The Principle of Virtual Power
Slide companion notes

Slide 2 In Modules 2 and 3 we have seen concepts of Statics and Kinematics in a separate way. In this module we shall see how the static and the kinematic analyses fit to each other, thanks to the Principle of Virtual Power, also called "of reciprocity". The coupling is so tight that we often speak of the "kinetostatic" behavior of a mechanism.

Slide 3 We start with a brief reminder of concepts that we have already seen in modules 2 and 3, stressing analogies and providing new insight.

Slide 4 Remember that in Statics we represent the resultant of a system of forces with a wrench $\hat{w}$, which encodes a force $\boldsymbol{f}$ acting on a line r. The wrench is formed by a vector $\boldsymbol{f}$ and its moment with respect to the origin $\boldsymbol{c}_o$. Analogously, in Kinematics we represent the velocity state of a lamina with a twist $\hat{t}$ encoding the angular velocity $\boldsymbol{\omega}$ and its action line m. The twist is formed by a vector $\boldsymbol{\omega}$ and its moment $\boldsymbol{v}_o$ with respect to the origin. The twist and the wrench have the same structure. They are both formed by a vector and its moment w.r.t the origin O.

Slide 5 It will be useful to see that there are two possible interpretations of a wrench $\hat{w} = \{\boldsymbol{f}; \boldsymbol{c}_o\}$. We can think that it encodes a force $\boldsymbol{f}$ applied along the line r defined by the wrench, or that it encodes a force $\boldsymbol{f}$ applied at the origin, plus a couple $\boldsymbol{c}_o$ applied on the lamina. The two force systems are *equivalent.* They have the same resultant force, and the same moment w.r.t the origin (and w.r.t any point).

Slide 6 A twist $\hat{t}$ also has two possible interpretations. So far, we are used to think of $\hat{t}$ as encoding the velocity field generated by an angular velocity $\boldsymbol{\omega}$ about a line m, being $\boldsymbol{v}_o$ the velocity of the origin point of the lamina (left figure). But now recall how we use $\hat{t}$ to compute the velocity $\boldsymbol{v}_P$ of a point P of the lamina:

$$\boldsymbol{v}_P = \boldsymbol{\omega} \times \overrightarrow{OP} + \boldsymbol{v}_o. \tag{1}$$

From this equation it becomes clear that we can equally think of $\hat{t}$ as encoding the sum of *two* velocity fields (right figure):

- A field generated by an angular velocity $\boldsymbol{\omega}$ acting on a line m' parallel to m, through O.
- A constant field of velocity $\boldsymbol{v}_o$.

These double interpretations of the wrench and the twist will be helpful soon, to easily compute the power generated by a wrench, under a given twist.

Slide 7 More analogies. Similarly to the static analysis of the 3-RPR manipulator (in which the platform wrench is a sum leg wrenches), in the kinematic analysis of the 3R robot the end-effector twist is a sum of relative link twists. Note that it remains to perform the kinematic analysis of the 3-RPR manipulator, and the static analysis of the 3R robot. We will perform them soon using the Principle of Virtual Power.

Slide 8 Finally, note that the static singularities of the 3-RPR manipulator arise when three lines are concurrent, whereas the kinematic singularities of the 3R robot appear when the three joints are collinear. In Projective Geometry, the concurrence condition of three lines is dual to the alignment condition of three points. The Principle of Virtual Power will also allow us to make the analysis of the kinematic and static singularities of the 3-RPR and 3R robots, respectively.

Slide 9 Let's see this principle. We will formulate it first for a *particle*, then for a *rigid body*, and finally for a multibody system, i.e., a *mechanism*.

Slide 10 Remember that in Mechanics the power generated by a force $\boldsymbol{f}$ applied at a particle with velocity $\boldsymbol{v}$ is defined as the product

$$P = \boldsymbol{f} \cdot \boldsymbol{v}.$$

P measures the kinetic energy that $\boldsymbol{f}$ transmits to the particle per unit of time. It is therefore a scalar quantity with units of power (Nm/s = J/s = W, in SI). We have:

- $P > 0 \Rightarrow$ the particle accelerates.
- $P = 0 \Rightarrow$ the particle maintains its velocity.
- $P < 0 \Rightarrow$ the particle slows down.

Slide 11 In the case of a unique particle, the principle says that the particle will be in equilibrium if, and only if, the power generated by the external forces acting on it is null **under any possible velocity of the particle**.

