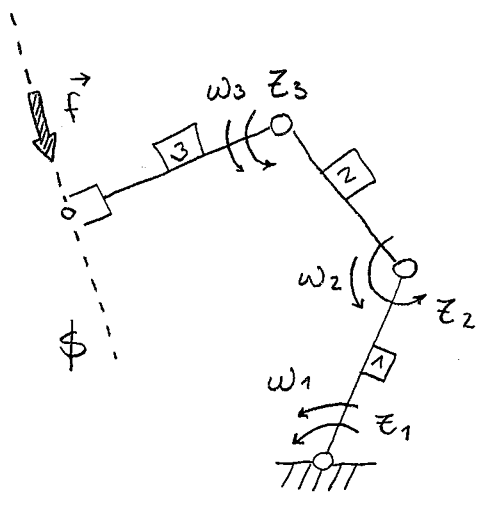
STATIC ANALYSIS OF A SERIAL MANIPULATOR

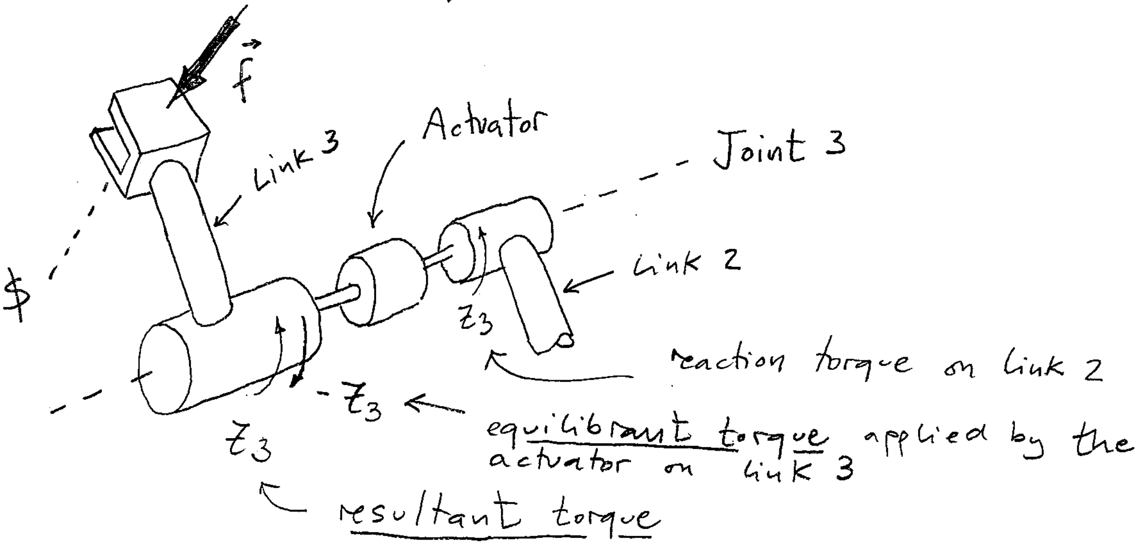
Consider the serial manipulator of the figure, where τ_1, τ_2, τ_3 are the resultant joint torques. That is:

τ_i = resultant torque at joint i
= signed magnitude of the moment of $\vec{f}$ with respect to joint i



The actuator in joint i will exert an equilibrant joint torque $-\tau_i$ on link i to keep the arm in equilibrium.

This is better understood with a 3D drawing of joint i. For example for joint 3:



Consider the wrench $\hat{w}$ of force $\vec{f}$, i.e.

$$\hat{w} = f \cdot \hat{s}$$

f ← signed magnitude of $\vec{f}$

$\hat{s}$ ← unit coords. of line \$

and let $\vec{\tau} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}$ be the vector of resultant joint torques. We wish to determine the relationship between $\vec{\tau}$ and $\hat{w}$.

To this end let us apply the principle of virtual power. For all $\vec{\gamma} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \neq 0$ it must be

$$\underbrace{-\tau_1 \omega_1 - \tau_2 \omega_2 - \tau_3 \omega_3}_{} + \underbrace{\overset{1}{T}{}^T \hat{W}}_{} = 0$$

Power generated by the equilibrant joint torques $-\tau_i$ under the angular velocity ω_i

Power generated by the wrench $\hat{w}$ acting on the end effector, under a given rotor $\hat{T}$ induced by $\vec{\gamma}$.

