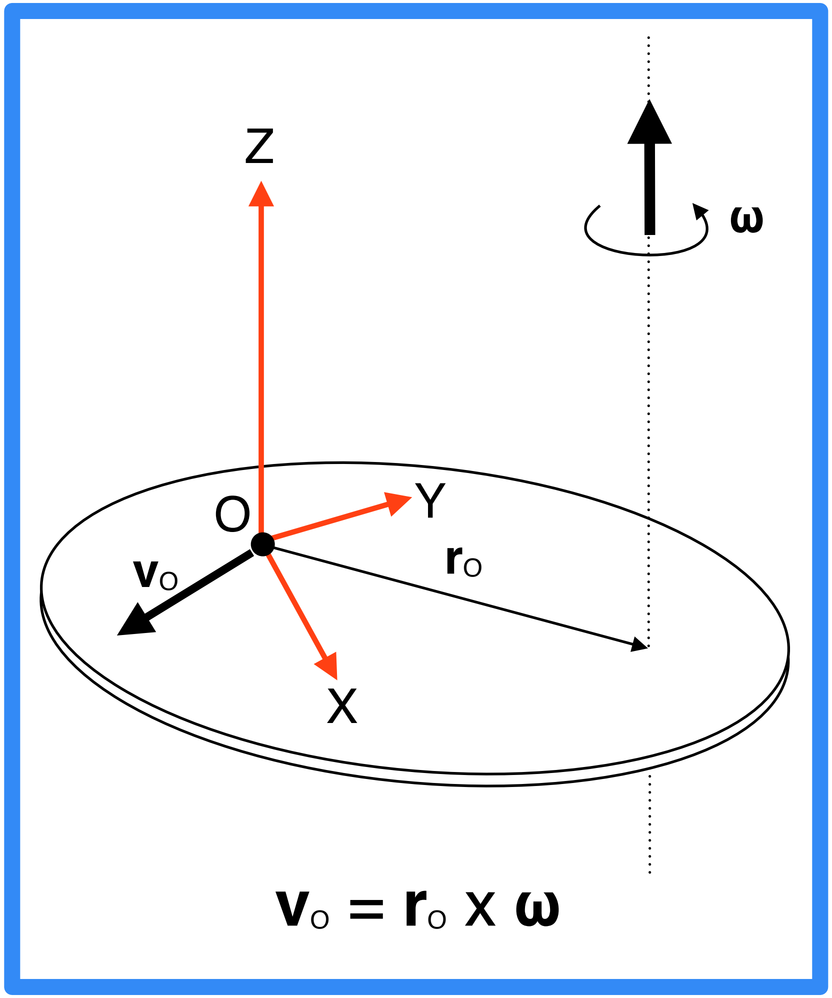


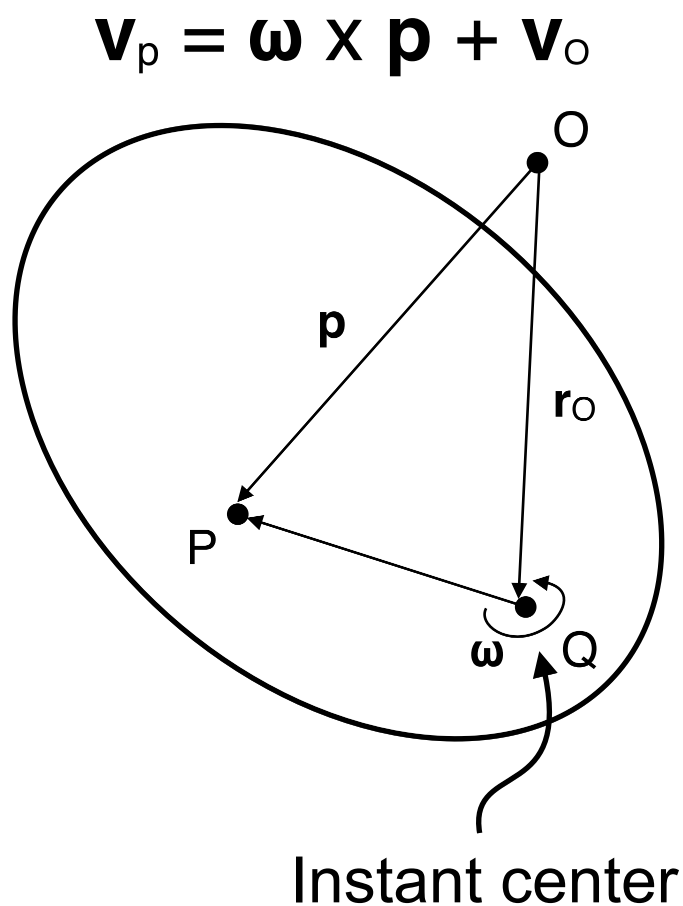
$v_p = \omega \times p + v_O$



Twist

$$\hat{t} = \{\boldsymbol{\omega};\mathbf{v}_O\}$$





Twist = multiple of vertical line

$$\hat{t} = \{\boldsymbol{\omega}; \mathbf{v}_o\}$$

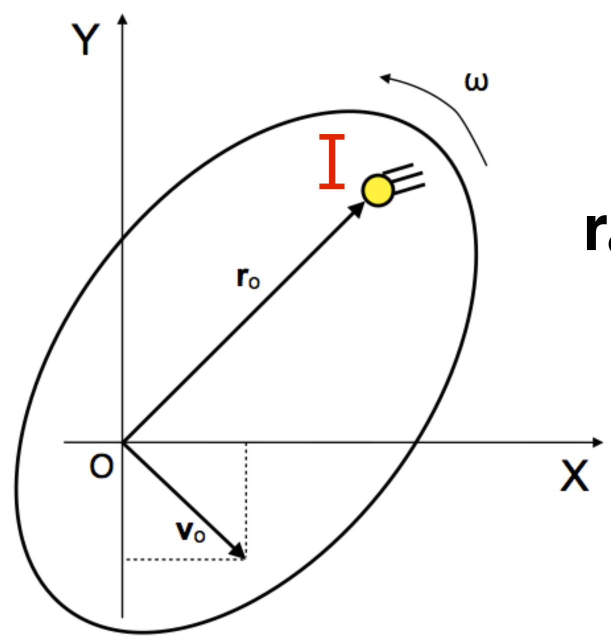
$$\boldsymbol{\omega} = \omega\, \mathbf{k}$$

$$\mathbf{v}_o = \mathbf{r}_o \times \boldsymbol{\omega}$$

$$\mathbf{v}_o = \begin{bmatrix} y_o\, \omega \\ -x_o\, \omega \\ 0 \end{bmatrix}$$

$$\hat{t} = \omega\, \{1;\, y_o, -x_o\}$$

$$\hat{t} = \omega \hat{s}$$



$$\mathbf{r}_o = \begin{bmatrix} x_o \\ y_o \\ 0 \end{bmatrix}$$

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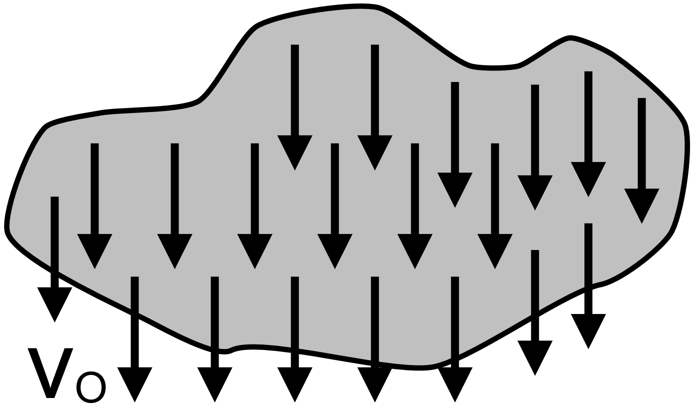
• Statics of manipulators	**[j]**		

Pure translation

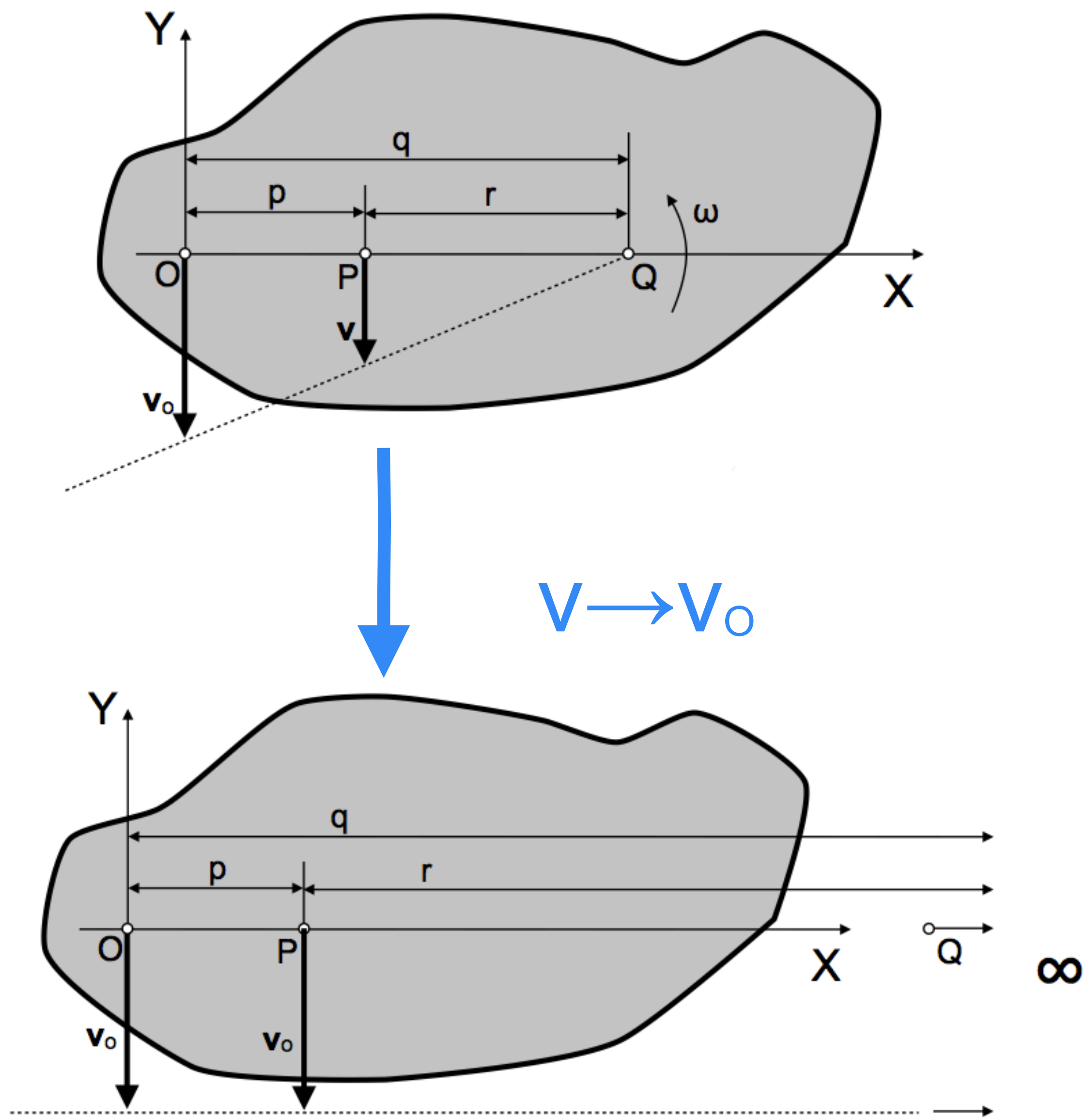


$$\hat{t} = \{0; v_O\}$$

unit vector

$$\hat{t} = v_O\{0; \underbrace{S_O\}}\}$$

unit coords of the vertical line at infinity in the direction orthogonal to $\mathbf{v}_0$



