Module 4

Series-parallel dualities

Kinetostatics = kinematics + statics

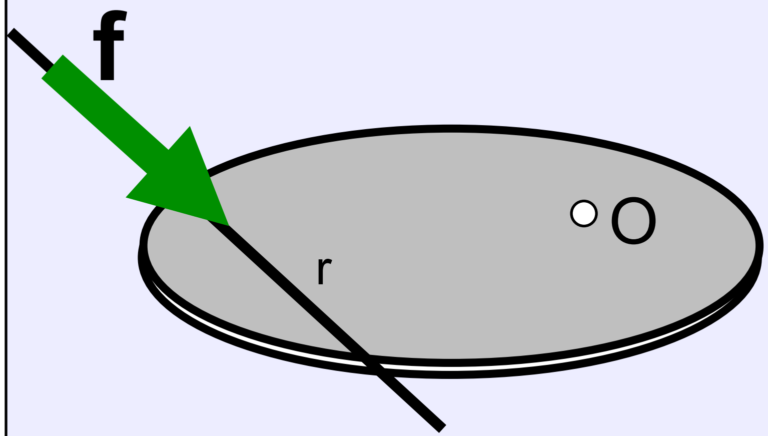
Fit thanks to the **principle of reciprocity**
(a.k.a. the principle of **virtual power**)

Overview of kinetostatic analogies

and new insights

Analogies statics - kinematics

Wrench

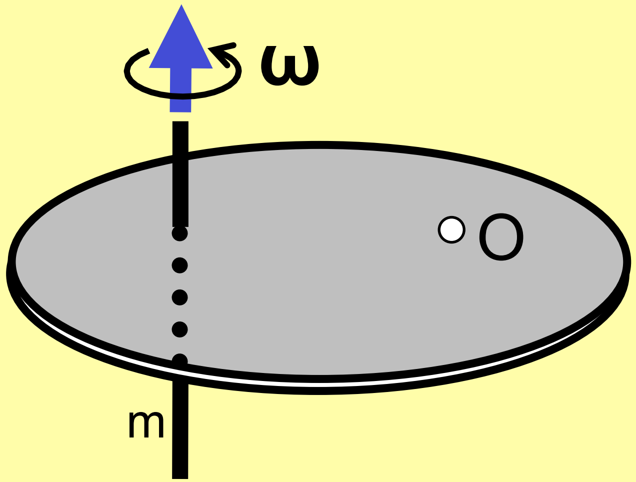


vector   moment w.r.t. O

$$\hat{w} = \{ \mathbf{f}; \mathbf{c}_o \}$$

$\hat{w}$ encodes a vector **f** sliding on a line **r**. Represents the resultant of a system of forces

Twist



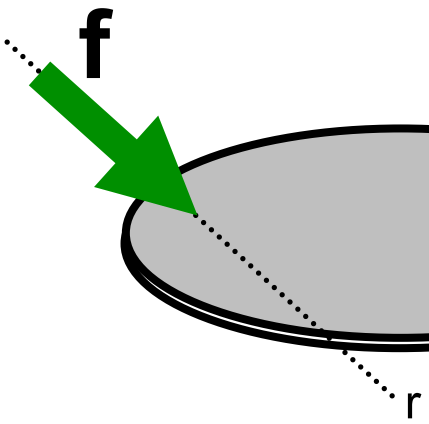
vector   moment w.r.t. O

$$\hat{t} = \{ \boldsymbol{\omega}; \mathbf{v}_o \}$$

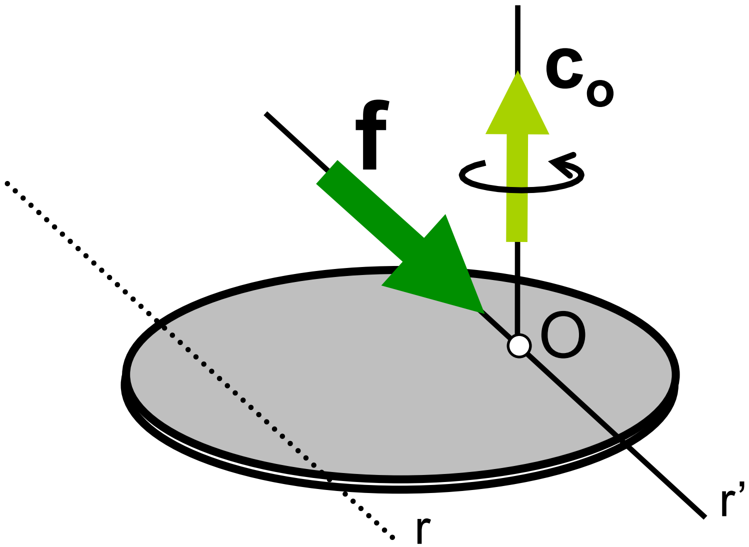
$\hat{t}$ encodes a vector **ω** sliding on a line **m**. Represents the velocity state

Two interpretations of a **wrench**

$$\hat{w} = \{ \mathbf{f}; \mathbf{c}_o \}$$



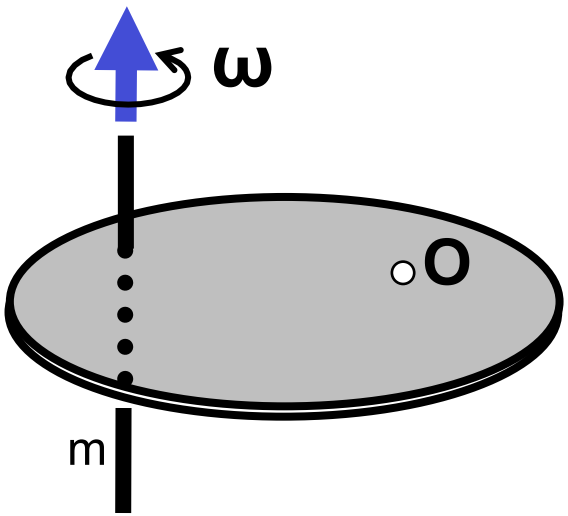
$\hat{w}$ encodes a vector **f** sliding on a line **r**. Represents the resultant of a system of forces



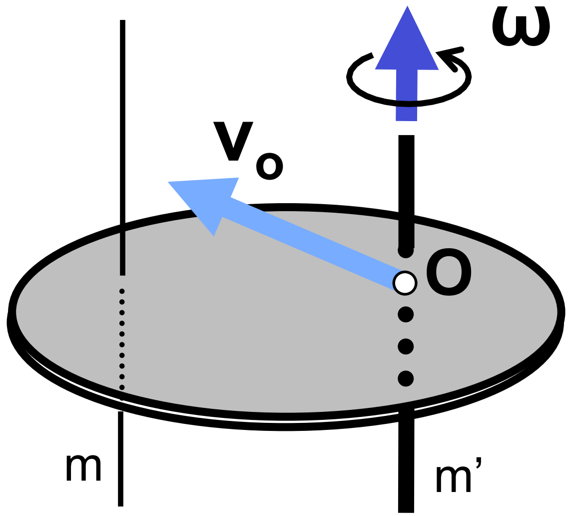
$\hat{w}$ encodes a force applied on a line r' paral·lel to r, through O, plus a compensation couple $\mathbf{c}_o$