Note: This term is actually the simplification of the following sum

$$-\tau_3 (\omega_3 + \omega_2 + \omega_1) + \tau_3 (\omega_2 + \omega_1)$$
$$-\tau_2 (\omega_2 + \omega_1) - \tau_2 (\omega_1)$$
$$-\tau_1 \omega_1$$

in which each torque is multiplied by the absolute angular velocity of the body on which it is acting on

> Observe how simple this derivation of $\vec{z} = J^T \hat{w}$ is, !! in comparison to the classic way of writing the static equilibrium equations for each separate body of the arm

In vector form, the previous equation is

$$-\underbrace{[\omega_1, \omega_2, \omega_3]}_{\vec{\gamma}} \underbrace{\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix}}_{\vec{z}} + \hat{T}^T \hat{w} = 0$$

We now can write:

$$-\vec{\gamma}^T \vec{z} + \hat{T}^T \hat{w} = 0$$

From the kinematic analysis of the arm, $\hat{T}$ and $\vec{\gamma}$ are related in this way

$$\hat{T} = J \vec{\gamma} \Leftrightarrow \hat{T}^T = \vec{\gamma}^T J^T$$

$$-\vec{\gamma}^T \vec{z} + \vec{\gamma}^T J^T \hat{w} = 0$$

$$\vec{\gamma}^T \left(-\vec{z} + J^T \hat{w}\right) = 0 \qquad \forall\, \vec{\gamma} \neq 0$$

Since the equality must hold $\forall\, \vec{\gamma} \neq 0$, it must be:

$$-\vec{z} + J^T \hat{w} = 0$$

$$\Downarrow$$

$$\boxed{\vec{z} = J^T \hat{w}}$$

which is the sought relationship $\vec{z} \leftrightarrow \hat{w}$

DEFINITION: When $\det J^T = 0$ we say that the arm is in a static singularity.

derived from $\begin{cases} \vec{\tau} = J^T \hat{w} \\ \hat{T} = J \cdot \vec{\gamma} \end{cases}$

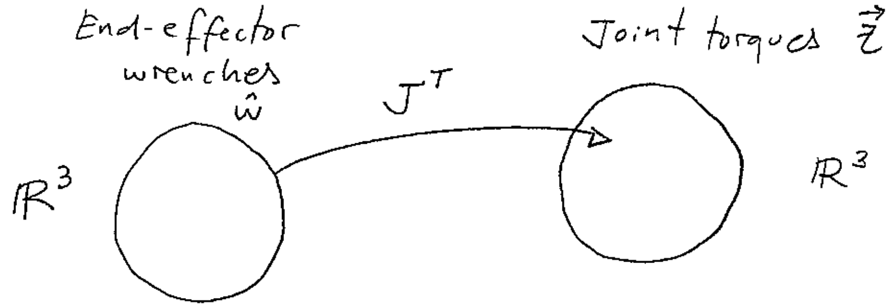
IMPORTANT CONCLUSION: since $\det J = \det J^T$ then a configuration is a static singularity if, and only if, it is a kinematic singularity

From the point of view of statics, there are two important problems to be solved:

- Inverse static problem: given $\hat{w}$, compute $\vec{\tau}$
- Forward " " : " $\vec{\tau}$, " $\hat{w}$

The inverse problem is always solvable, because $\vec{\tau} = J^T \hat{w}$. The forward problem is not always solvable. Let us see this in some detail.

J^T is a linear transformation from the space of end-effector wrenches to the space of joint torques:

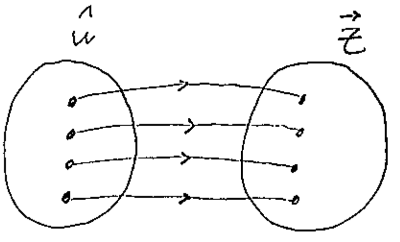


There are two situations:

When $\det J^T \neq 0$ The linear transformation is bijective

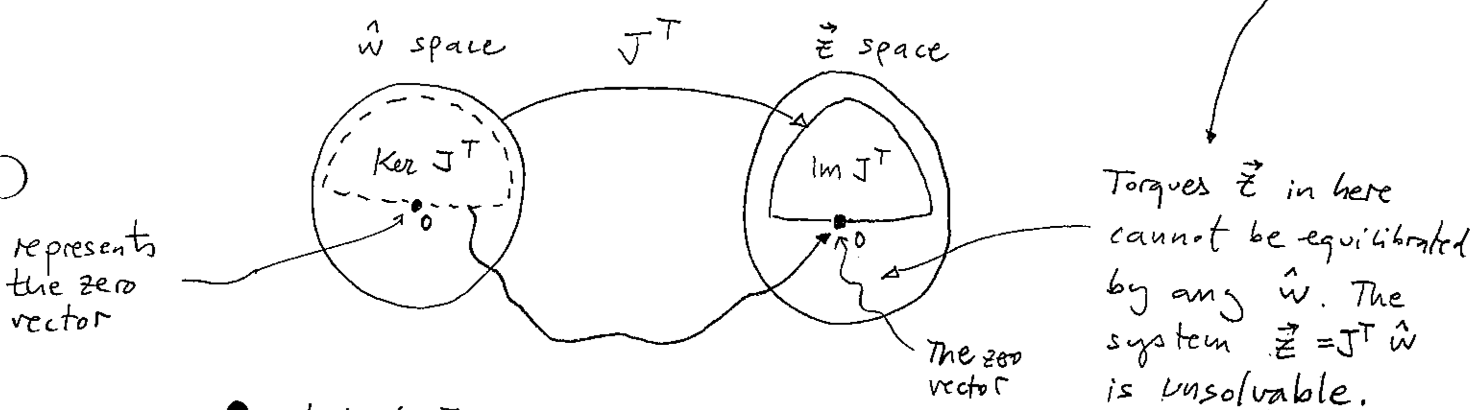
There is one, and only one $\hat{w}$ corresponding to a given $\vec{\tau}$, and vice versa.

So $\vec{\tau} = J^T \hat{w}$ is solvable for all $\vec{\tau}$



When $\det J^T = 0$ Then we have:

- $\det J^T = 0 \Rightarrow \text{rank } J^T < 3 \Rightarrow \text{Dim}\left[\text{Im}(J^T)\right] < 3$
  That is, there exist joint torques $\vec{\tau}$ that do not correspond to any $\hat{w}$. If we inject those torques no possible wrench exists to equilibrate the torques.

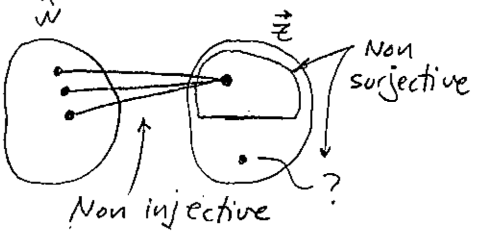


- $$\text{Null}(J^T) = \underbrace{\text{Dim}(\hat{w}\text{ space})}_{3} - \underbrace{\text{Rank}(J^T)}_{<3}$$
  $\Downarrow$
  $\text{Null}(J^T) \geq 1$ (The kernel of J^T is non-trivial)
  $\Downarrow$
  A feasible $\vec{\tau}$ (one in $\text{Im } J^T$) corresponds to infinitely-many $\hat{w}$.

  proof:

  > Let $\hat{w}$ and $\vec{\tau}$ be such that $\vec{\tau} = J^T \hat{w}$
  > Let $\hat{w}_o$ be such that $J^T \hat{w}_o = 0$ (in the Kernel)
  > Then:
  > $$J^T(\hat{w} + \hat{w}_o) = J^T \hat{w} + \underbrace{J^T \hat{w}_o}_{=0} = J^T \hat{w} = \vec{\tau}$$
  > To any $\hat{w}$, we can add a vector $\hat{w}_o$ from the Kernel of J^T, and we obtain the same $\vec{\tau}$

- The linear transformation is non-surjective ($\det J^T = 0$) and non-injective ($\text{Null } J^T \geq 1$)



How do we solve the forward static problem in the 2 situations?