Analogy table

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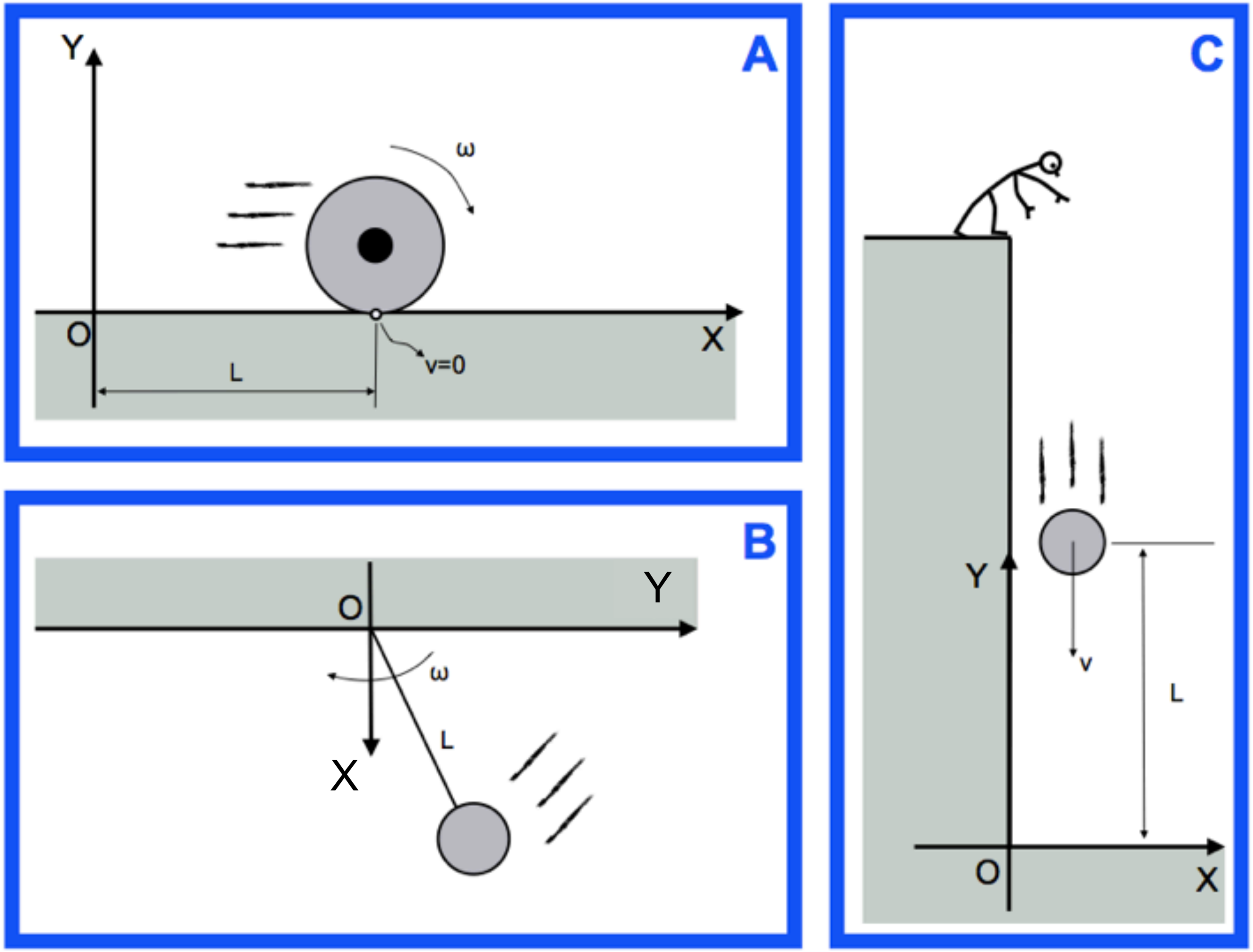

A
Y
ω
O
X
L
v=0
B
O
Y
ω
X
L
C
v
Y
L
O
X

Ray and axis coordinates

Order	Order name	Preferred for
{vector ; moment}	Ray coords.	Wrenches, lines in XY plane
{moment ; vector}	Axis coords.	Twists, lines $\perp$ XY plane

From now on

Wrenches in ray	Twists in axis
$\hat{w} = \{\mathbf{f};\mathbf{c}_O\} = \begin{bmatrix} \mathbf{f} \\ \mathbf{c}_O \end{bmatrix}$	$\hat{T} = \{\mathbf{v}_O;\boldsymbol{\omega}\} = \begin{bmatrix} \mathbf{v}_O \\ \boldsymbol{\omega} \end{bmatrix}$
lowercase	uppercase

In a given coordinate system, an observer sees the following twist for the end effector of a robot

$\hat{\mathbf{T}} = \{10 \text{ m/s} , -5 \text{ m/s} ; 5 \text{ rad/s}\}$

Which of the following statements are true?

A The instantaneous center of rotation is in point (1,2), and the angular velocity of the end effector is of 5 rad/s counterclockwise.

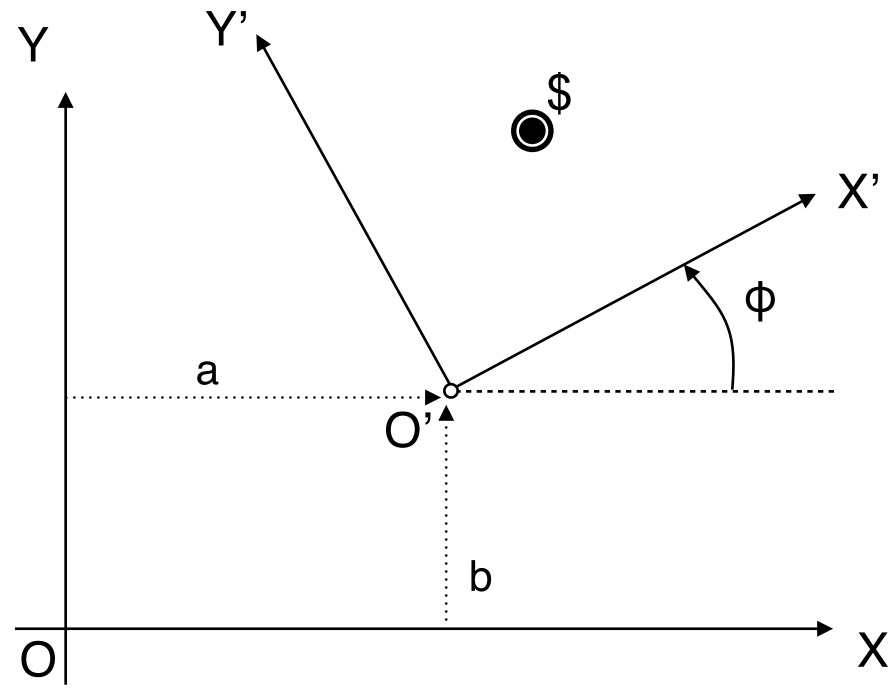
B The instantaneous center of rotation is in point (5,10), and the angular velocity of the end effector is of 5 rad/s clockwise.

C The velocity of the origin point of the end effector is $\mathbf{v}_O = (10,-5)$ m/s.

D The velocity of the end effector point P=(2,1) is $\mathbf{v}_P = (5,5)$ m/s.

Changing the coordinate system of lines and twists

All lines or twists in axis coords!



General change

$$\begin{bmatrix} P \\ Q \\ N \end{bmatrix} = [E] \begin{bmatrix} P' \\ Q' \\ N' \end{bmatrix} \quad \text{where} \quad [E] = \begin{bmatrix} c & -s & b \\ s & c & -a \\ 0 & 0 & 1 \end{bmatrix}$$

In OXY

In O'X'Y'

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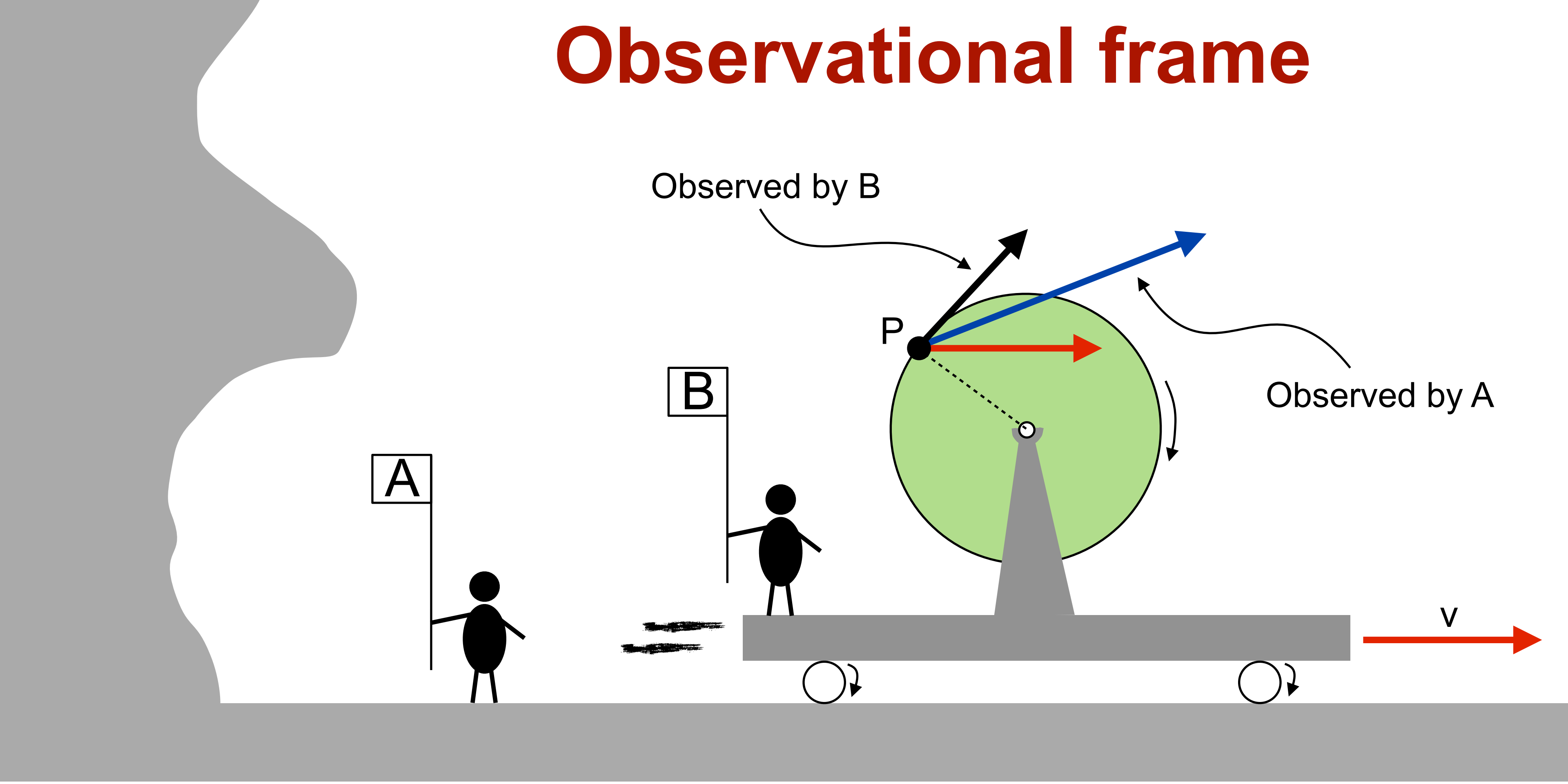
• Pure couple:	$\hat{w} = \{\mathbf{0};\mathbf{c}_O\}$	• Pure translation:	$\hat{t} = \{\mathbf{0};\mathbf{v}_O\}$
• Translation+rotation:	**[e]**	• Translation+rotation:	**[E]**
• Statics of manipulators	**[j]**		