Proof:

> ($\Rightarrow$) Since the particle is in equilibrium, it must be $\boldsymbol{r} = \sum_{i=1}^{n} \boldsymbol{f}_i = \boldsymbol{0}$, and therefore it will be
>
> $$\boldsymbol{r}\boldsymbol{v} = 0, \forall \boldsymbol{v}.$$
>
> ($\Leftarrow$) If $\boldsymbol{r}\boldsymbol{v} = 0\ \forall \boldsymbol{v}$, in particular it has to be $\boldsymbol{r}\boldsymbol{v} = 0$ when $||\boldsymbol{v}|| \neq 0$, and this implies that $\boldsymbol{r} = 0$. In other words, the particle has to be in equilibrium.

The power is said to be "virtual" because there is no need for the particle to actually *move* to apply the principle. It only needs to be in equilibrium, either moving, or at rest. The principle of virtual power, hence, is more regarded as an analytical mathematical tool in itself, useful to compute forces in equilibrium conditions. We will illustrate the tool with examples by the end of these slides.

Slide 12 The power of a force $\boldsymbol{f}$ applied to a rigid body on a line r is defined as the product

$$P = \boldsymbol{f} \cdot \boldsymbol{v}_Q,$$

where $\boldsymbol{v}_Q$ is the velocity of any point Q of r.

The value of P does not depend on the specific point Q of r we choose. To prove this, we see that the power obtained at a point Q equals the power obtained at any other point $Q' \neq Q$ of r.

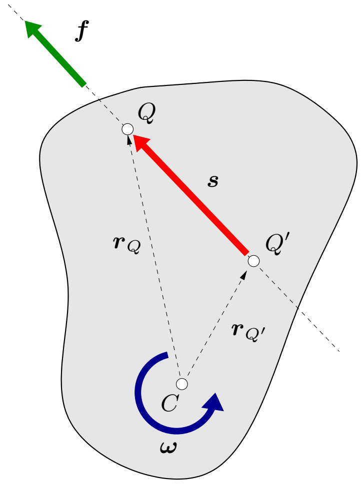
If we define:

- C as the instantaneous center of rotation of the body.
- $\boldsymbol{r}_Q = Q - C$.
- $\boldsymbol{r}_{Q'} = Q' - C$.
- $\boldsymbol{s} = Q - Q'$.

Then:

$$\begin{aligned} P = \boldsymbol{f} \cdot \boldsymbol{v}_Q &= \boldsymbol{f} \cdot (\boldsymbol{\omega} \times \boldsymbol{r}_Q) = \\ &= \boldsymbol{f} \cdot (\boldsymbol{\omega} \times (\boldsymbol{r}_{Q'} + \boldsymbol{s})) = \\ &= \boldsymbol{f} \cdot (\boldsymbol{\omega} \times \boldsymbol{r}_{Q'}) + \boldsymbol{f} \cdot (\boldsymbol{\omega} \times \boldsymbol{s}) \\ &= \boldsymbol{f} \cdot \boldsymbol{v}_{Q'} \end{aligned}$$

because $\boldsymbol{\omega} \times \boldsymbol{s}$ is orthogonal to $\boldsymbol{f}$:



Slide 13 As in the case of a particle, P measures the kinetic energy transmitted by the force to the body. The example shows different situations where P is positive, negative, or zero.

Slide 14 The power of a couple $\boldsymbol{\pi}$ applied on a rigid body is defined as the product

$$P = \boldsymbol{\pi} \cdot \boldsymbol{\omega},$$

where $\boldsymbol{\omega}$ is the angular velocity of the body.

This definition is consistent with the one given in Slide 12 for forces. If we see the couple $\boldsymbol{\pi}$ as the resultant of a system of two antiparallel forces $-\boldsymbol{f}$ and $\boldsymbol{f}$ respectively applied at two points A and B of the body, where $\boldsymbol{r} = B - A$ makes $\boldsymbol{r} \times \boldsymbol{f} = \boldsymbol{\pi}$, it is easy to see that the power generated by these forces has to be $P = \boldsymbol{\pi} \cdot \boldsymbol{\omega}$. Indeed, if C is the instantaneous centre of rotation, and $\boldsymbol{r}_A = A - C$, $\boldsymbol{r}_B = B - C$, $\boldsymbol{r} = B - A$, we have:

$$\begin{aligned} P &= -\boldsymbol{f} \cdot \boldsymbol{v}_A + \boldsymbol{f} \cdot \boldsymbol{v}_B = \\ &= -\boldsymbol{f} \cdot (\boldsymbol{\omega} \times \boldsymbol{r}_A) + \boldsymbol{f} \cdot (\boldsymbol{\omega} \times \boldsymbol{r}_B) = \\ &= \boldsymbol{f} \cdot (\boldsymbol{\omega} \times (\boldsymbol{r}_B - \boldsymbol{r}_A)) \\ &= \boldsymbol{f} \cdot (\boldsymbol{\omega} \times \boldsymbol{r}) = \\ &= \boldsymbol{\omega} \cdot (\boldsymbol{r} \times \boldsymbol{f}) = \\ &= \boldsymbol{\omega} \cdot \boldsymbol{\pi} \end{aligned}$$