When $\det J^T \neq 0$ Trivial, the solution is simply

$$\hat{w} = (J^T)^{-1} \vec{z}$$

When $\det J^T = 0$ In this case J^T is singular, we first check whether $\vec{z} \in \operatorname{Im} J^T$, using

$$\vec{z} \in \operatorname{Im} J^T \iff \operatorname{rank} J^T = \operatorname{rank} [J^T \,\vdots\, \vec{z}]$$

If $\vec{z} \in \operatorname{Im} J^T$, then $J^T \hat{w} = \vec{z}$ has a whole vector space of dimension ≥ 1 of solutions. We can use the following result to obtain a parametric representation of this space

<u>Proposition</u>: Consider the system $\underset{m\times n}{A}\,\underset{n\times 1}{x} = \underset{m\times 1}{b}$, where $b \in \operatorname{Im} A$, then

$$x = A^+ b + (I - A^+A)\, y, \quad y \in R^n$$

(A^+: $n\times m$, b: $m\times 1$, A^+b: $n\times 1$; $(I - A^+A)$: $n\times n$; A^+A: $n\times n$; y: $m\times 1$)

is a parametrization of the solution set of $Ax=b$, where A^+ is the Moore-Penrose pseudoinverse of A.

A^+ always exists regardless of the rank of A, and can be obtained from the singular value decomposition of A.

The vector $y \in R^n$ in the previous expression is the vector of free parameters of the parametrization. Varying y we obtain all feasible x.

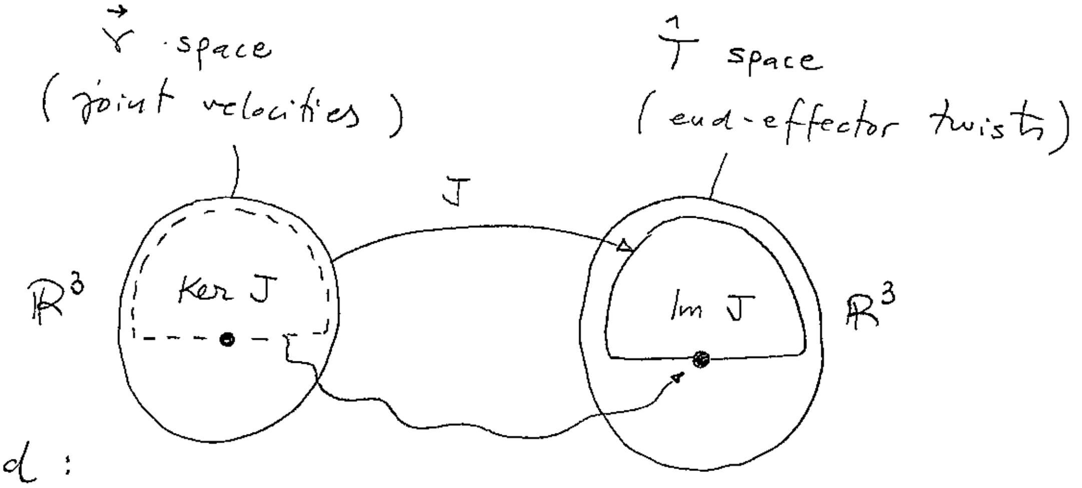
A COMPARISON OF THE STATIC AND KINEMATIC ANALYSES

The static analysis of a serial arm is analogous to its kinematic analysis. In the latter case we had

$$\hat{T} = J \vec{\gamma}$$

end-effector twist ($\hat{T}$); joint angular velocities ($\vec{\gamma}$) $\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$

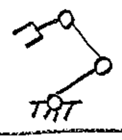
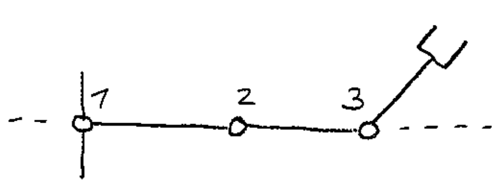
which represents a linear transformation as well:



We had:

- If $\det J \neq 0 \Rightarrow \mathrm{Im}\, J = \mathbb{R}^3 \Rightarrow \mathrm{Ker}\, J = \{0\} \Rightarrow$ Inverse kinematic problem always solvable. Bijective relationship between $\vec{\gamma}$ and $\hat{T}$.
- If $\det J = 0 \Rightarrow \mathrm{Dim}(\mathrm{Im}(J)) < 3 \Rightarrow \mathrm{Null}\, J \geq 1$ Inverse kinematic problem only solvable for a given $\hat{T}$ if $\mathrm{rank}\, J = \mathrm{rank}[J \,\vdots\, \hat{T}]$. There are unreachable twists $\hat{T}$, and there is a many $\vec{\gamma}$ to one $\hat{T}$ relationship for the feasible $\hat{T}$.