Relationship between [e] and [E]

$$[e]^{-1} = [E]^T$$

STATICS	KINEMATICS
Lines / wrenches (ray coords)	Lines / twists (axis coords)

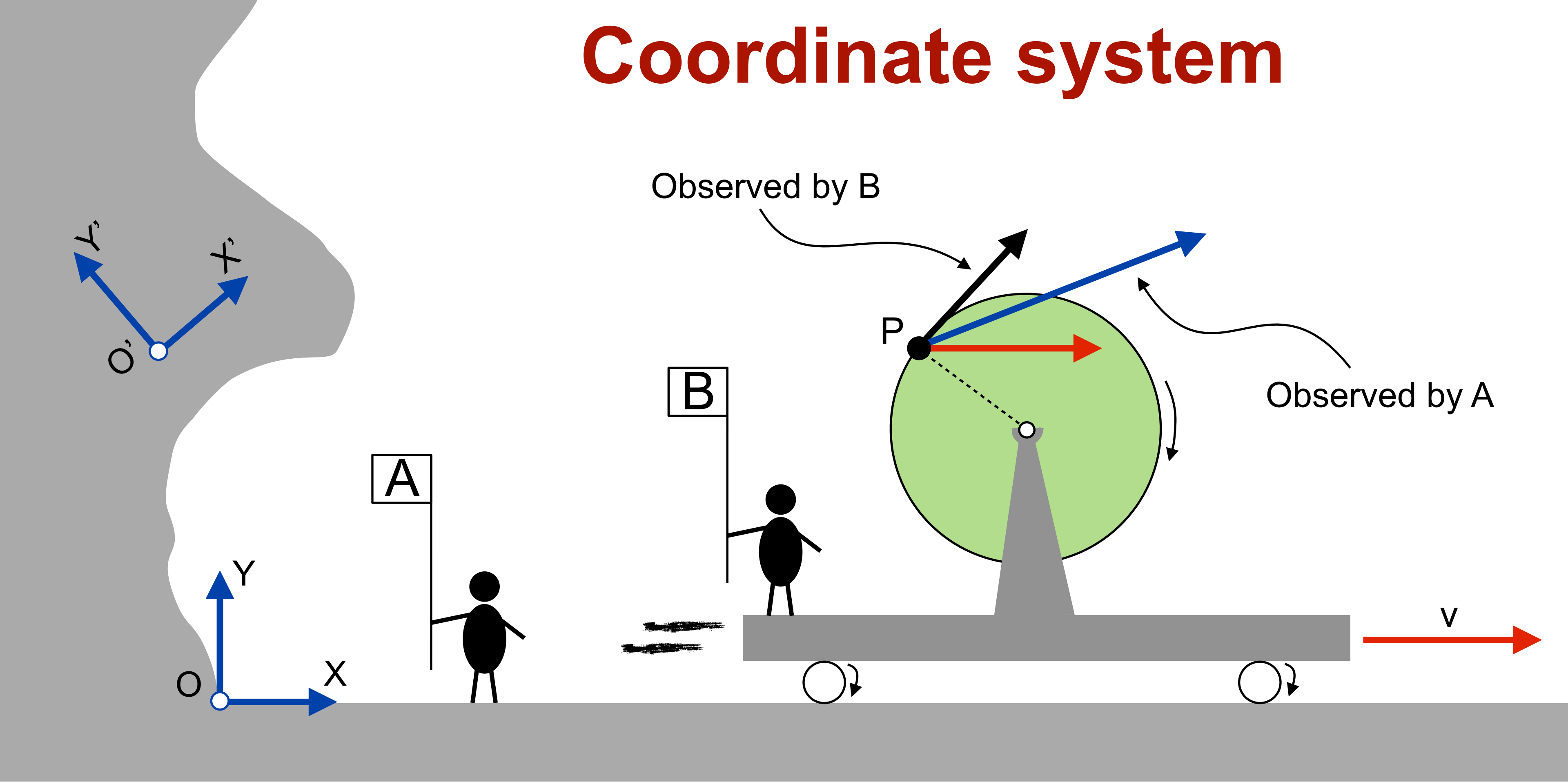
Observational frame



Observational frame = An observer and all points fixed to that observer

The velocity field is always **relative** to a chosen observational frame

Coordinate system



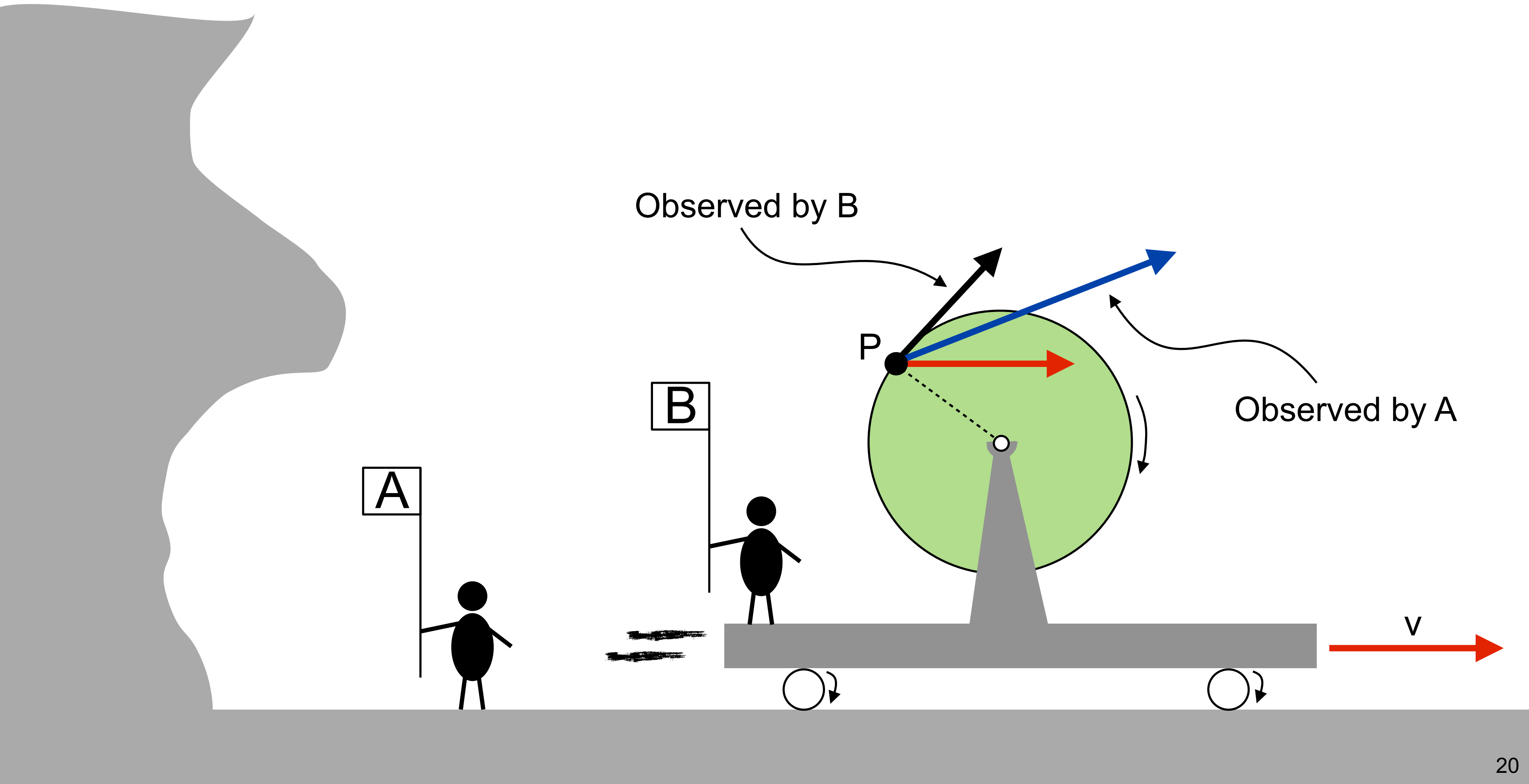
Coordinate system = A point and three spatial directions attached to an observational frame, used to describe points and velocities seen from that frame

Sentence

- Orthodox → Twist of the wheel relative to **observational frame A**, expressed in **coordinate system OXY**
- Simplified → Twist of the wheel relative to **coordinate system OXY**

Law of composition of velocities

Velocity of P seen from frame A	=	Velocity of P seen from frame B	+	Velocity of P seen from frame A, assuming P is fixed in frame B



