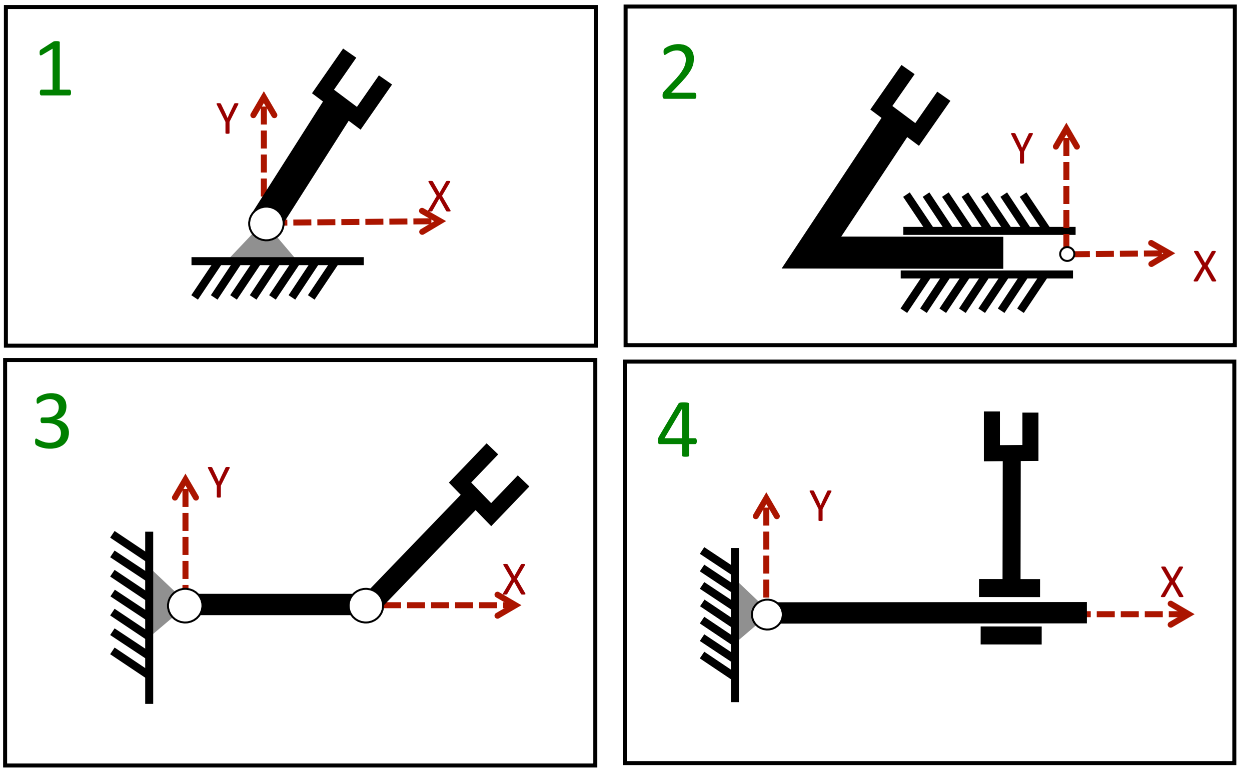
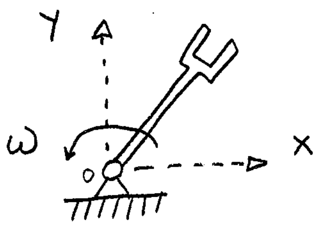
# Exercise

Provide the spaces of twists of freedom and wrenches of constraint of these robots. I.e., give a basis of such spaces in the indicated reference frames.