Summary table
Kinetostatic analysis of the serial 3R manipulator

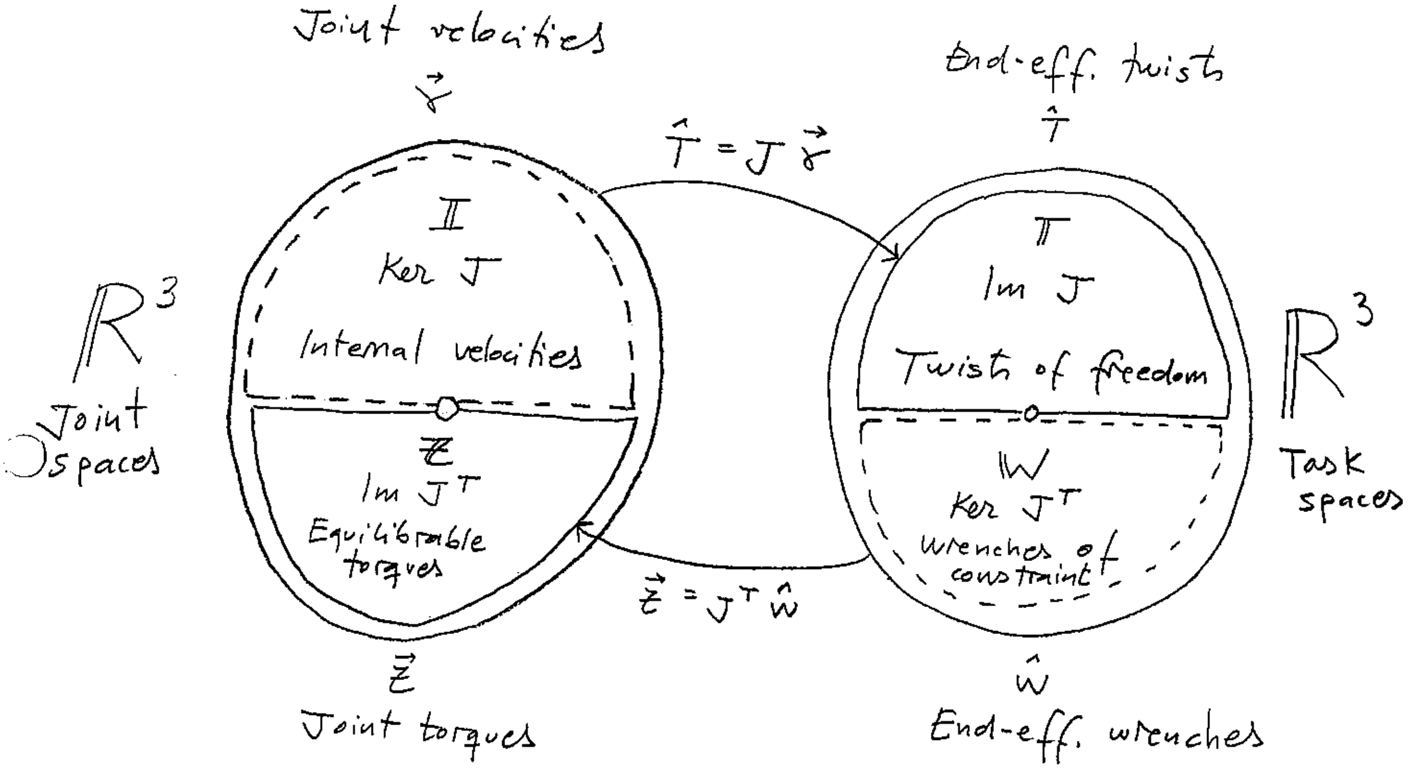
	Kinematic analysis	Static analysis
Input	$\vec{\gamma} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$ Joint angular velocities	$\vec{\tau} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}$ Joint torques
Output	$\hat{T}$ End-effector twist	$\hat{w}$ End-effector wrench
Linear transformation	$\hat{T} = J \vec{\gamma}$ ↑ By columns, the line coords, of the joint axes	$\vec{\tau} = J^T \hat{w}$ ↑ By rows, the line coords. of the joint axes
Always-solvable problem	Forward kinematic $\vec{\gamma} \rightarrow \hat{T}$	Inverse static $\hat{w} \longrightarrow \vec{\tau}$
Not always a solvable problem	Inverse kinematic $\hat{T} \rightarrow \vec{\gamma}$ If det $\neq 0 \Rightarrow$ always solvable If det $= 0 \Rightarrow$ solvable when $\text{rank}[J] = \text{rank}[J \vdots \hat{T}]$	Forward static $\vec{\tau} \rightarrow \hat{w}$ If det $J^T \neq 0 \Rightarrow$ always solvable If det $J^T = 0 \Rightarrow$ solvable when $\text{rank}[J^T] = \text{rank}[J^T \vdots \vec{\tau}]$
Singularity when...	det J = det $J^T = 0 \Leftrightarrow$ 1, 2, 3 aligned 	