3R serial manipulator

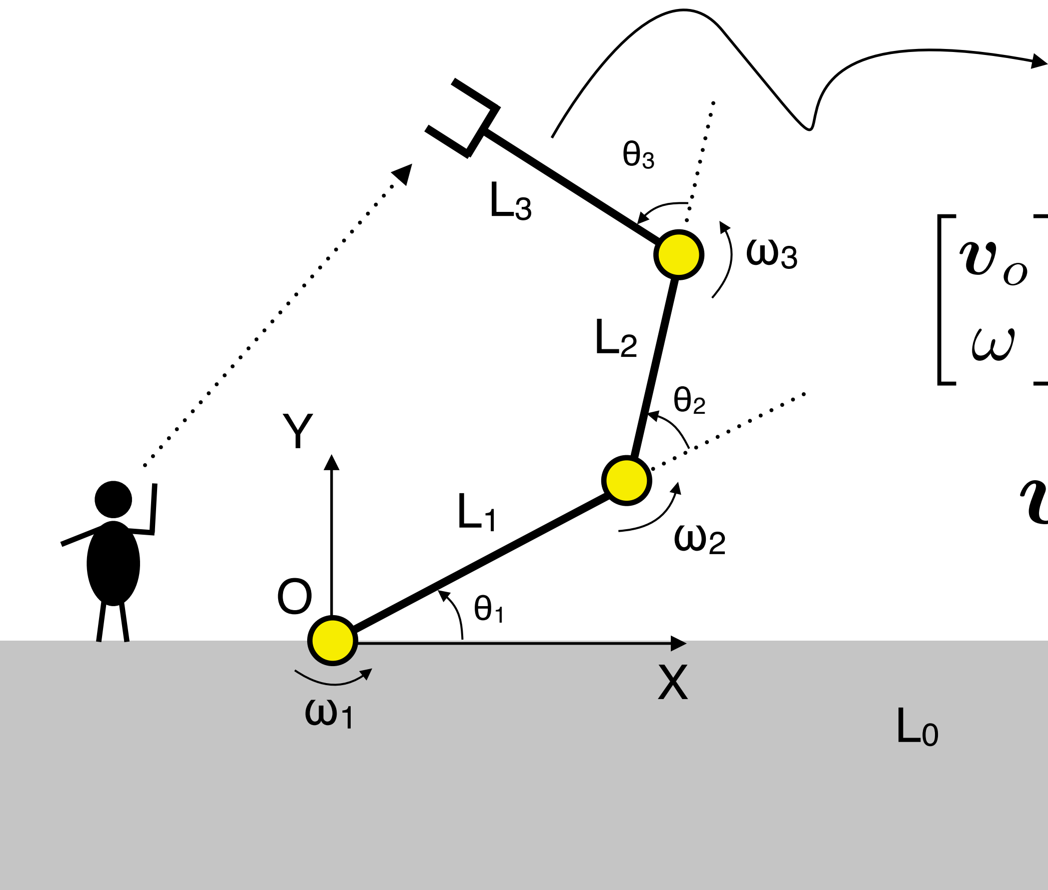
$$\gamma = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

Vector of scalar joint angular velocities

Instantaneous kinematic problems	
FORWARD	$\gamma \rightarrow \hat{T}$
INVERSE	$\hat{T} \rightarrow \gamma$

$$\hat{T} = \begin{bmatrix} \boldsymbol{v}_o \\ \omega \end{bmatrix}$$

End-effector twist relative to a ground observer, using OXY



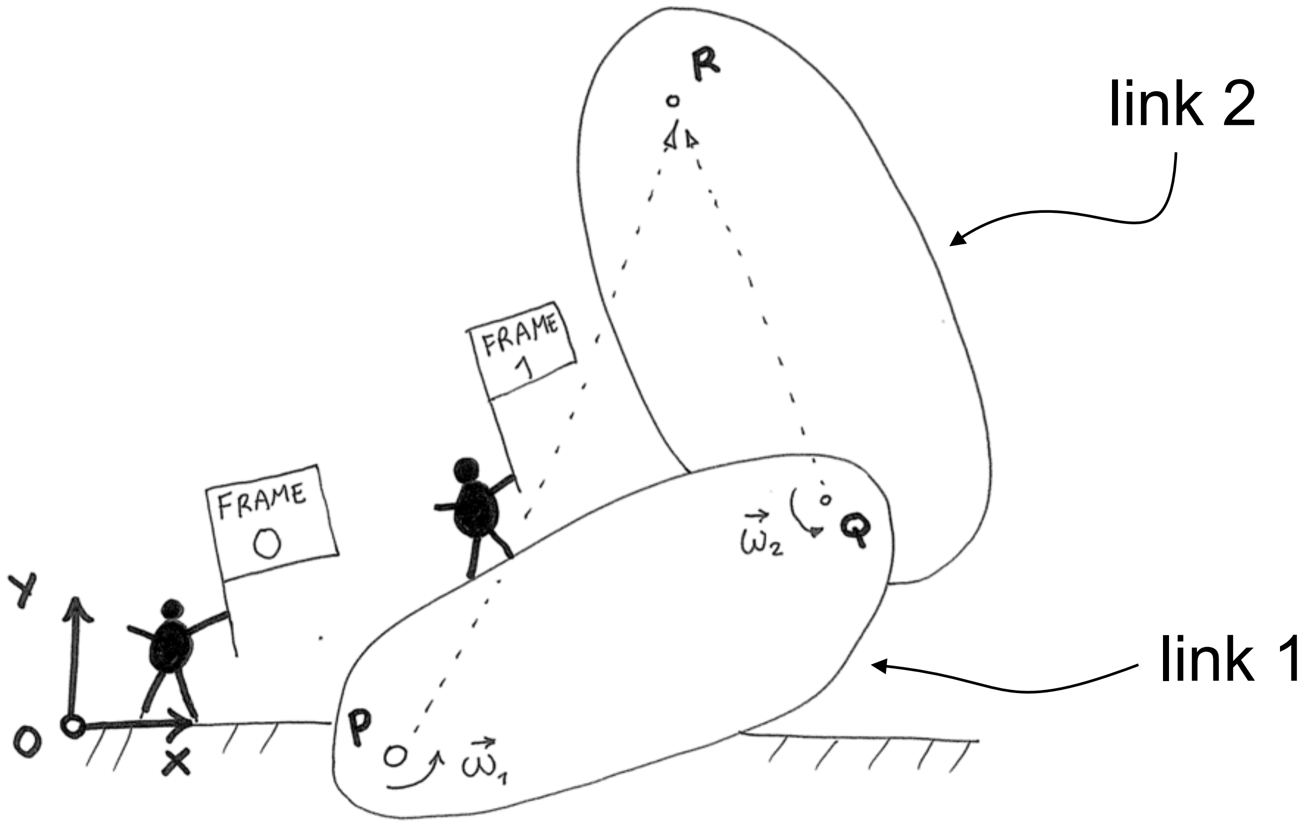
$$\hat{T} = \hat{T}_1 + \hat{T}_2 + \hat{T}_3$$

$$\begin{bmatrix} \boldsymbol{v}_o \\ \omega \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_{o1} \\ \omega_1 \end{bmatrix} + \begin{bmatrix} \boldsymbol{v}_{o2} \\ \omega_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{v}_{o3} \\ \omega_3 \end{bmatrix}$$

$\boldsymbol{v}_{oi} \rightarrow$ Velocity of the "origin point" of link i assuming that all joints are locked except joint i

Why does the velocity sum hold?

$$\begin{bmatrix} \boldsymbol{v}_o \\ \omega \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_{o1} \\ \omega_1 \end{bmatrix} + \begin{bmatrix} \boldsymbol{v}_{o2} \\ \omega_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{v}_{o3} \\ \omega_3 \end{bmatrix}$$



Apply the law of composition of velocities

The velocity equation

$$\hat{T} = \hat{T}_1 + \hat{T}_2 + \hat{T}_3$$

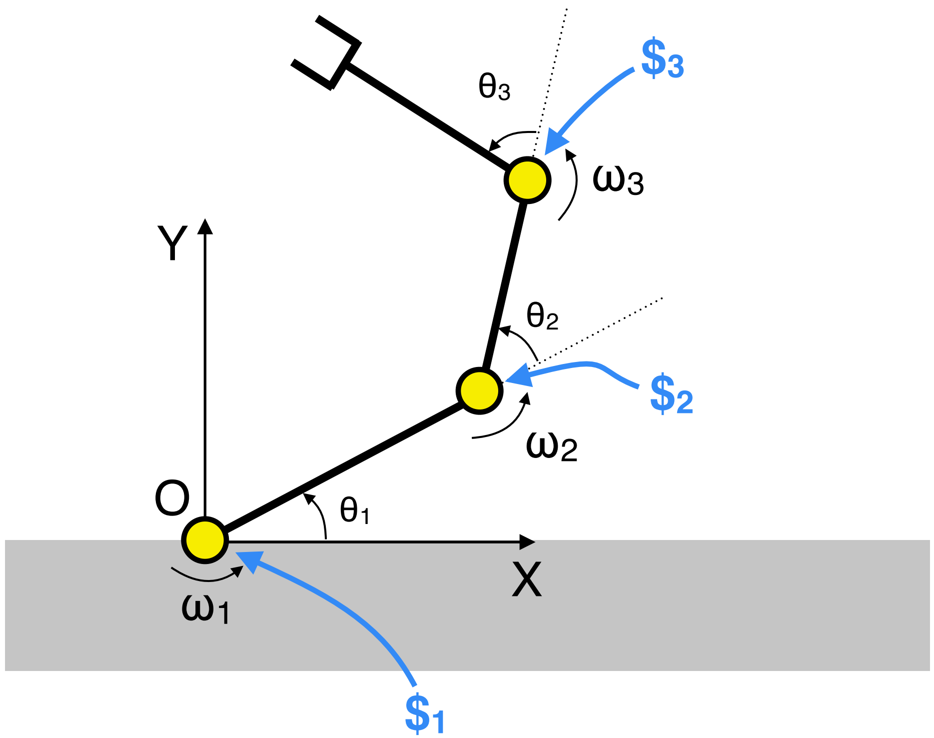
$$\hat{T} = \begin{bmatrix} \boldsymbol{v}_{o1} \\ \omega_1 \end{bmatrix} + \begin{bmatrix} \boldsymbol{v}_{o2} \\ \omega_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{v}_{o3} \\ \omega_3 \end{bmatrix}$$

$$\hat{T} = \omega_1 \begin{bmatrix} y_1 \\ -x_1 \\ 1 \end{bmatrix} + \omega_2 \begin{bmatrix} y_2 \\ -x_2 \\ 1 \end{bmatrix} + \omega_3 \begin{bmatrix} y_3 \\ -x_3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} v_{ox} \\ v_{oy} \\ \omega \end{bmatrix} \qquad \boxed{\hat{T} = \boldsymbol{J} \cdot \gamma} \qquad \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$$\begin{bmatrix} y_1 & y_2 & y_3 \\ -x_1 & -x_2 & -x_3 \\ 1 & 1 & 1 \end{bmatrix}$$