Two interpretations of a **twist**

$$\hat{t} = \{\, \boldsymbol{\omega};\ \mathbf{v}_o \,\}$$



$\hat{t}$ encodes the velocity field of an angular velocity about line m

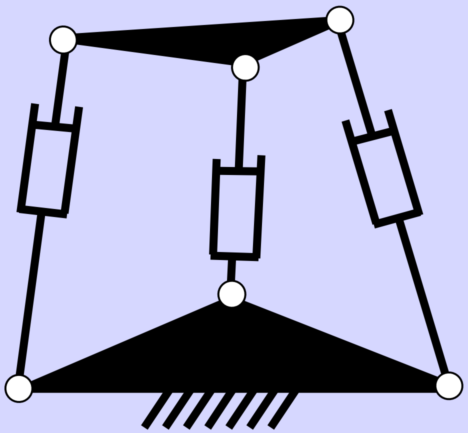


$\hat{t}$ encodes the sum of two fields:

- The field of an ang. velocity about a line m' parallel to m, through O
- A constant field of velocity $\mathbf{v}_o$

More analogies…

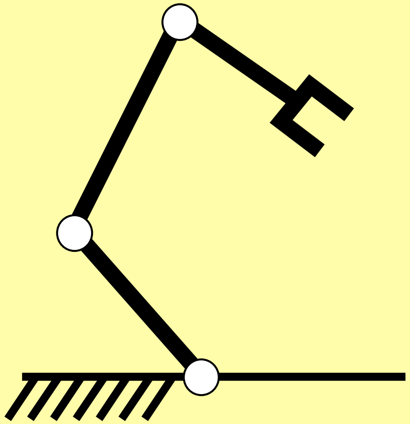
Static analysis of the 3RPR



$$\hat{w} = f_1\,\hat{s}_1 + f_2\,\hat{s}_2 + f_3\,\hat{s}_3$$

$$\hat{w} = \boldsymbol{j}\,\boldsymbol{\lambda}$$

Kinematic analysis of the 3R



$$\hat{T} = \omega_1\,\hat{S}_1 + \omega_2\,\hat{S}_2 + \omega_3\,\hat{S}_3$$

$$\hat{T} = \boldsymbol{J}\,\boldsymbol{\gamma}$$

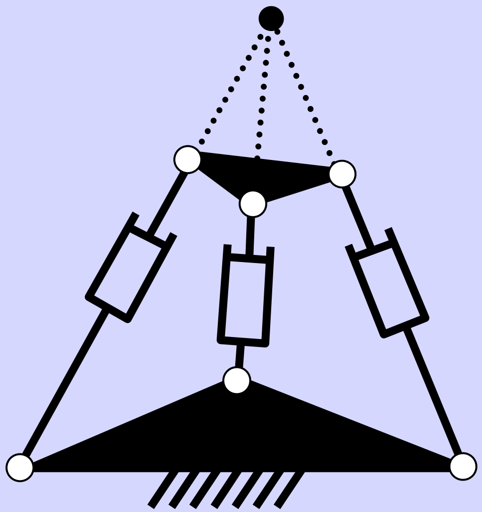
Pending: kinematic analysis

Pending: static analysis

With the principle of virtual power

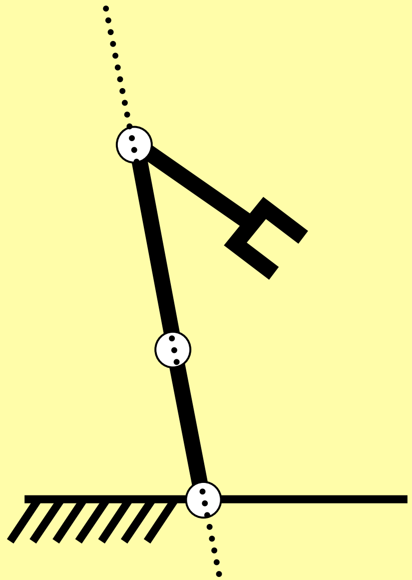
and more …

Static singularities of the 3-RPR



det j = 0

Kinematic singularities of the 3R



det J = 0

Pending: kinematic singularities

Pending: static singularities

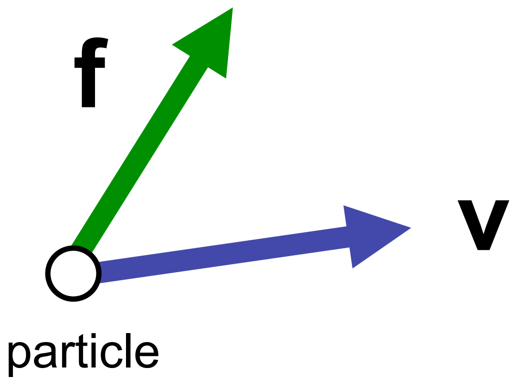
With the principle of virtual power

Principle of virtual power

- Particle
- Rigid body
- Mechanism

Power of a force applied to a particle

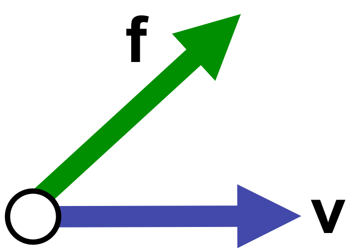
(w.r.t. a velocity)



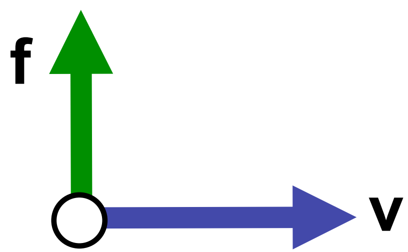
$$P = \mathbf{f} \cdot \mathbf{v}$$

- P is a scalar magnitude
- Units of power: Nm/s = J/s = W
- Measures the kinetic energy transferred to the particle per unit of time
- If **f** orthogonal to **v**, then P = 0