Slide 15 Now we have the necessary elements to compute the power generated by a wrench $\hat{w} = \{\boldsymbol{f}; \boldsymbol{c}_o\}$ on a body moving under a twist $\hat{t} = \{\boldsymbol{\omega}; \boldsymbol{v}_o\}$. The wrench can be seen as a force $\boldsymbol{f}$ applied at the origin, plus a couple $\boldsymbol{c}_o$ orthogonal to the lamina. Since the velocity of the origin is $\boldsymbol{v}_o$, and the angular velocity of the body is $\boldsymbol{\omega}$, the power generated by $\boldsymbol{f}$ and $\boldsymbol{c}_o$ is

$$P = \boldsymbol{f} \cdot \boldsymbol{v}_o + \boldsymbol{c}_o \cdot \boldsymbol{\omega} = \hat{w}^{\mathsf{T}} \cdot \hat{T},$$

where $\hat{T} = \{\boldsymbol{v}_0; \boldsymbol{\omega}\}$, i.e., it is $\hat{t}$ written in axis coordinates. Now we see the interest of using ray coordinates for wrenches, and axis coordinates for twists. They allow to compute P as the product $\hat{w}^{\mathsf{T}} \cdot \hat{T}$. When $\hat{w}^{\mathsf{T}} \cdot \hat{T} = 0$ one says that $\hat{w}$ and $\hat{T}$ are **reciprocal**. This means that, seen as abstract vectors, $\hat{w}$ and $\hat{T}$ are orthogonal.

Slide 16 Taking into account that

$$\begin{aligned} \hat{w} &= f\hat{s} \\ \hat{T} &= \omega\hat{S} \end{aligned}$$

where $\hat{s}$ and $\hat{S}$ are the unit coordinates of the respective lines, and f and ω the corresponding signed magnitudes, we can write

$$P = f\omega \cdot \hat{s}^{\mathsf{T}}\hat{S}.$$

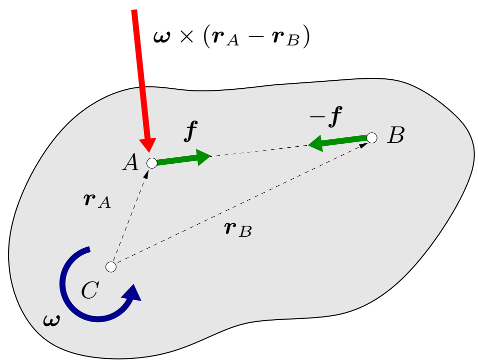
The product $\hat{s}^{\mathsf{T}}\hat{S}$ is called the *mutual moment* of the mentioned lines and is equal to the signed distance r between them (see Duffy's book for a proof). If $\boldsymbol{s}$ and $\boldsymbol{S}$ are, respectively, the unit vectors corresponding to $\hat{s}$ and $\hat{S}$, then $\hat{s}^{\mathsf{T}}\hat{S}$ is positive if $\overrightarrow{PQ} \times \boldsymbol{S}$ points in the same direction than $\boldsymbol{s}$.

An important conclusion is that $P = 0$ if, and only if, the lines of $\hat{w}$ and $\hat{T}$ intersect. This fact will shortly be used in the kinetostatic analyses of forthcoming robots.

Note that the power P is invariant to changes in the coordinate system. Independently of the system chosen to express $\hat{w}$ and $\hat{T}$, its product P will always be the same.

Slide 17 We have shown that the Principle of Virtual Power is true for a particle, but it is also true for rigid bodies, as these can be seen as aggregates of particles. In the case of a rigid body, however, it is only necessary to count in P the power of all forces and couples applied from the outside on the body, as the power generated by the interior forces is globally null.

Indeed, let $\boldsymbol{f}$ be the force that a particle B exerts on another particle A. This force is the resultant of the gravitational and electrostatic forces. By the action-reaction principle, the force that the particle A will exert on B is $-\boldsymbol{f}$.