OBSERVATION: Note that the technique used to compute $\mathbb{W}$ and $\mathbb{T}$ is valid for any serial manipulator

Robot 1




In the shown reference frame, the twist of the end effector has the form

$$\hat{T}_e = \underbrace{\omega}_{\substack{\text{scalar} \\ (\text{rad/s})}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{(in } \underline{\text{axis}} \text{ coords.)}$$

Therefore, $\mathbb{T}$ is the vector space spanned by $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$:

$$\mathbb{T} = \left\langle \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$

$\langle\ \rangle$ means "spanned by"

Since $\mathbb{W}$ is its reciprocal complement, it must be

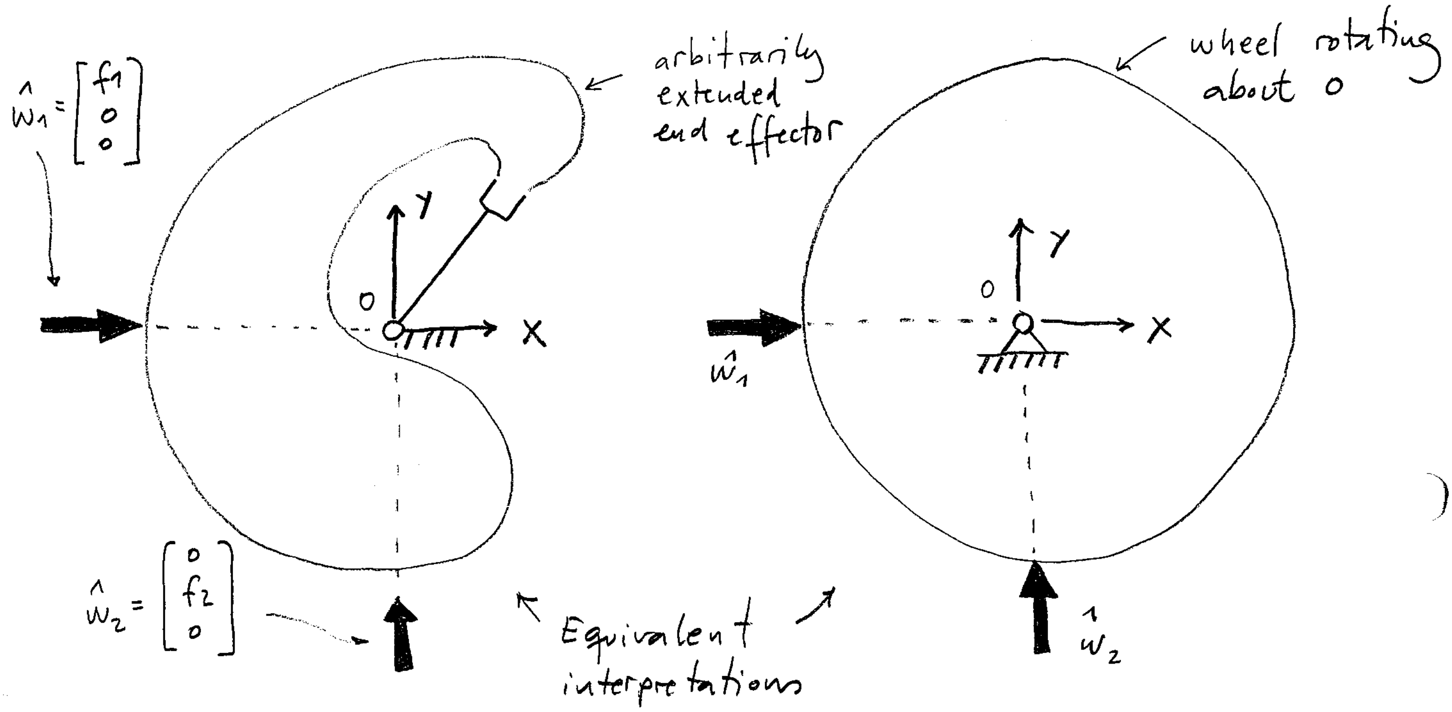
$$\mathbb{W} = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\rangle \qquad \text{(in } \underline{\text{ray}} \text{ coords)}$$

Interpretation: Since $\mathbb{W} = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\rangle$, the wrenches of constraint take the parametric form