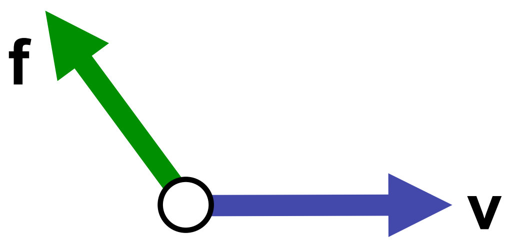
Examples where the particle …



… accelerates



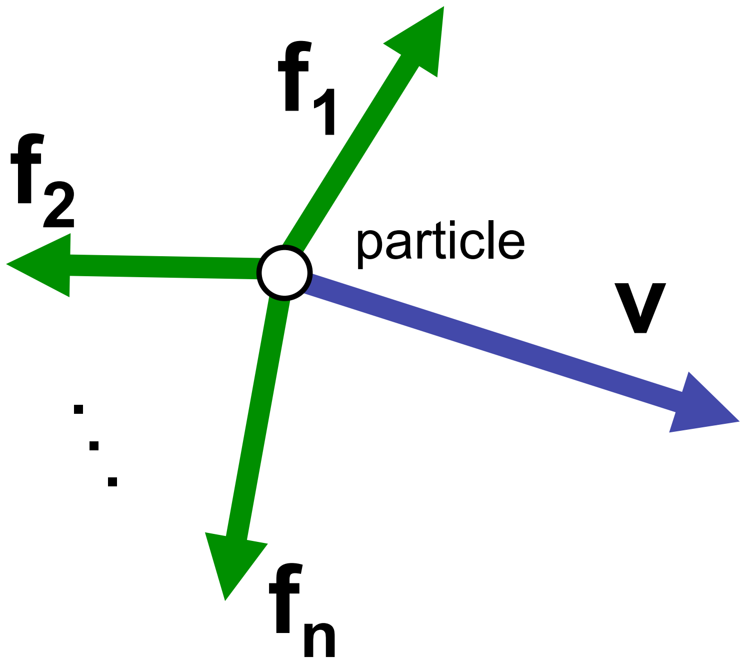
... keeps its speed



… slows down

Principle of virtual power for a **particle**

"A particle is in equilibrium **if, and only if,**
the power generated by all external forces applied to it
is null, under **any possible velocity** for the particle"



Power generated by a force applied to a rigid body

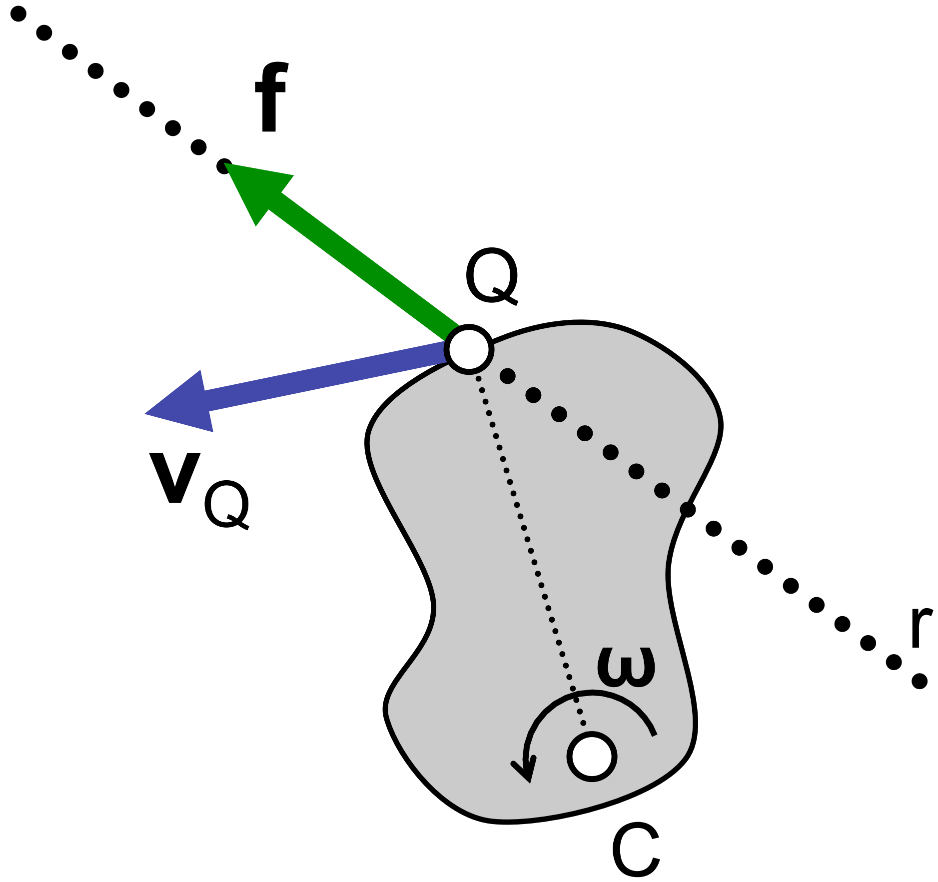
It is defined as

$$P = \mathbf{f} \cdot \mathbf{v}_Q$$

where Q is any point of the action line of **f**

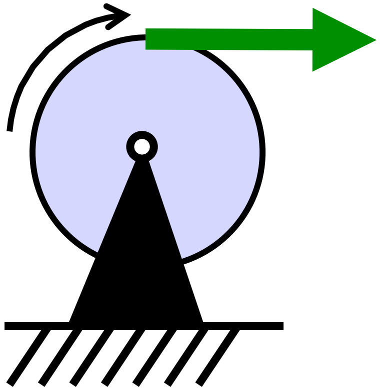
Measures kinetic energy transferred to the body

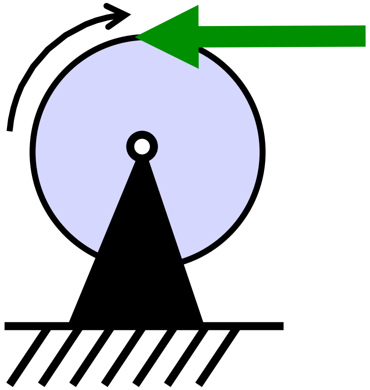
P is invariant to the chosen point Q



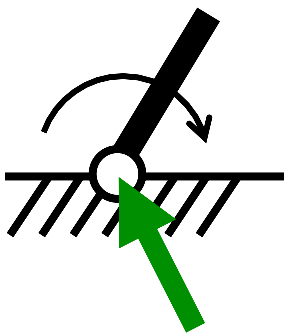
Examples

… in which the force accelerates (P>0) or slows down (P<0) the body:

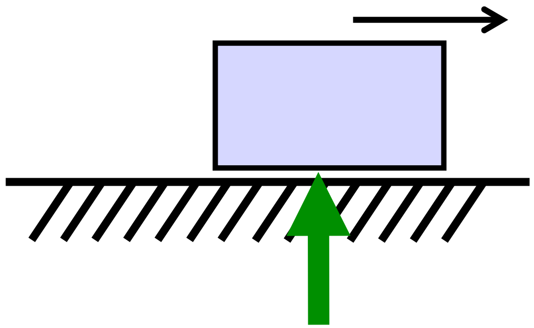




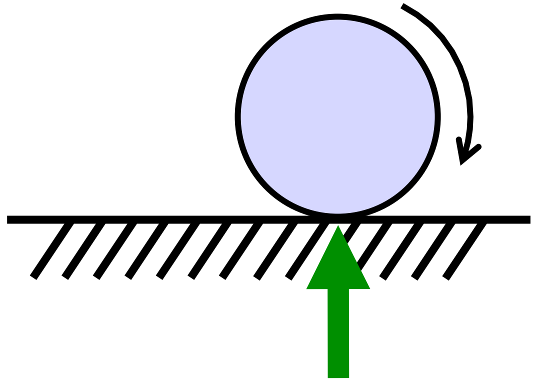
… in which the force can neither accelerate nor slow down the body (P=0):