Duality diagram of a serial manipulator

The kinetostatic performance of a serial manipulator is described by the linear maps

$$\hat{T} = J \cdot \vec{\gamma} \qquad \text{(joint velocities} \to \text{end-eff. twists)}$$
$$\vec{\tau} = J^T \cdot \hat{w} \qquad \text{(end-eff. wrenches} \to \text{joint torques)}$$

The two maps take values in $\mathbb{R}^3$ and return images in $\mathbb{R}^3$. We can sumarize the connections of the two maps with the following duality diagram:



The following can be proved

PROP. (1) $\text{Dim}\ \mathbb{T} + \text{Dim}\ \mathbb{W} = 3$

" (2) $\text{Dim}\ \mathbb{I} + \text{Dim}\ \mathbb{Z} = 3$

" (3) $\hat{w}^T \cdot \hat{T} = 0 \quad \forall \hat{w} \in \mathbb{W}, \forall \hat{T} \in \mathbb{T}$

" (4) $\vec{\gamma}^T \cdot \vec{\tau} = 0 \quad \forall \vec{\gamma} \in \mathbb{I}, \forall \vec{\tau} \in \mathbb{Z}$

PROOFS OF PROP (1) and (2)

We know that

$$\overbrace{\mathrm{Dim}\,(\mathrm{Im}\,J)}^{\mathrm{Dim}\,\mathbb{T}} = \mathrm{rank}\,J = \mathrm{rank}\,J^T = \overbrace{\mathrm{Dim}(\mathrm{Im}\,J^T)}^{\mathrm{Dim}\,\mathbb{Z}}$$

and by the rank-nullity theorem of linear algebra it is

$$3 = \mathrm{Dim}\,\mathbb{T} + \mathrm{Dim}\,\mathbb{I} \qquad (1)$$

$$3 = \mathrm{Dim}\,\mathbb{Z} + \mathrm{Dim}\,\mathbb{W} \qquad (2)$$

From (1) we can write

$$3 = \mathrm{Dim}\,\mathbb{I} + \underbrace{\mathrm{Dim}\,\mathbb{T}}_{\mathrm{Dim}\,\mathbb{Z}} = \mathrm{Dim}\,\mathbb{I} + \mathrm{Dim}\,\mathbb{Z} \quad \leftarrow \text{Proves PROP (1)}$$

From (2)

$$3 = \underbrace{\mathrm{Dim}\,\mathbb{Z}}_{\mathrm{Dim}\,\mathbb{T}} + \mathrm{Dim}\,\mathbb{W} = \mathrm{Dim}\,\mathbb{T} + \mathrm{Dim}\,\mathbb{W} \quad \leftarrow \text{Proves PROP (2)}$$

(QED)

PROOF OF PROP. (3)