3R serial manipulator



$$\begin{array}{c} \begin{array}{ccc} \hat{\$}_1 & \hat{\$}_2 & \hat{\$}_3 \end{array} \\ \boldsymbol{J} = \begin{bmatrix} y_1 & y_2 & y_3 \\ -x_1 & -x_2 & -x_3 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$\boldsymbol{J}^{-1} = \frac{1}{det\ \boldsymbol{J}} \begin{bmatrix} a_{23}\ \hat{s}_{23}^T \\ a_{31}\ \hat{s}_{31}^T \\ a_{13}\ \hat{s}_{12}^T \end{bmatrix}$$

$$det\ \boldsymbol{J} = a_{23}\ \hat{s}_{23}^T\ \hat{S}_1 = a_{31}\ \hat{s}_{31}^T\ \hat{S}_2 = a_{12}\ \hat{s}_{12}^T\ \hat{S}_3$$

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• Translation+rotation:	**[e]**	• Translation+rotation:	**[E]**
• Statics of manipulators	**[j]**	• Kinematics of manipulators	**[J]**

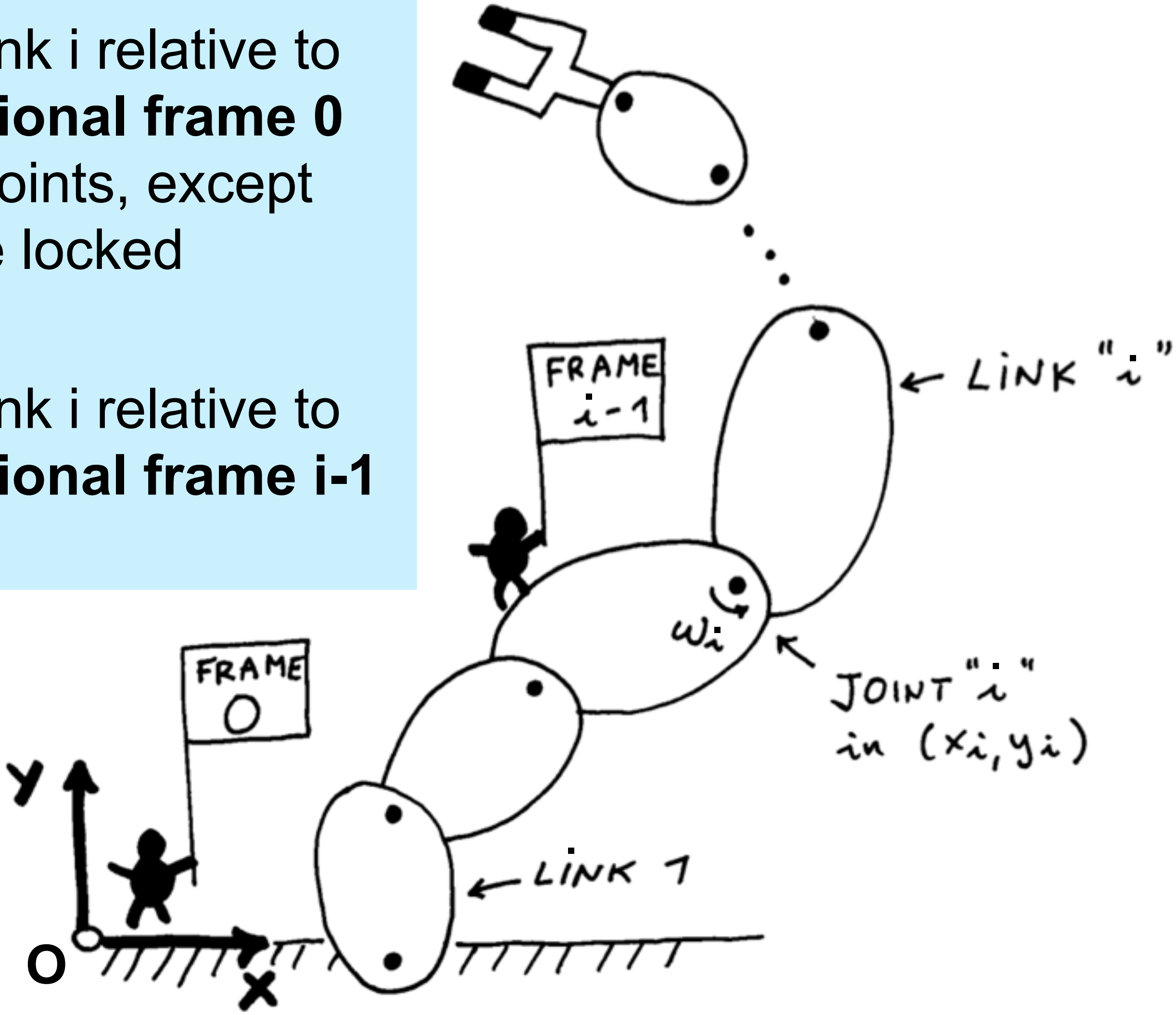
Velocity equation in general chains

$$\hat{T} = \hat{T}_1 + \ldots + \hat{T}_n$$

$\hat{T}_i$

- Twist of link i relative to **observational frame 0** when all joints, except joint i, are locked
- Twist of link i relative to **observational frame i-1**

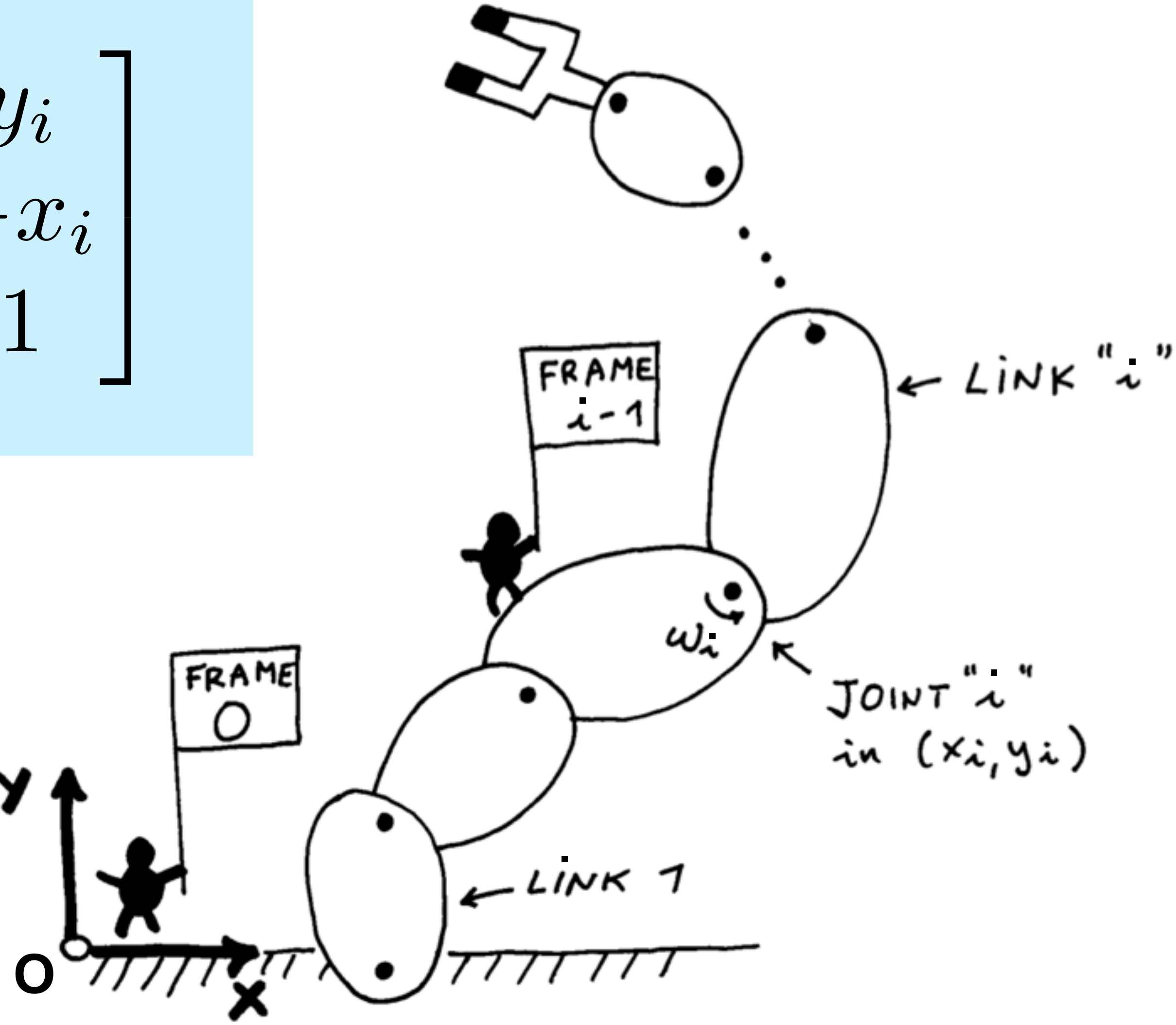
In both interpretations $\hat{T}_i$ is expressed in OXY