Reaction in an
R joint

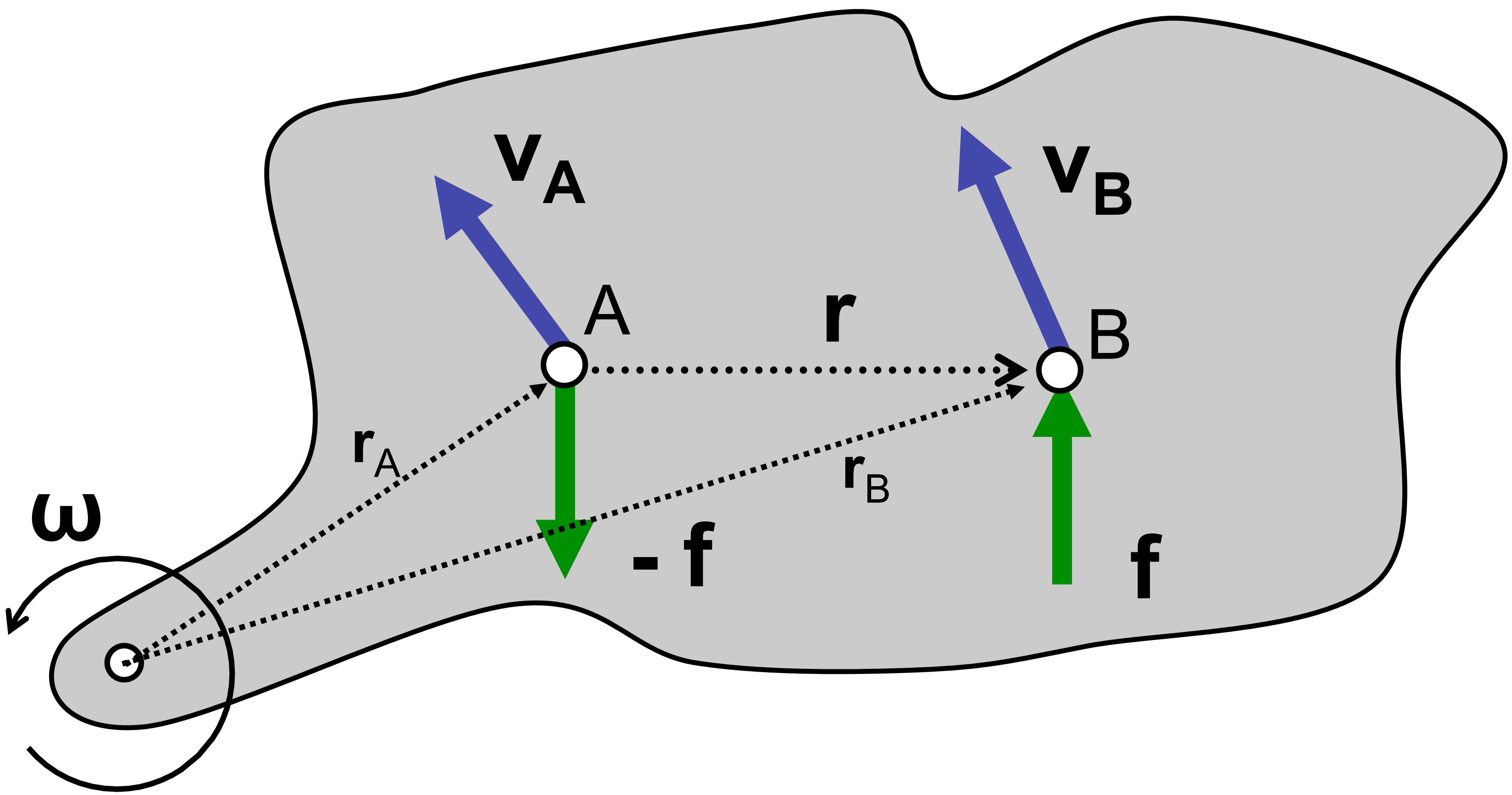


Reaction in a sliding
contact without friction



Reaction in a
rolling contact

Power of a couple applied to a rigid body

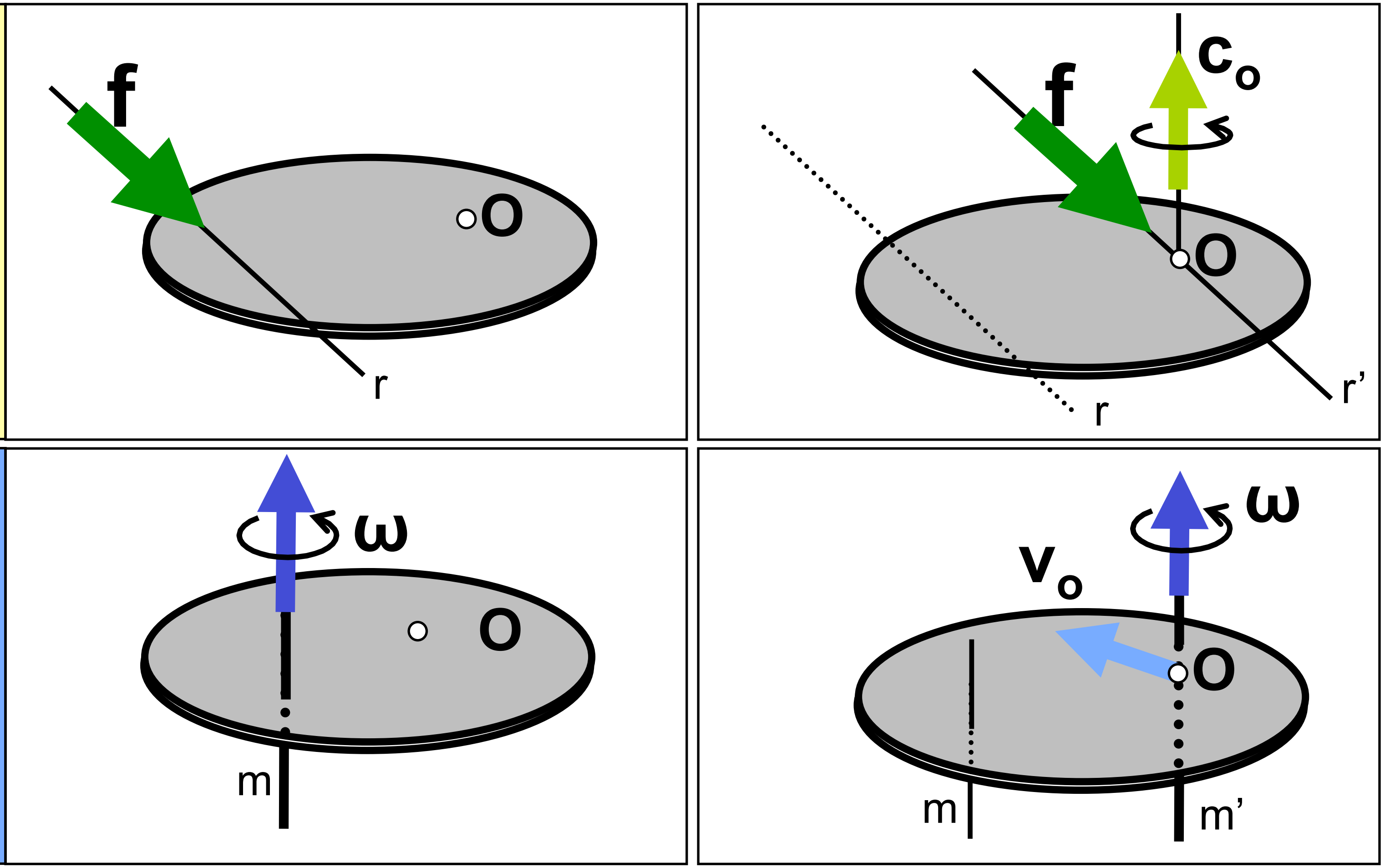


$$P = \boldsymbol{\omega} \cdot (\mathbf{r} \times \mathbf{f}) = \boldsymbol{\omega} \cdot \boldsymbol{\pi}$$

Power of a wrench under a given twist

$\hat{w} = \{\, \mathbf{f};\ \mathbf{c}_o \,\}$

$\hat{t} = \{\, \boldsymbol{\omega};\ \mathbf{v}_o \,\}$



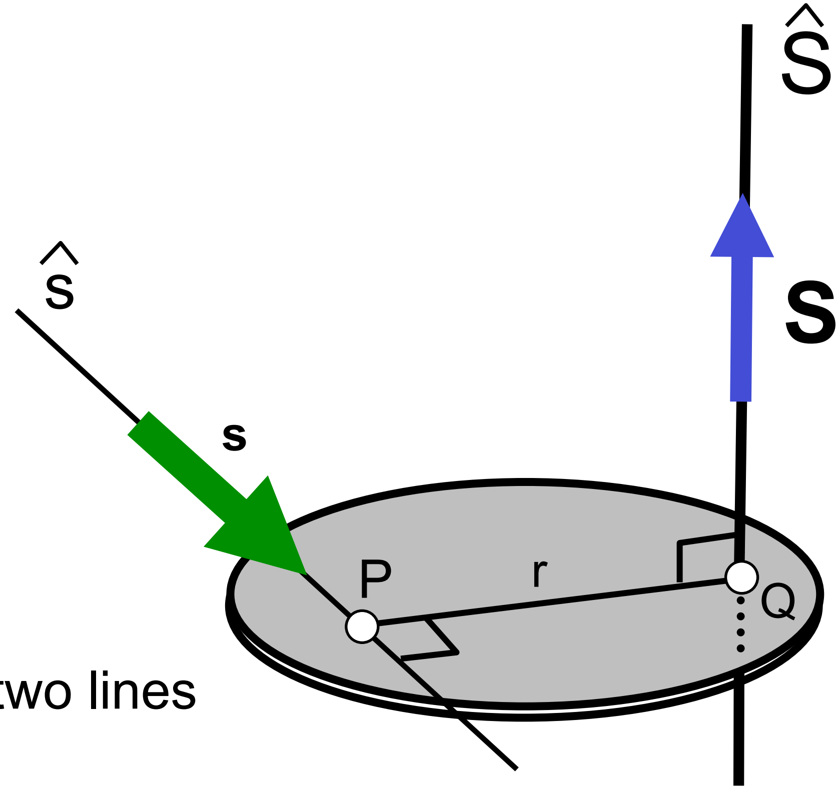
$$P = \mathbf{f} \cdot \mathbf{v}_o + \mathbf{c}_o \cdot \boldsymbol{\omega} \;\rightarrow\; P = \hat{W}^T \cdot \hat{T}$$

Geometric interpretation of the power using the "mutual moment"

$$P = \hat{w}^T \cdot \hat{T}$$