If $\hat{T} \in \mathrm{Im}\,J$, $\exists\, \vec{\gamma}$ such that $\hat{T} = J\vec{\gamma}$, thus:

$$\underset{\mathrm{Ker}\,J^T}{\underset{\cap}{\hat{w}^T}} \cdot \underset{\mathrm{Im}\,J}{\underset{\cap}{\hat{T}}} = \hat{w}^T J \vec{\gamma} = (\underbrace{J^T \hat{w}}_{\vec{0}})^T \cdot \vec{\gamma} = 0 \quad \text{(QED)}$$

$\vec{0}$, because $\hat{w} \in \mathrm{Ker}\,J^T$

PROOF OF PROP (4)

Analogous. If $\vec{z} \in \mathrm{Im}\,J^T$, $\exists\, \hat{w}$ such that $\vec{z} = J^T\hat{w}$, thus:

$$\underset{\mathrm{Ker}\,J}{\underset{\cap}{\vec{\gamma}^T}} \cdot \underset{\mathrm{Im}\,J^T}{\underset{\cap}{\vec{z}}} = \vec{\gamma}^T \cdot J^T \cdot \hat{w} = (\underbrace{J \cdot \vec{\gamma}}_{\vec{0}})^T \cdot \hat{w} = 0 \quad \text{(QED)}$$

$\vec{0}$, because $\vec{\gamma} \in \mathrm{Ker}\,J$

COROLLARY Propositions (1) – (4) prove that

$\mathbb{I}$ and $\mathbb{Z}$ are orthogonal complements w.r.t. $\mathbb{R}^3$
$\mathbb{T}$ and $\mathbb{W}$ " " " " $\mathbb{R}^3$

NOTE: When we say "orthogonal complements" we mean in an abstract way, as vector spaces. However, the vectors in $\mathbb{I}$ (resp. $\mathbb{T}$) have different units than those of $\mathbb{Z}$ (resp $\mathbb{W}$). It is more correct to say that the two spaces are reciprocal complements.

Physical interpretation of the diagram

A if $\det J \neq 0$ then $\begin{cases} \text{Null } J = \text{Null } J^T = 0 \\ \text{Dim Im } J = \text{Dim Im } J^T = 3 \end{cases}$

The two kernels reduce to the trivial zero vector
The two image spaces span the whole $\mathbb{R}^3$
There are no internal velocities nor wrenches of constraint (except the trivial zero ones).

Both J and J^T are bijective maps

B If $\det J = 0$ then $\begin{cases} \text{Null } J = \text{Null } J^T \geq 1 \\ \text{Dim Im } J = \text{Dim Im } J^T \leq 2 \end{cases}$

and we are in a singular configuration

Relative to **A**

$\mathbb{I}$ and $\mathbb{W}$ "inflate"
$\mathbb{Z}$ and $\mathbb{T}$ "deflate"

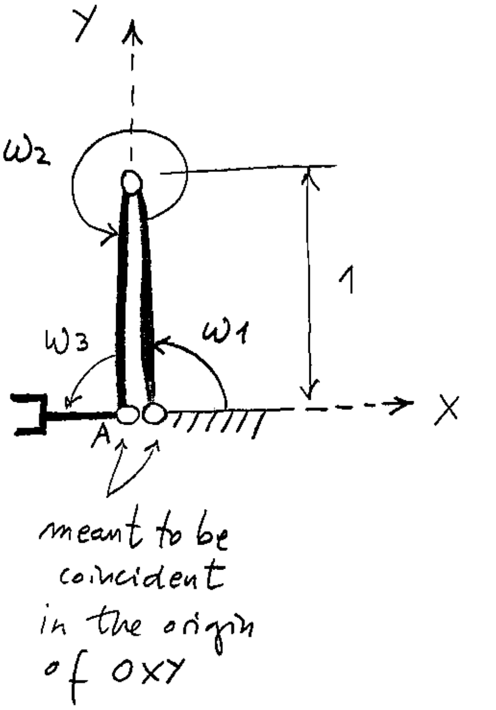
The following table summarizes all physical consequences.