If $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$ are the position vectors of A and B w.r.t the instant center of rotation C, the power of the two forces is

$$
\begin{aligned}
P &= \boldsymbol{f}\cdot\boldsymbol{v}_A - \boldsymbol{f}\cdot\boldsymbol{v}_B = \\
&= \boldsymbol{f}\cdot(\boldsymbol{\omega}\times\boldsymbol{r}_A) - \boldsymbol{f}\cdot(\boldsymbol{\omega}\times\boldsymbol{r}_B) \\
&= \boldsymbol{f}\cdot(\boldsymbol{\omega}\times(\boldsymbol{r}_A-\boldsymbol{r}_B)) = 0
\end{aligned}
$$

because $\boldsymbol{\omega}\times(\boldsymbol{r}_A-\boldsymbol{r}_B)$ is orthogonal to $\boldsymbol{f}$:

Slide 18 The principle is also true for a mechanism. The mechanism is in equilibrium if, and only if, the power generated by all forces and couples applied from the outside is null, under any velocity state of the mechanism **compatible with its joint assembly constraints**.

Note that the principle holds for any one of the bodies separately, and this means that it will hold for the global set of bodies. When we compute the total balance of power, however, the contribution of the connector forces is globally null. This can be seen clearly in the 4-bar mechanism of the slide, subject to three forces applied from the outside, one at each bar (left figure). If we analize the forces that each body receives separately (right figure) we see that the global force balance at the joints is null. The reactions of the ground do not generate power, because they are applied at points that do not move. The forces at the mobile joints generate also a null sum of power, because if $\boldsymbol{f}$ is the force that bar i exerts on a neighbor bar j, the force that j will exert on i will be $-\boldsymbol{f}$. As we only consider motions compatible with the joints, $\boldsymbol{f}$ and $-\boldsymbol{f}$ are applied at the same point with a given velocity $\boldsymbol{v}$. The contribution of these two forces will therefore be $\boldsymbol{f}\boldsymbol{v} + (-\boldsymbol{f})\boldsymbol{v} = 0$.

Slides 19 and 20 In the remaining slides we shall illustrate how the principle of virtual power allows us to determine

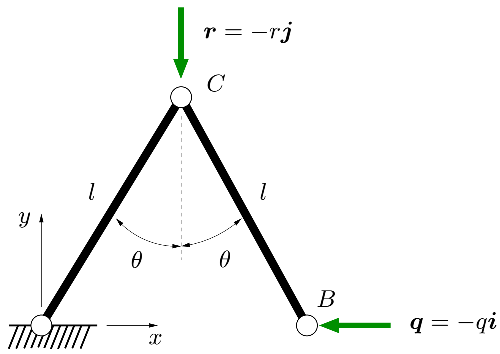
- The unknown forces in input/output problems.
- The forces that can be structurally supported by the end-effector of a robotic mechanism.

By a "structurally supportable" force we mean one that the mechanism can equilibrate without having to actuate any one of its joints, under the assumption that the mass of the links is negligible.

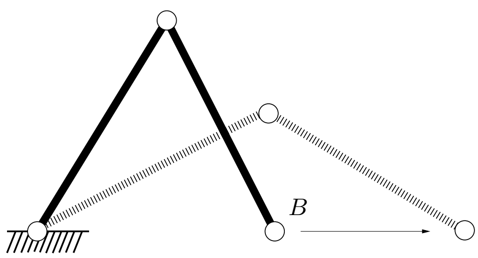
The green force acting on the shown robot arm is structurally supportable, because its moment relative to the joints is null. No torque at the joints is needed to counteract it. Accordingly, a climber can withstand its weight with relaitve ease in the shown position, because his arm is fully extended!

Anticipating such forces is useful, not only because the robot consumes less power when counteracting them, but also because they are associated, as we shall see soon, with the presence of singularities.

Slide 21 Let us see an example of application of the principle to the determination of unknown forces in input/output problems. The figure displays a 3R mechanism with negligible mass, used to compress a wooden block (brown) by means of a piston (grey). The piston maintains a frictionless planar contact with the ground. Given the force $\boldsymbol{r} = -r\boldsymbol{j}$ applied at C, we want to know the compression force applied on the wooden block. The problem boils down to computing the reaction $\boldsymbol{q} = -q\boldsymbol{i}$ that the block will exert against the piston:



Let us now consider this movement



which is compatible with the assembly constraints of the mechanism, and compute the power P generated by all of the externally-applied forces. The positions of B and C as a function of θ are

$$(x_B, y_B) = (2l \sin\theta, 0),$$
$$(x_C, y_C) = (l \sin\theta, l \cos\theta),$$

so that the velocities of B and C are

$$\boldsymbol{v}_B = (\dot{x}_B, \dot{y}_B) = (2l\dot{\theta} \cos\theta, 0),$$
$$\boldsymbol{v}_C = (\dot{x}_C, \dot{y}_C) = (l\dot{\theta} \cos\theta, -l\dot{\theta} \sin\theta),$$

and thus

$$\begin{aligned} P &= \boldsymbol{q} \cdot \boldsymbol{v}_B + \boldsymbol{r} \cdot \boldsymbol{v}_C = \\ &= rl\dot{\theta} \sin\theta - 2ql\dot{\theta} \cos\theta. \end{aligned}$$

Since the mechanism is in equilibrium, $P = 0$,

$$rl\dot{\theta} \sin\theta - 2ql\dot{\theta} \cos\theta = 0,$$

and we obtain that the relation between r and q is

$$q = \frac{r}{2} \tan\theta.$$

Slide 22 Now, let us derive a condition characterizing the structurally supportable end-effector wrenches of an arbitrary robotic mechanism. For concreteness the slide shows a 3R robotic arm, but the analysis applies to *any* mechanism. Just substitute the interior of the dashed line for your preferred one.

Let us apply the Principle of Virtual Power to the mechanism. We break the connection with the ground, substituting its action by a wrench $\hat{w}_g$. Since we want to compute the structurally supportable forces, we assume that no joint is actuated, so the only wrenches acting from the outside are $\hat{w}_g$ on the base link, and $\hat{w}_e$ on the end effector. By the Principle of Virtual Power we have

$$P = \hat{w}_g{}^{\mathsf{T}} \cdot \hat{T}_g + \hat{w}_e{}^{\mathsf{T}} \cdot \hat{T}_e = 0$$

where $\hat{T}_g$ is the twist of the base link, and $\hat{T}_e$ is the end-effector twist. The previous equation must hold for any twist $\hat{T}_e$ compatible with the joint assembly constraints of the robot. These twists are also called the **twists of freedom** of the end effector.

Since $\hat{T}_g = 0$,

$$P = \hat{w}_e{}^{\mathsf{T}} \cdot \hat{T}_e = 0$$

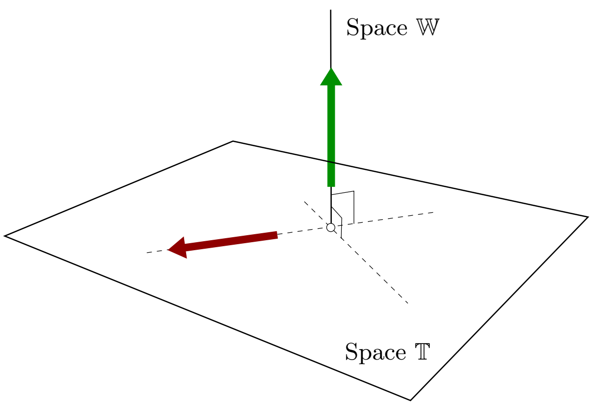
for all twists of freedom $\hat{T}_e$. Therefore, the structurally supportable wrenches $\hat{w}_e$ are those that are reciprocal **to all** twists of freedom $\hat{T}_e$ of the end effector (recall slide 15 for a definition of the term "reciprocal"). The structurally supportable wrenches are called the **wrenches of constraint** of the end-effector.

Slide 23 The space $\mathbb{T}$ of twists of freedom, and the space $\mathbb{W}$ of wrenches of constraint are vector spaces, and they are reciprocal complements from one another, since

$$\hat{w}_e{}^{\mathsf{T}} \cdot \hat{T}_e = 0, \qquad \forall \hat{w}_e \in \mathbb{W}, \qquad \forall \hat{T}_e \in \mathbb{W},$$
$$\mathrm{Dim}(\mathbb{T}) + \mathrm{Dim}(\mathbb{W}) = 3$$

That is, seen as abstract vector spaces, $\mathbb{T}$ and $\mathbb{W}$ are orthogonal complements from one another. Strictly speaking, we cannot use the word "orthogonal", because the vectors in $\mathbb{T}$ and $\mathbb{W}$ are of a different physical nature.

As this figure shows



any vector of $\mathbb{T}$ is orthogonal to any vector of $\mathbb{W}$, and the dimensions of $\mathbb{T}$ and $\mathbb{W}$ have to add up to the dimension of the whole space $\mathbb{R}^3$:

$$\dim(\mathbb{T}) + \dim(\mathbb{W}) = 3.$$

Slide 24 This is an exercise for you to practice. Provide the spaces of twists of freedom and wrenches of constraint of the shown robots. I.e., give a basis of such spaces, in the indicated coordinate systems.