$$P = (f\ \hat{s}^T)\cdot(\omega\ \hat{S})$$

$$P = f\ \omega\ \hat{s}^T \cdot \hat{S}$$



Mutual moment of the two lines

=

signed distance r between the lines

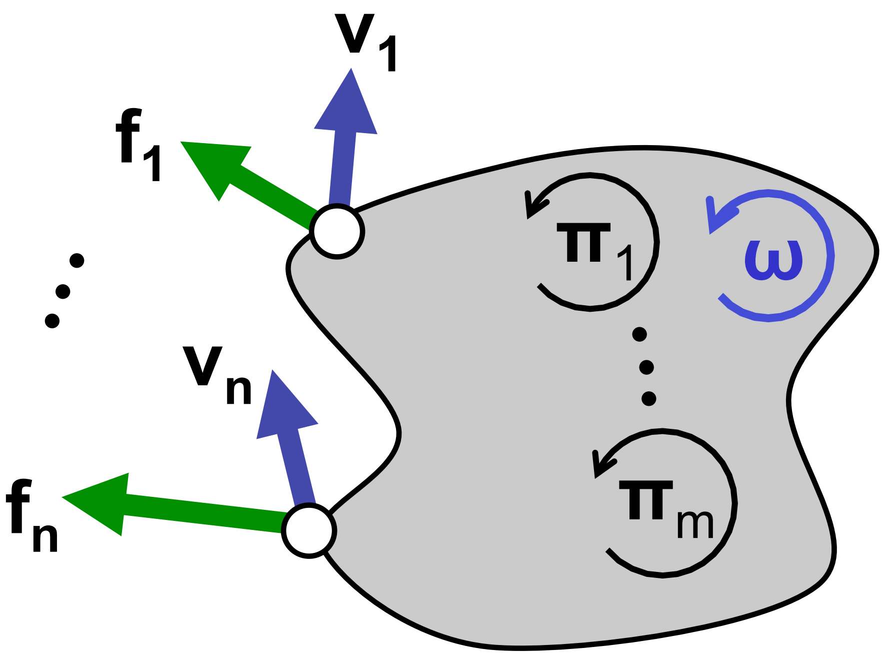
(**+** if $\overrightarrow{PQ}$ x **S** points in the same direction than **s**)

Thus, P = 0 if, and only if, the two lines $\hat{s}$ and $\hat{S}$ intersect

Principle of virtual power for a **rigid body**

"A rigid body is in equilibrium **if, and only if,** the power generated by all externally-applied forces and couples is null, **under any feasible velocity for the body**"

$$P = f_1 v_1 + \ldots + f_n v_n + \pi_1 \omega + \ldots + \pi_m \omega = 0$$



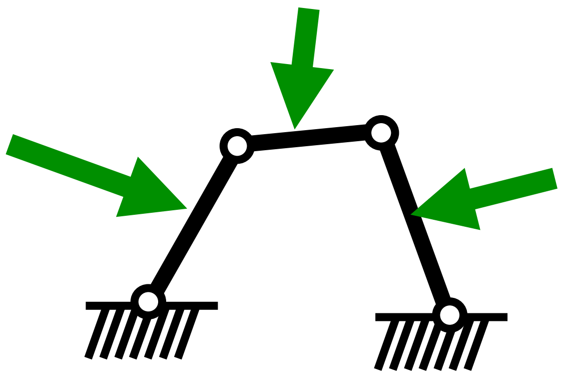
Only the power of the externally-applied forces and couples needs to be counted.