If i-th joint is revolute

$$\hat{T} = \hat{T}_1 + \ldots + \hat{T}_n$$

$$\hat{T}_i = \omega_i \begin{bmatrix} y_i \\ -x_i \\ 1 \end{bmatrix}$$



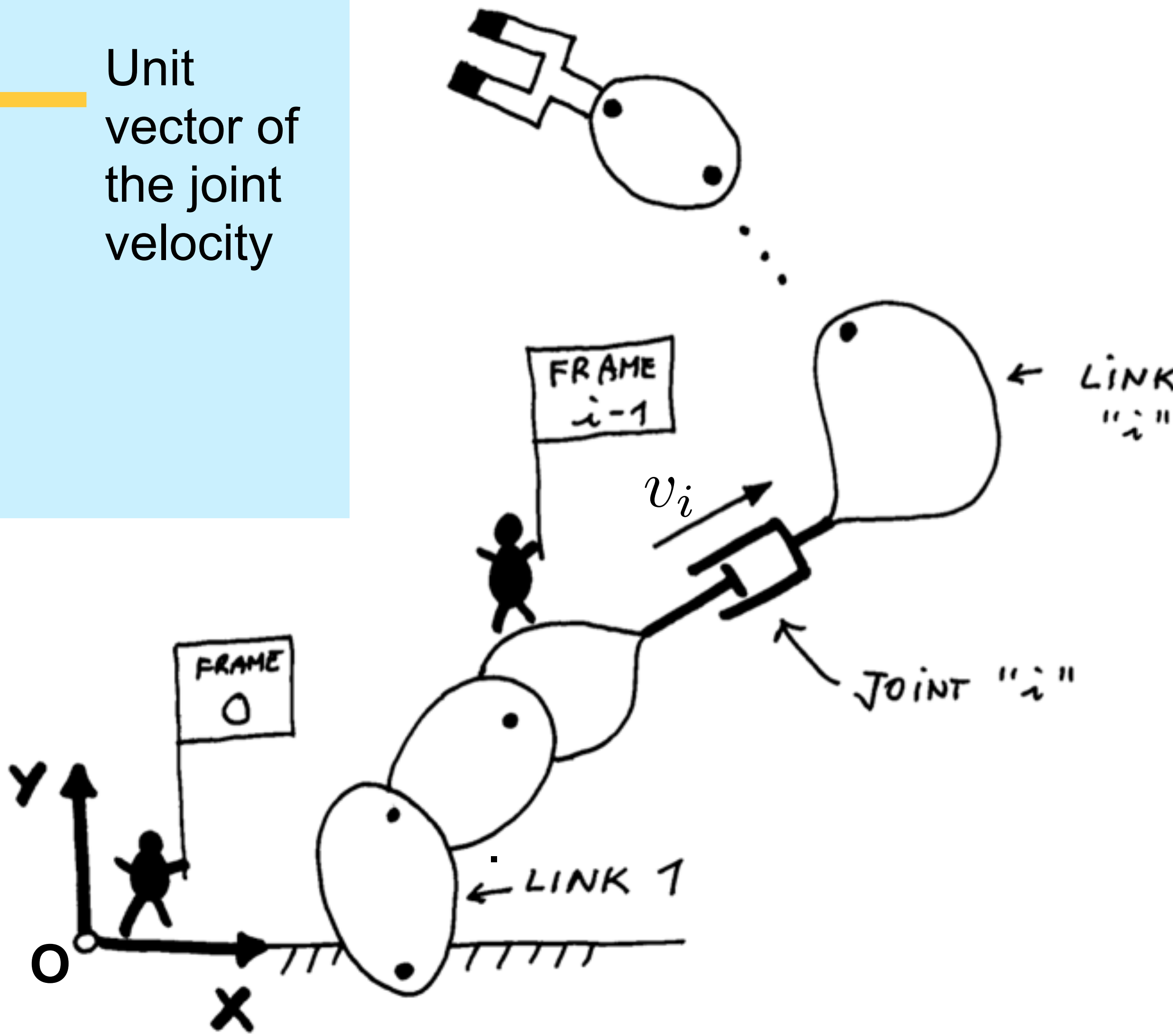
If i-th joint is prismatic

$$\hat{T} = \hat{T}_1 + \ldots + \hat{T}_n$$

$$\hat{T}_i = v_i \begin{bmatrix} a_i \\ b_i \\ 0 \end{bmatrix}$$

Unit vector of the joint velocity

Magnitude of the joint velocity

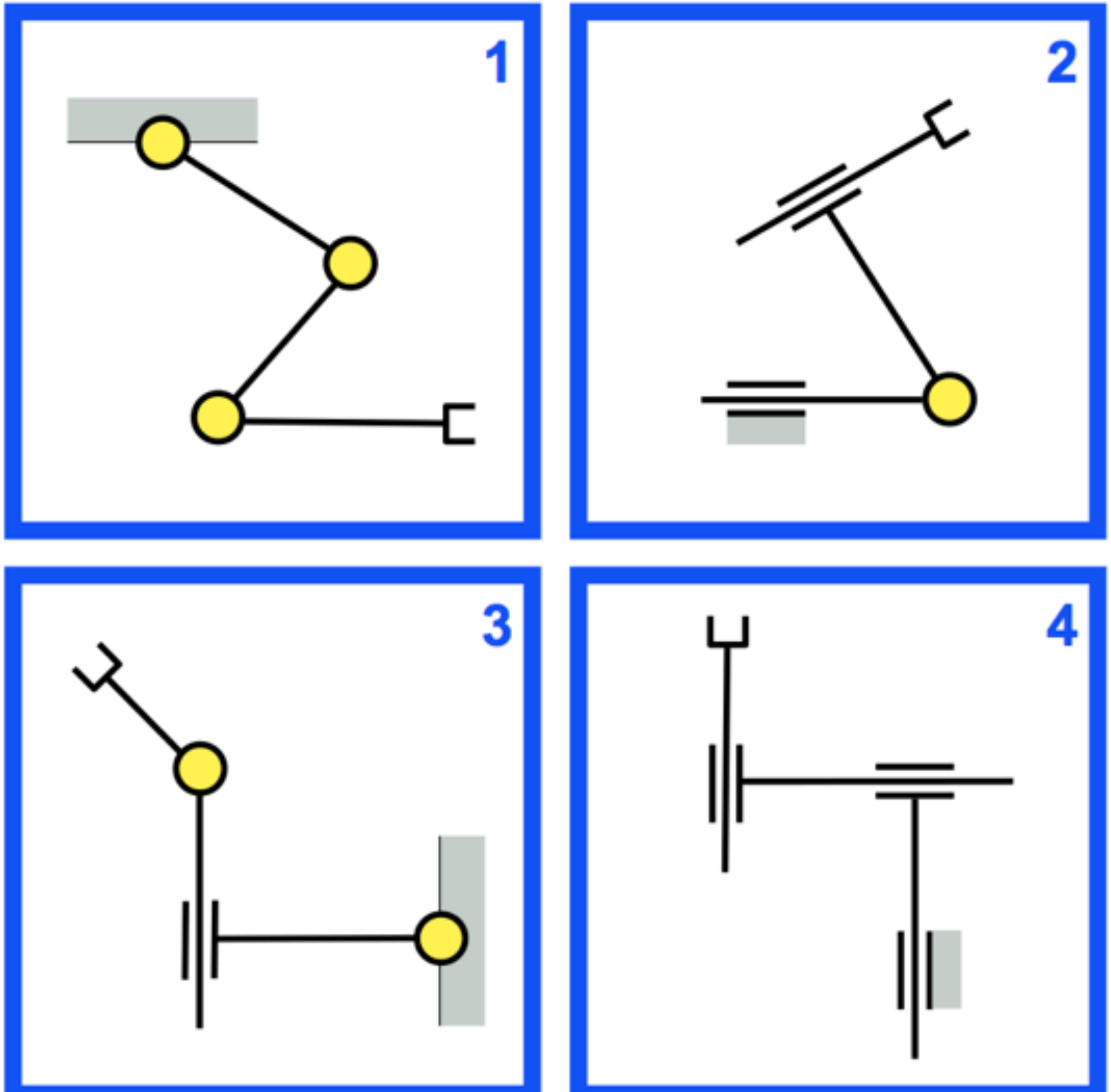


General form of the velocity equation

$$\hat{T} = \begin{bmatrix} & y_i & & a_j & \\ \dots & -x_i & \dots & b_j & \dots \\ & 1 & & 0 & \end{bmatrix} \cdot \begin{bmatrix} \vdots \\ \omega_i \\ \vdots \\ v_j \\ \vdots \end{bmatrix}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \text{If joint i revolute} & \text{If joint j prismatic} \end{array}$$

Match every manipulator with its Jacobian



$$A = \begin{bmatrix} y_1 & 0 & y_3 \\ -x_1 & 1 & -x_3 \\ 1 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} y_1 & y_2 & y_3 \\ -x_1 & -x_2 & -x_3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & \cos\gamma \\ 0 & -x_2 & \sin\gamma \\ 0 & 1 & 0 \end{bmatrix}$$

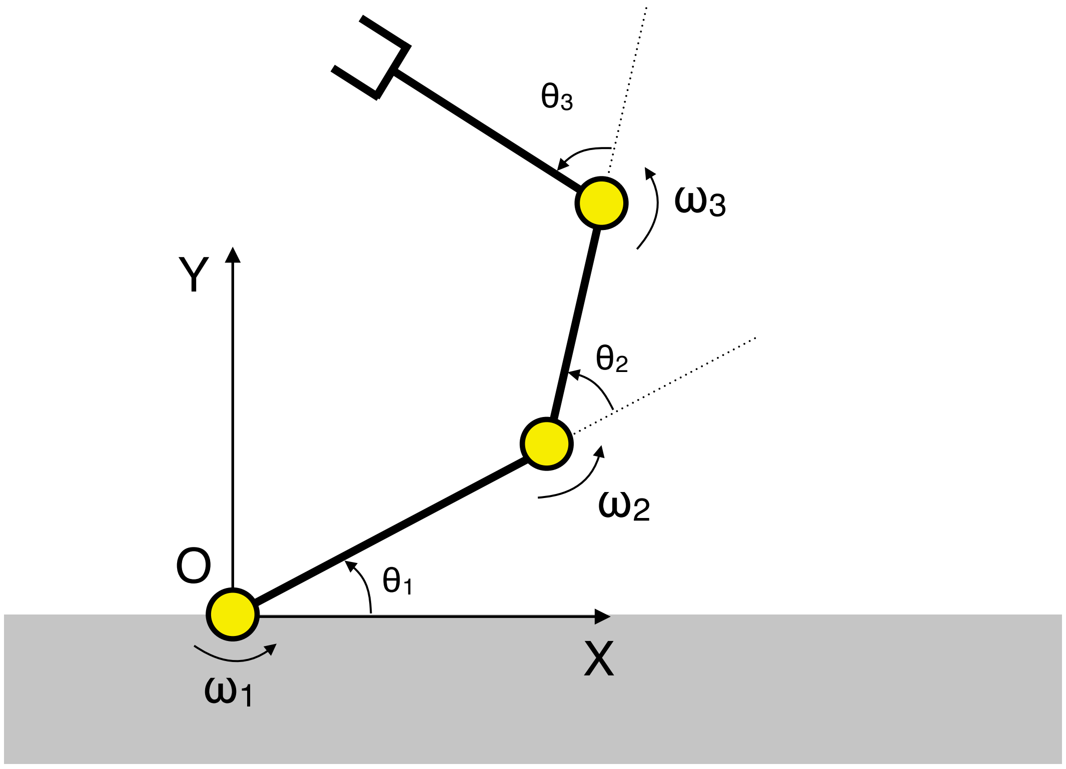
Singularities

(in general mechanisms)

Forward Inst. Kinem. Prob.	Inverse Inst. Kinem. Prob.
γ $\downarrow$ Whole velocity state, obtaining $\hat{T}$ in particular	$\hat{T}$ $\downarrow$ Whole velocity state, obtaining γ in particular

Nonsingular configuration	**When the two problems have unique solutions for any γ or $\hat{T}$**
Singular configuration	**Otherwise** (one of the problems unsolvable or undetermined)