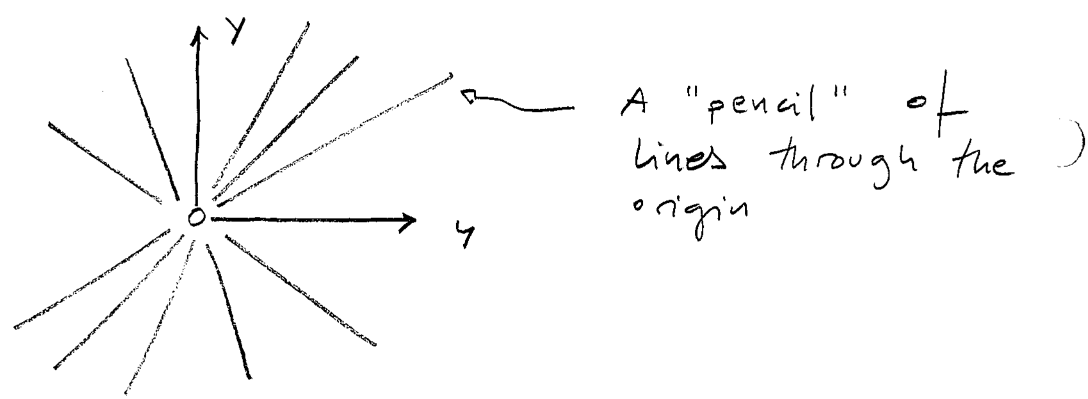
$$\hat{w}_e = \overbrace{f_1}^{\text{Newton}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \overbrace{f_2}^{\text{Newton}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ 0 \end{bmatrix} \qquad \text{(} \underline{\text{ray}} \text{ coords}$$

The wrenches of the form $\begin{bmatrix} f_1 \\ f_2 \\ 0 \end{bmatrix}$ have their line of action through the origin. A direction vector of the line is given by $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$. These are the forces that the robot can equilibrate without activating its joints.

Grafically, we imagine the end effector as a lamina of arbitrary extension. The structurally supportable forces are the linear combinations of $\hat{w}_1$ and $\hat{w}_2$:

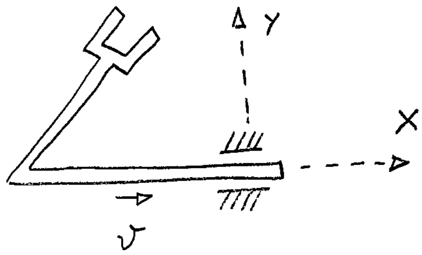



The linear combinations of $\hat{w}_1$ and $\hat{w}_2$ span a 2-system of wrenches, whose lines of action go through the origin:




This result could have been anticipated by geometric analysis. Note that since it must be $\hat{w}_e^T \cdot \hat{T}_e = 0$ $\forall\ \hat{T}_e \in \mathbb{T}$, $\hat{w}_e \in \mathbb{W}$, and $\hat{w}_e^T \cdot \hat{T}_e = f \cdot \omega \cdot \hat{s}_e^T \cdot \hat{S}_e$, the determination of $\mathbb{W}$ reduces to finding the lines $\hat{s}_e$ whose mutual moment with respect to $\hat{S}_e$ is null. Since in this case $\hat{T}_e = \omega \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \hat{S}_e = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$ the only lines with a null mutual moment relative to $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are those of the previous pencil.

$\boxed{\text{Robot 2}}$




In the indicated frame, the end-effector twists take the form

$$\hat{T}_e = v\begin{bmatrix}1\\0\\0\end{bmatrix}$$

Thus:

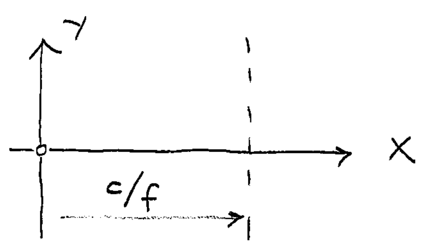
$$\mathbb{T} = \Big\langle \underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{\text{axis}} \Big\rangle \quad \Rightarrow \quad \mathbb{W} = \Big\langle \underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}}_{\text{ray}} \Big\rangle$$

and the wrenches of constraint take the form:

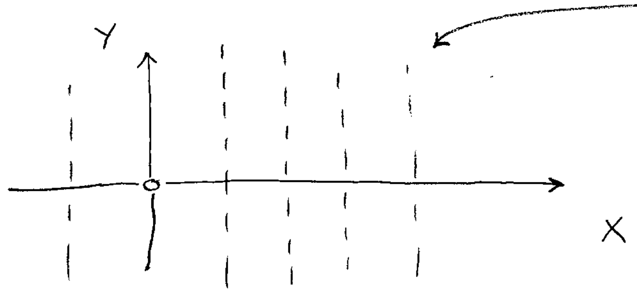
$$\hat{w}_e = \underbrace{f}_{\text{Newton}}\begin{bmatrix}0\\1\\0\end{bmatrix} + \underbrace{c}_{\text{Newton}\cdot\text{meter}}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\f\\c\end{bmatrix}$$

where $f$ and $c$ are scalar signed magnitudes with force and moment units respectively.

The coordinates $\begin{bmatrix}0\\f\\c\end{bmatrix}$ describe a line parallel to the Y axis at a distance $c/f$ from the origin:

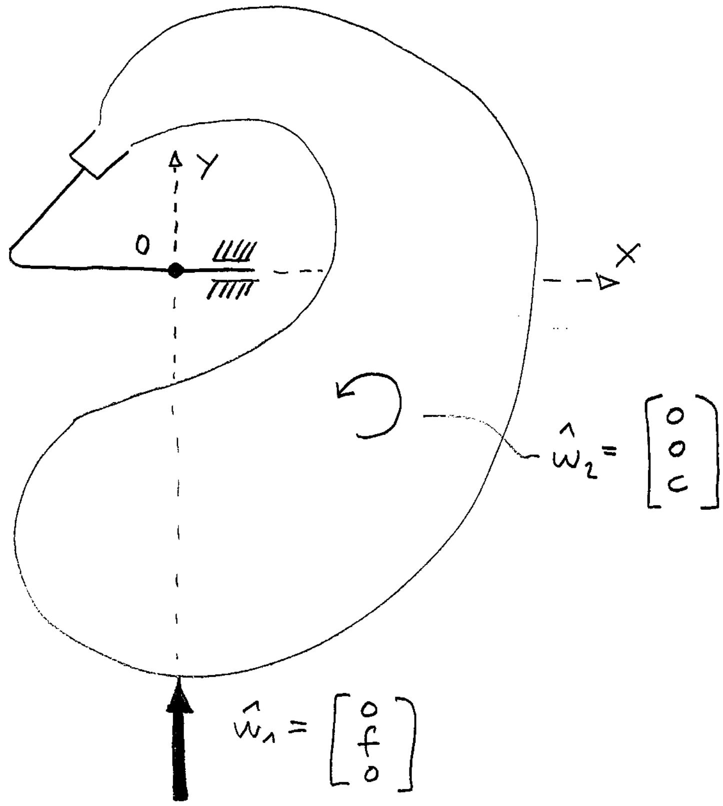



Thus, the structurally-supportable forces are those whose line of action belongs to the line bundle:



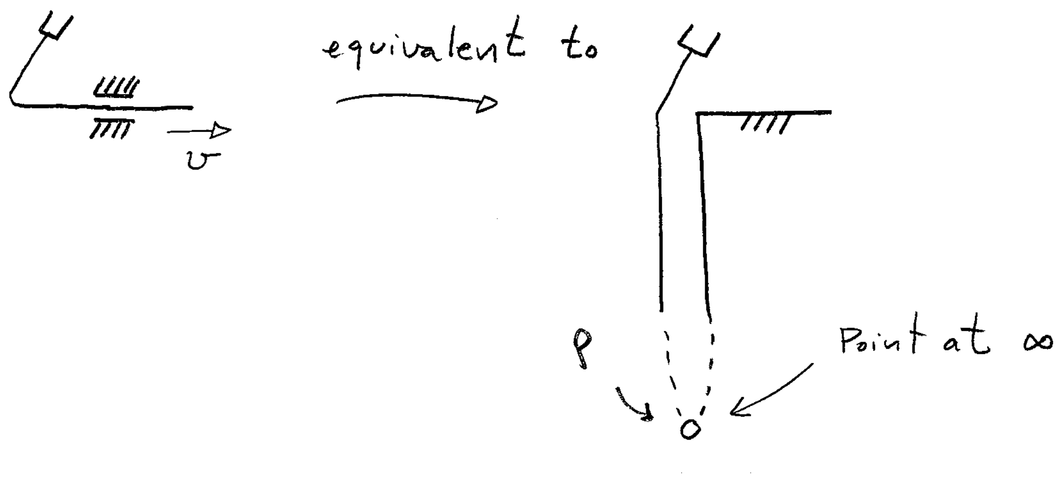

When $f = 0$, $\hat{w}_e = \begin{bmatrix}0\\0\\c\end{bmatrix}$ and the line of action is the line at infinity. In this case, $\hat{w}_e$ is a pure couple applied to the end effector

Sumarizing, the end effector can structuralley support any linear combination of the vertical forces $\hat{w}_1 = \begin{bmatrix} 0 \\ f \\ 0 \end{bmatrix}$ and pure couples $\hat{w}_2 = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$



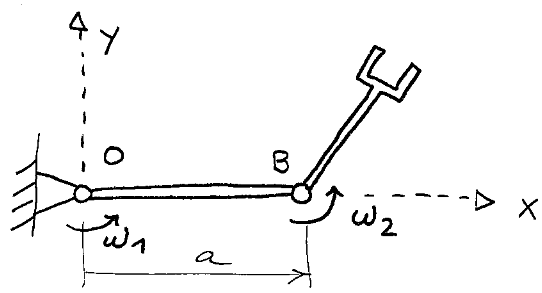

Pure couple, which we interpret as a force of null magnitude, applied to the line at ∞.

We could have anticipated (from the very beginning) that the structurally supportable forces are those of a line bundle parallel to the Y axis, because the prismatic joint is equivalent to a revolute joint at infinity, in the direction orthogonal to the joint velocity:



The end-effector twist, thus, has a line of support r orthogonal to the XY plane at point P. The only lines whose mutual moment with respect to r is null are those that meet P. That is, the lines parallel to the Y axis.

Robot 3




The end-effector twists take the form:

$$\hat{T}_e = \underbrace{\omega_1}_{\frac{rad}{s}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \underbrace{\omega_2}_{\frac{rad}{s}} \begin{bmatrix} 0 \\ -a \\ 1 \end{bmatrix}$$

Thus:

$$\mathbb{T} = \left\langle \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -a \\ 1 \end{bmatrix} \right\rangle \qquad \text{(in axis coords)}$$

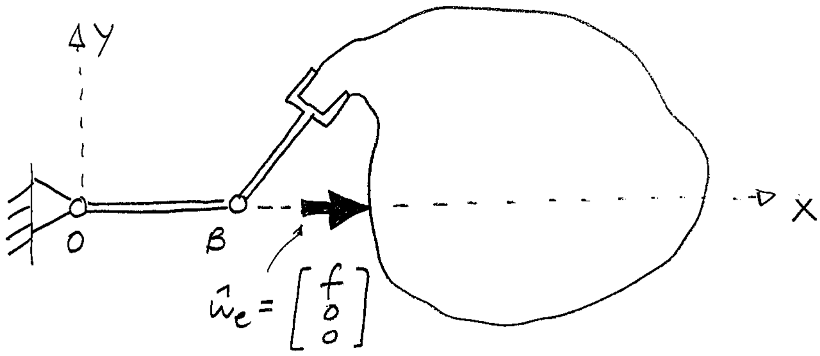
$\mathbb{W}$ must be the reciprocal complement of $\mathbb{T}$. Since $\dim(\mathbb{T}) = 2 \Rightarrow \dim(\mathbb{W}) = 1$. A basis vector for $\mathbb{W}$ is the cross product $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ -a \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$. Thus:

$$\mathbb{W} = \left\langle \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle \qquad \text{(in ray coords)}$$