Why?

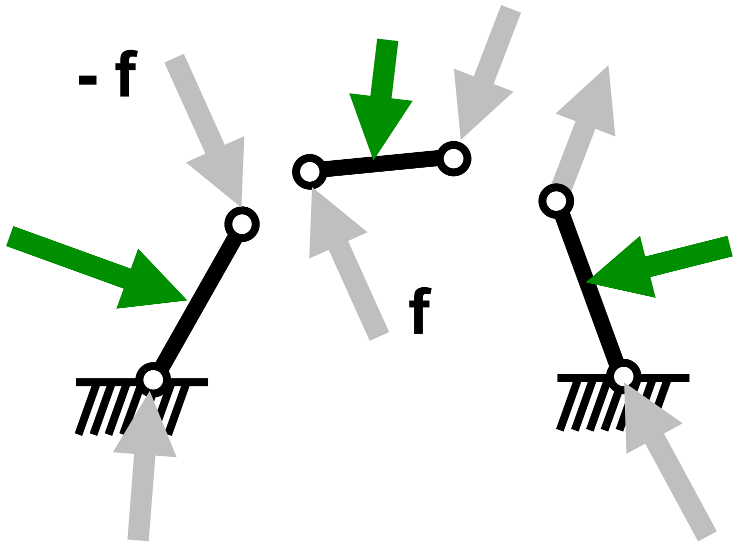
Principle of virtual power for a **mechanism**

"A mechanism is in equilibrium **if, and only if,** the power generated by all externally-applied forces and couples is null, under **any feasible velocity state of the mechanism**"

(feasible = compatible with the joint assembly constraints of the mechanism)



Only the external forces count ...

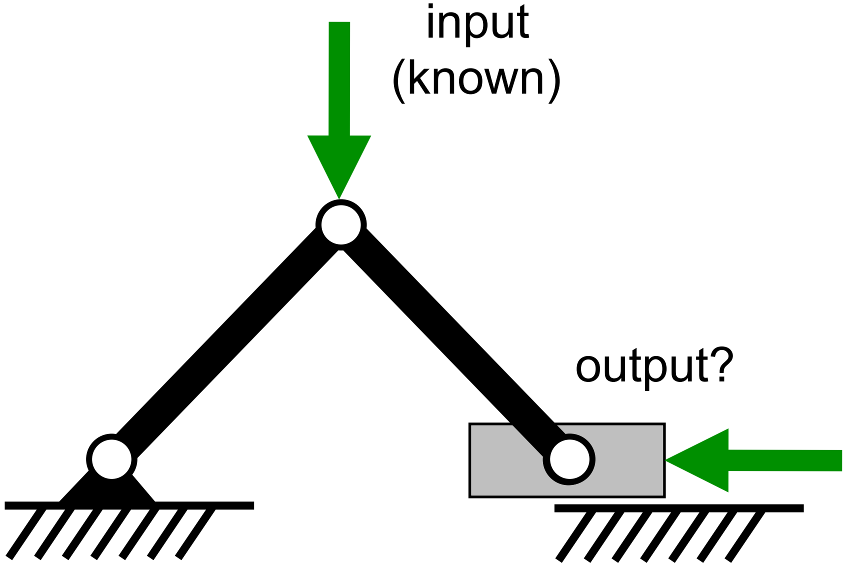


... because the power of the connector forces is globally null

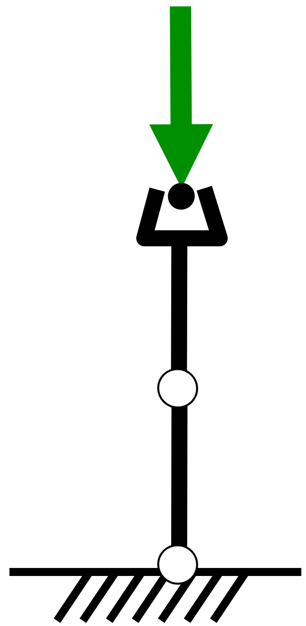
The principle of virtual power

allows us to determine ...

The unknown forces in input/output problems



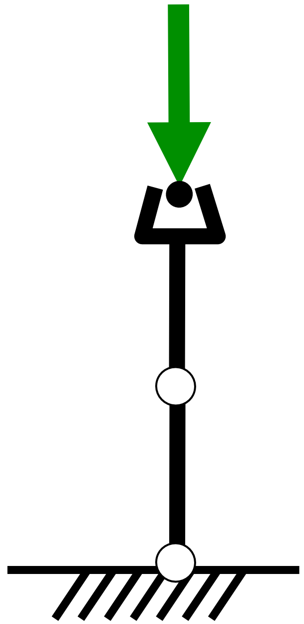
The forces that can be structurally supported by the end-effector of a robotic mechanism