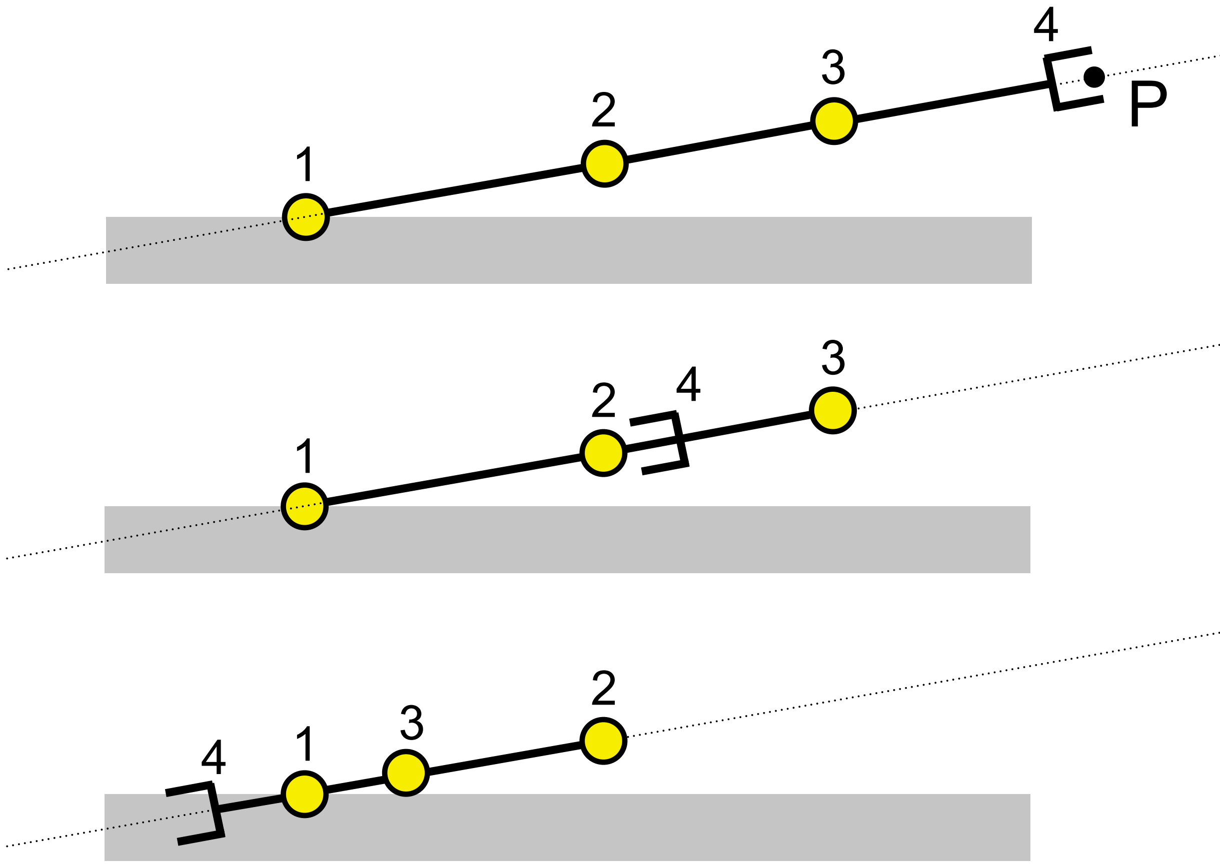
Singularities of a serial manipulator



$$\hat{T} = \boldsymbol{J} \cdot \gamma$$

Singularities of a serial manipulator

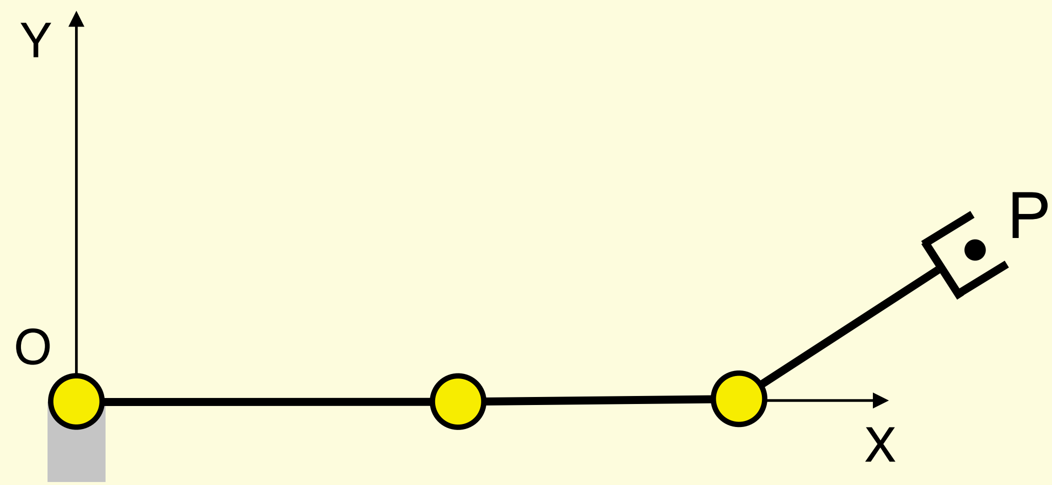
Unsolvable / undetermined INVERSE inst. kinem. problem



Homework

Find the end-effector twists for which the IIKP has no solution or infinitely-many solutions

Apparent paradox



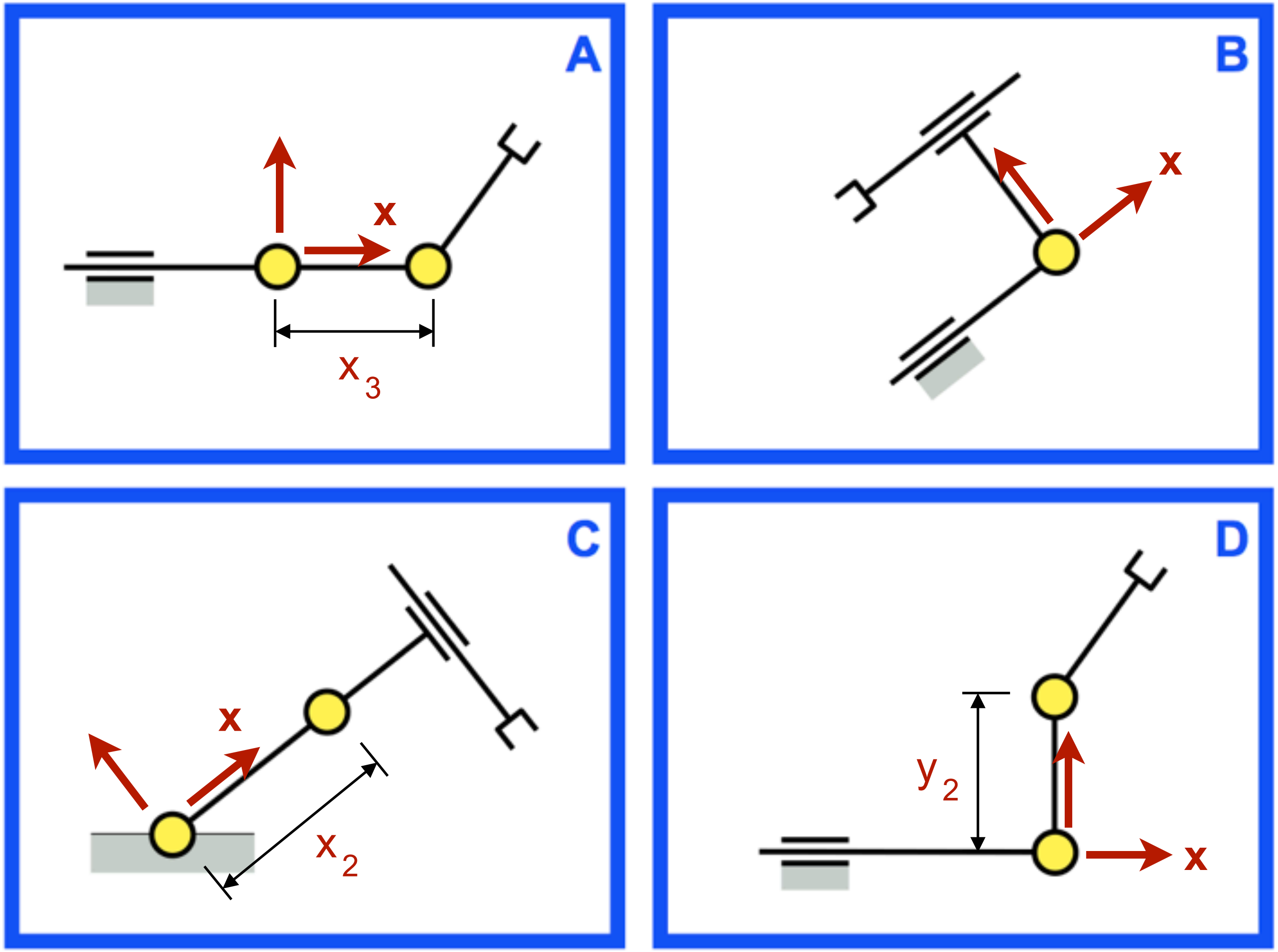
This a singular configuration but ...

Prove that P can move under any velocity

Determine the location of the instant center of the end effector

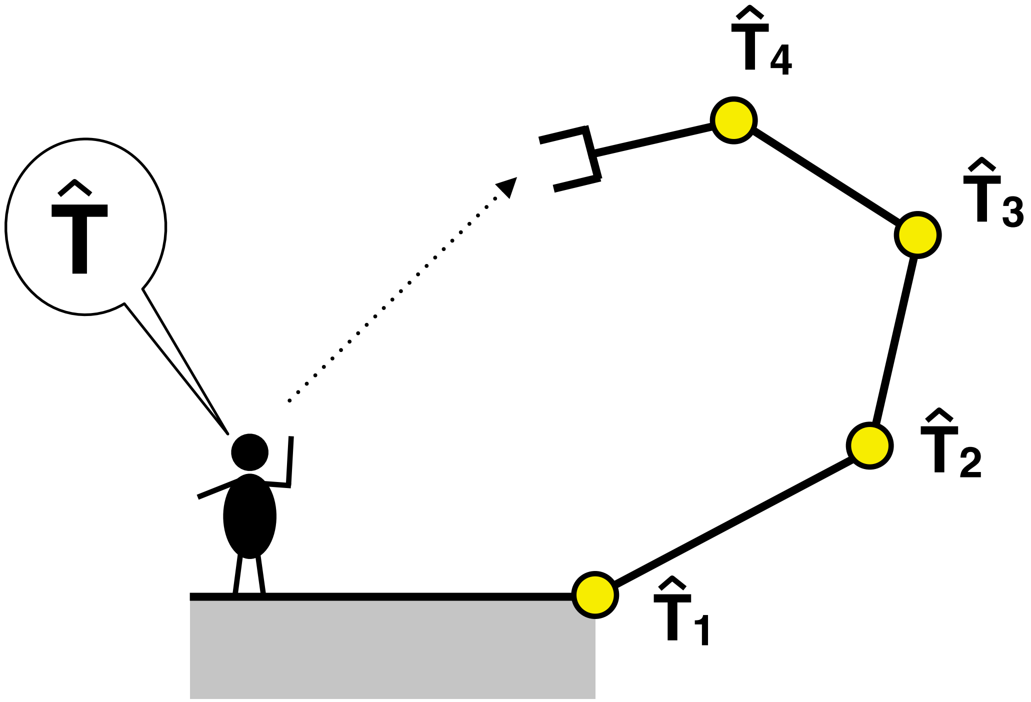
Give an explanation of the apparent paradox

Which of these configurations are singular?