The wrenches of constraint thus take the form

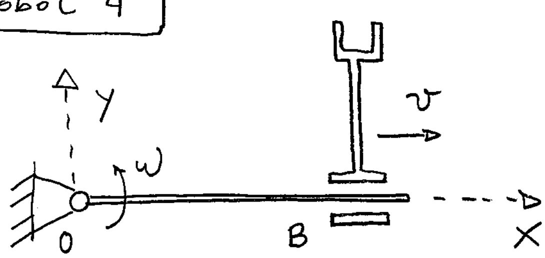
$$\hat{w}_e = \underbrace{f}_{\text{Newton}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

where $f$ is an arbitrary scalar. Thus, the end effector can equilibrate any force on the OB line, without actuating its joints:




We could have anticipated this result by pure geometric reasoning. The only line with a null mutual moment w.r.t. the lines $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -a \\ 1 \end{bmatrix}$ is line OB.

Robot 4




The end-effector twists take the form

$$\hat{T}_e = \underbrace{\omega}_{\text{rad/s}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \underbrace{v}_{\text{m/s}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

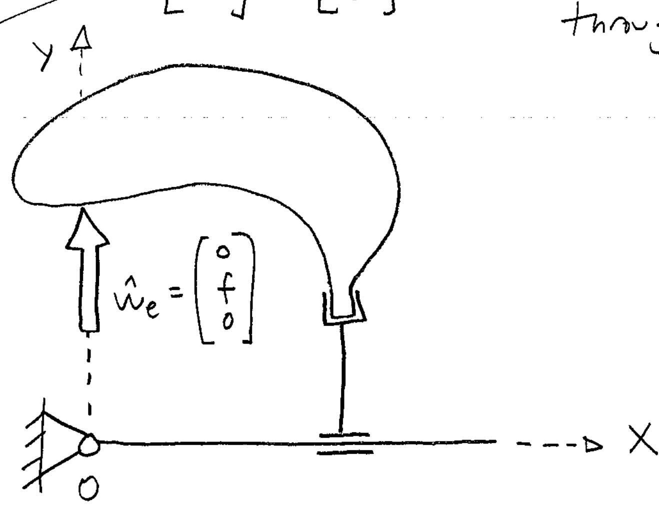
and thus:

$$\mathbb{T} = \left\langle \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle \Rightarrow \mathbb{W} = \left\langle \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\rangle$$

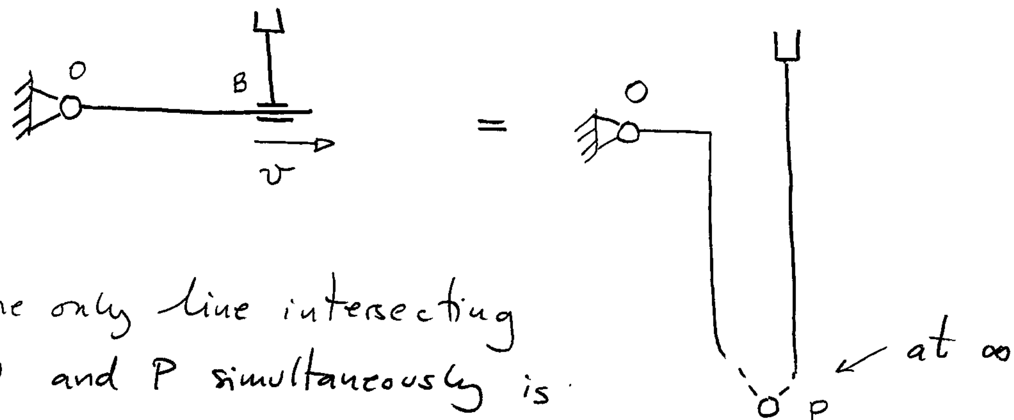
The wrenches of constraint adopt the form:

$$\hat{w}_e = \underbrace{f}_{\text{Newton}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ f \\ 0 \end{bmatrix} \Rightarrow$$

The end effector can structurally support any vertical force through point O



A geometric argument also anticipates this result.
The prismatic joint in B is equivalent to a revolute joint at infinity, in the direction orthogonal to the velocity of P:




The only line intersecting O and P simultaneously is the line of the Y axis.