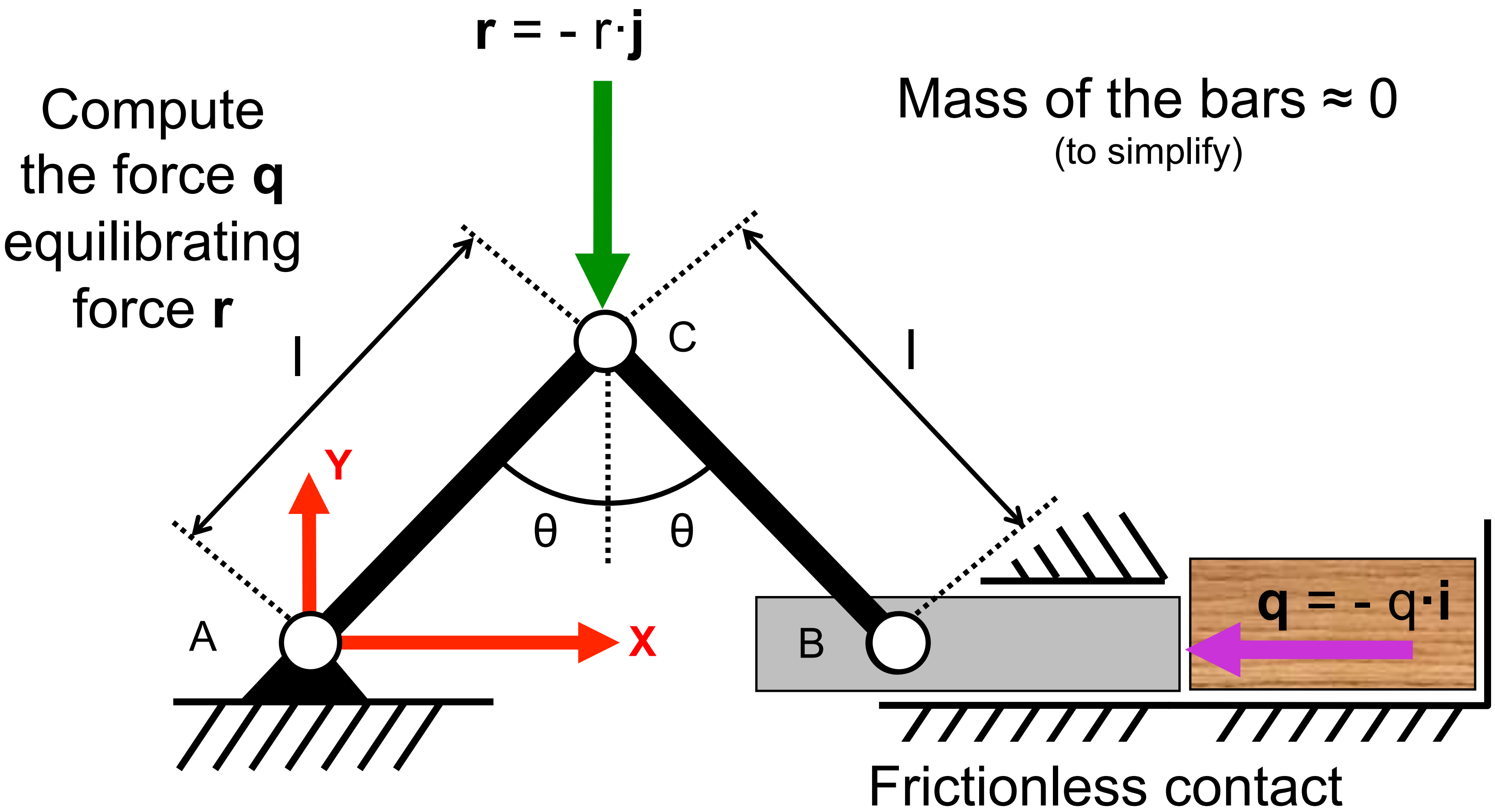
The principle of virtual power

allows us to determine ...

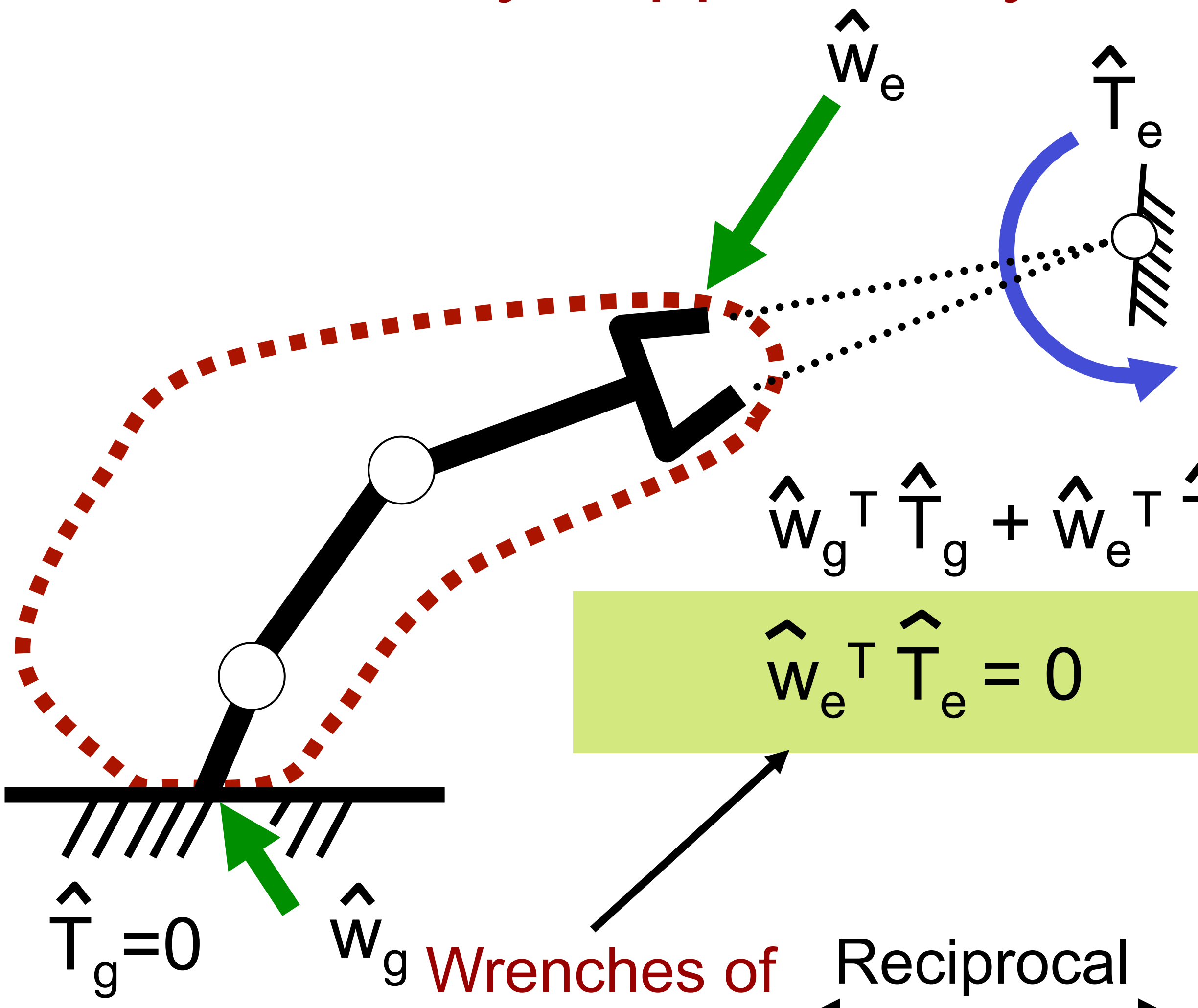
The forces that can be structurally supported by the end-effector of a robotic mechanism



Unknown forces in input/output problems



Which end-effector forces can be structurally supported by the robot?