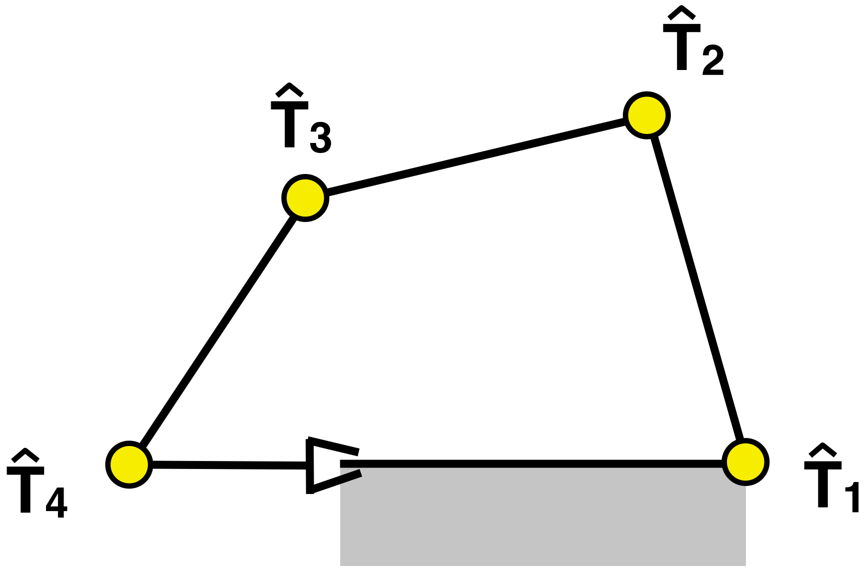
In the singular ones:
how are the instant centers of the relative link twists?

Singularities of a closed kinematic chain



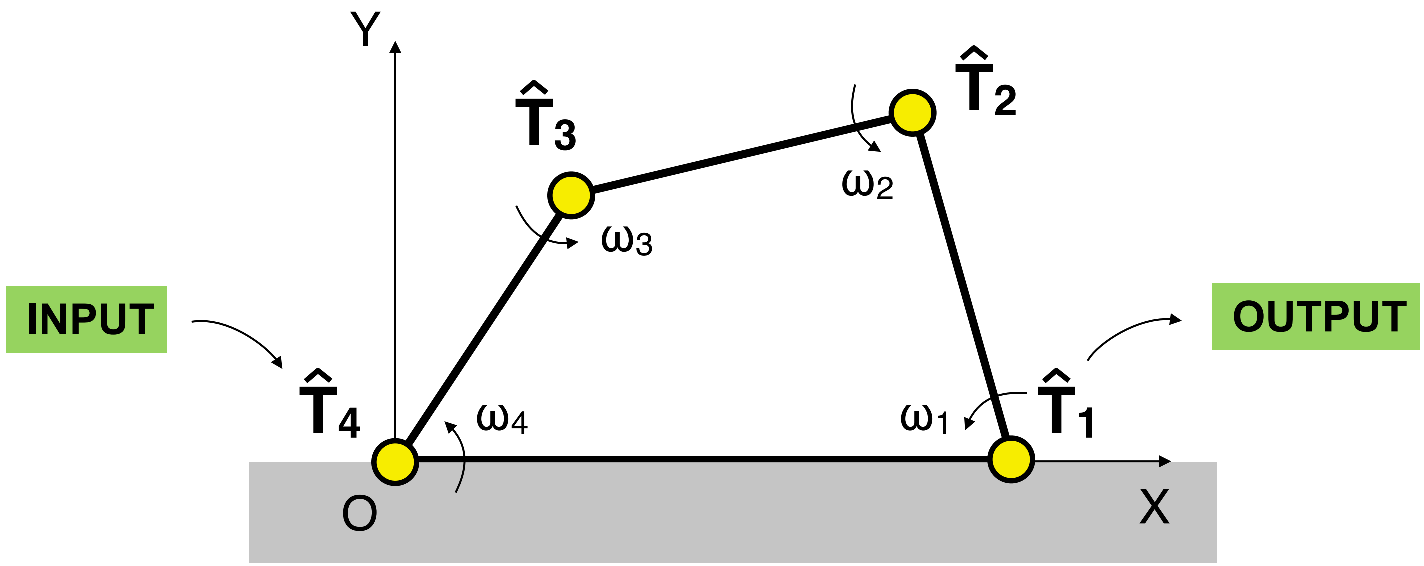
$$\hat{T} = \hat{T}_1 + \hat{T}_2 + \hat{T}_3 + \hat{T}_4$$

Singularities of a closed kinematic chain



$$\hat{T} = \hat{T}_1 + \hat{T}_2 + \hat{T}_3 + \hat{T}_4 = \mathbf{0}$$

Singularities of a closed kinematic chain

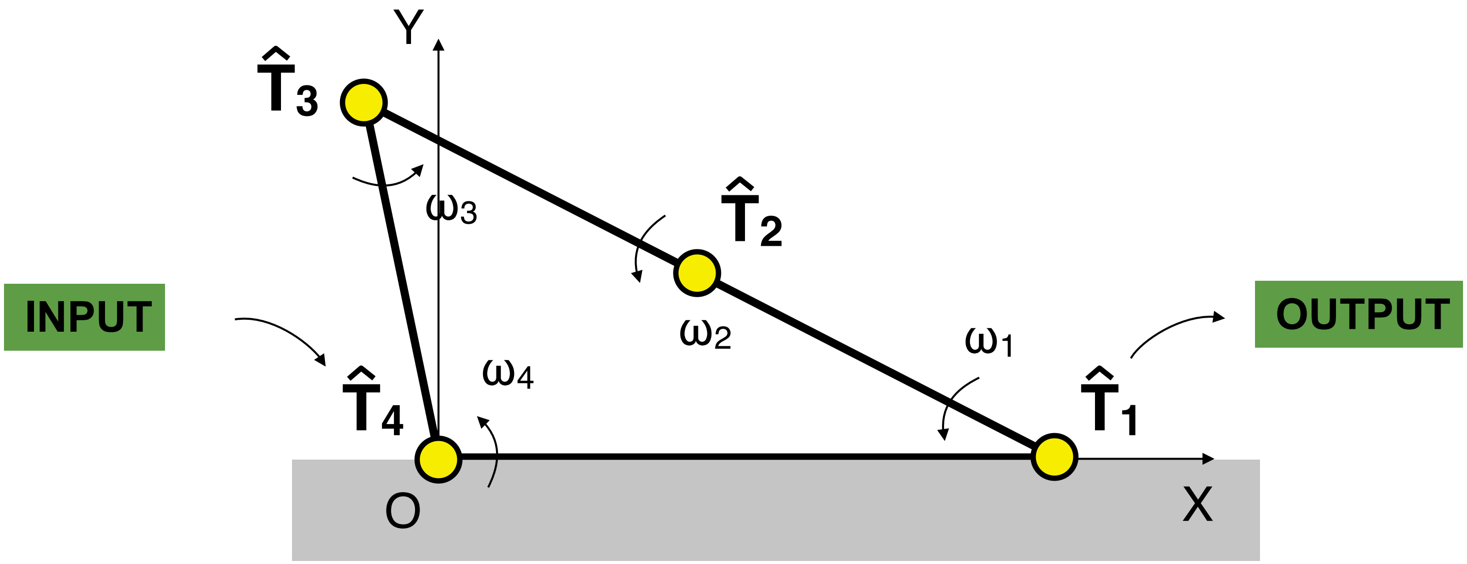


$$\mathbf{0} = \hat{T}_1 + \hat{T}_2 + \hat{T}_3 + \hat{T}_4$$

FIKP: Given the input velocities → compute the whole velocity state

IIKP: Given the output velocities → compute the whole velocity state

Forward singularities



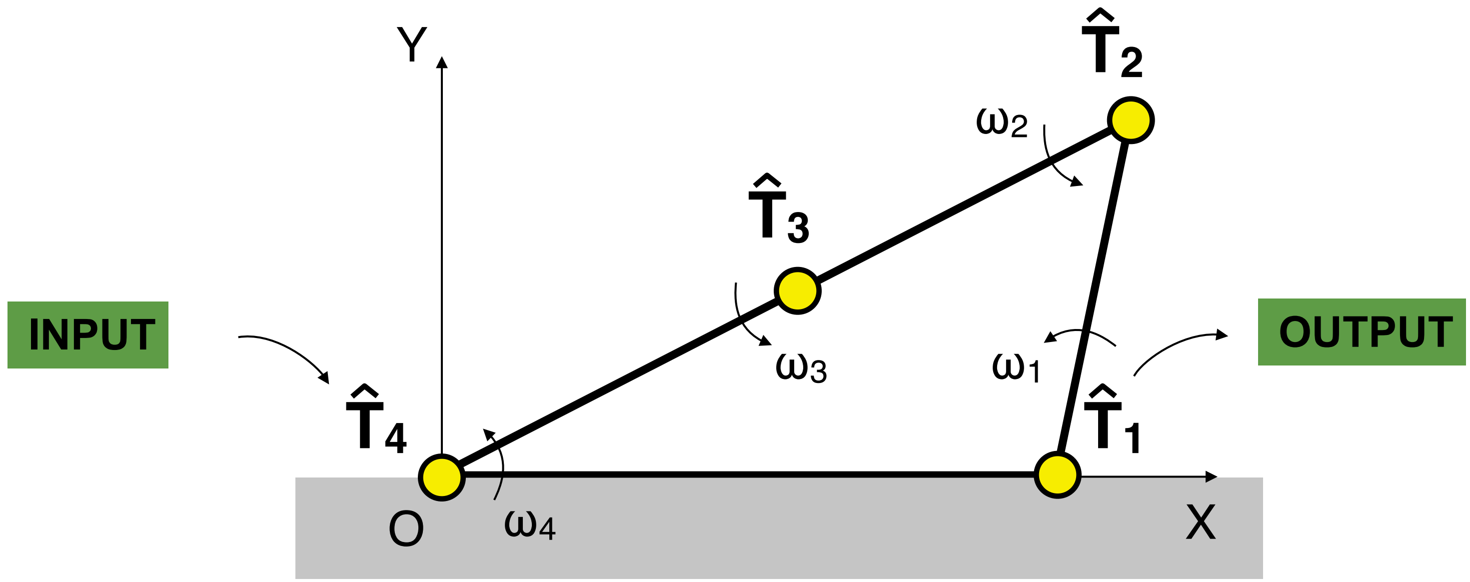
Unsolvable or undetermined

FORWARD inst. kinematic problem

Forward singularities

https://youtu.be/_AIbarxcwkA

Inverse singularities



Unsolvable or undetermined

INVERSE inst. kinematic problem

Almost end of module

An appendix on infinitesimal displacements next week