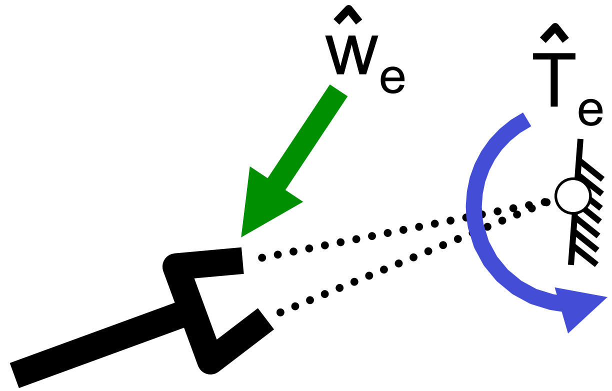
$$\hat{w}_g^T \hat{T}_g + \hat{w}_e^T \hat{T}_e = 0$$

$$\hat{w}_e^T \hat{T}_e = 0 \quad \forall \hat{T}_e$$ compatible with the joints

Wrenches of constraint ⟷ Reciprocal ⟷ Twists of freedom

Twists of freedom and wrenches of constraint

For a given mechanism link
(e.g., its end effector) we define:



Its space of "twists of freedom":

$\mathbb{T}$ = { twists $\hat{T}_e$ under which the link can move }

Its space of “wrenches of constraint”:

$\mathbb{W}$ = { wrenches $\hat{w}_e$ that the link can structurally support }

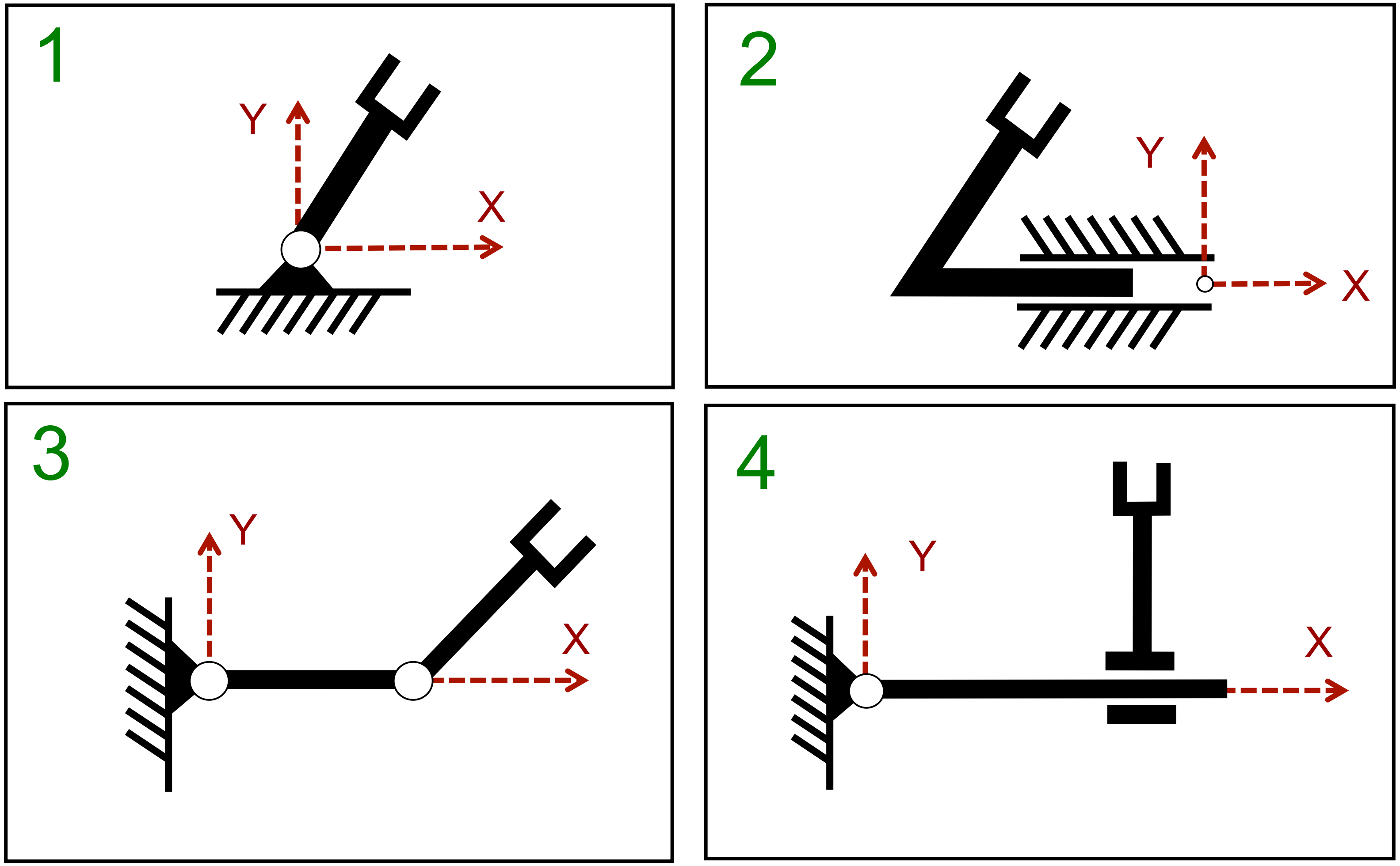
$\mathbb{T}$ and $\mathbb{W}$ are vector spaces. They are reciprocal complements, since:

$$\hat{w}_e^T \, \hat{T}_e = \hat{0} \qquad \forall \, \hat{w}_e \, , \, \hat{T}_e$$

$$\text{Dim } \mathbb{T} + \text{Dim } \mathbb{W} = 3$$

Exercise

Provide the spaces of twists of freedom and wrenches of constraint of these robots. I.e., give a basis of such spaces in the indicated coordinate systems.

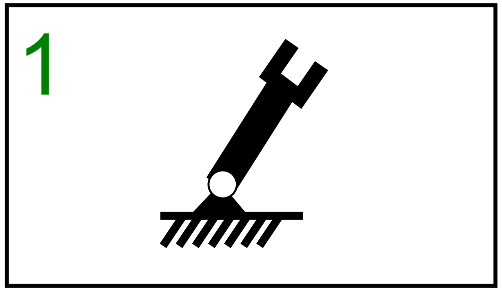


Twist and wrench n-systems

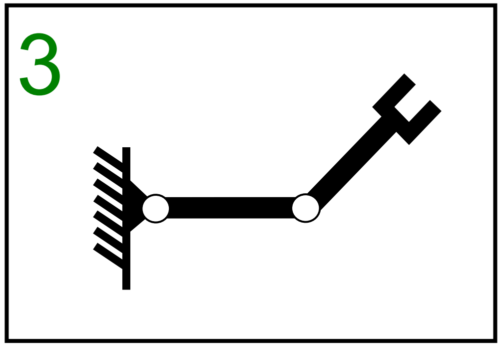
An **n-system of twists** is the space of twists spanned by the linear combinations of n **independent** twists

An **n-system of wrenches** is the space of wrenches spanned by the linear combinations of n **independent** wrenches

Examples:



1-system of twists of freedom
2-system of wrenches of constraint



2-system of twists of freedom
1-system of wrenches of constraint

Historical note

- Several forms of the principle of virtual power where stated by Johann Bernouilli (1667-1748) and his son Daniel Bernouilli (1700-1782).
- The principle was later on generalised to take dynamic effects into account, giving rise to the so-called D'Alembert principle, which forms the basis of Lagrangian mechanics.

Johann Bernouilli

Daniel Bernouilli