Serial manipulators
Physical behaviour at a singularity

	Kinematic behaviour	Static behaviour
Kernel elements	**a** ∃ internal joint velocities ($\vec{\gamma}$ that produce $\hat{T}=0$)	**e** ∃ wrenches of constraint ($\hat{w}$ that produce $\vec{z}=0$, also called "structurally supportable" forces)
Many-to-one relationships	**b** A same twist $\hat{T}$ can be produced by infinitely many joint velocities $\vec{\gamma}$ (it is a consequence of [a])	**f** A same joint torque $\vec{z}$ can be equilibrated by infinitely-many wrenches $\hat{w}$ (a consequence of [b])
Impossible input or output	**c** ∃ impossible $\hat{T}$ such $\hat{T}$ cannot be produced by any $\vec{\gamma}$ (let's call them $\hat{T}_{imp}$) Loss of dexterity	**g** ∃ impossible $\vec{z}$ such $\vec{z}$ can't be equilibrated by any $\hat{w}$ (let's call them $\vec{z}_{neq}$) Loss of equilibrability
Near the singularity	**d** The $\hat{T}_{imp}$ require very large $\vec{\gamma}$ ⇓ overspeeding in resolved motion rate control	**h** The $\vec{z}_{neq}$ may equilibrate very large end-effector forces ⇓ Gain of mechanical advantage when transporting heavy loads. Dangerous in human-robot interactions

Let us illustrate the meaning of each table cell with examples and further insights. We proceed column-wise. First the column on the kinematic behaviour:

[a] Note for example that in the following configuration



$$\vec{\gamma} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \omega \\ 0 \\ -\omega \end{bmatrix}$$

is an internal velocity for any ω. The first two links can rotate freely, keeping the end effector fixed

[b] The statement is clearly true for $\hat{T} = 0$ and $\vec{\gamma}$ as in [a], in the shown configuration. But it is also true for any twist of freedom $\hat{T}$ because if for some $\vec{\gamma}$ we have

$$J \vec{\gamma} = \hat{T}$$

then

$$J(\vec{\gamma} + \vec{\gamma}_{int}) = \hat{T}$$

as well, for any $\vec{\gamma}_{int} \in \ker J$.

[c] Let us show one such $\hat{T}_{imp}$ in the configuration of [a]. In OXY, the Jacobian J is

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow \text{Im } J = \left\langle \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$

and one $\hat{T}_{imp}$ is, e.g.,

$$\hat{T}_{imp} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Under such a twist, all end effector points should have velocities parallel to the OY axis, but note that point A can only have horizontal velocities (parallel to OX).

We say there is a <u>loss of dexterity</u> of the end effector.

[d] We don't prove this rigorously, but note that in a non-singular configuration

$$\vec{\gamma} = J^{-1} \hat{T} = \frac{\text{Adj}(J)}{\det(J)} \cdot \hat{T}$$

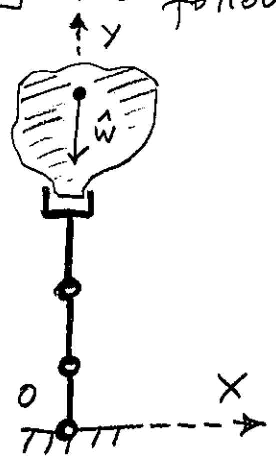
Adjugate, or adjoint of J (the transpose of the cofactor matrix).

If $\det \to 0$, then $\vec{\gamma} \to \infty$

The overspeeding effect can be seen visually on a Geogebra animation posted on the course web. Such overspeeding may be dangerous in human-robot interactions (e.g. robot surgery). We don't want the arm to <u>suddenly</u> speed up for even small displacements of the end effector.

Now the column on the static behaviour:

[e] The following wrench is a <u>wrench of constraint</u> $\forall a$:



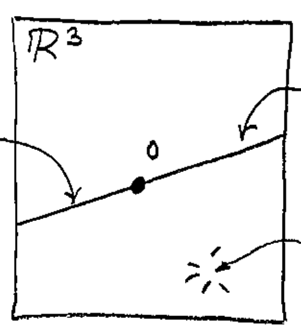
$$\hat{w} = \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix}$$

(i.e. a "structurally supportable" one)

Produces a resultant torque $\vec{\tau} = 0$. Can be equilibrated without the help of the actuators

[f] From [a] we clearly see, e.g., that $\vec{\tau} = 0$ corresponds to infinitely many wrenches $\hat{w} = \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix}$. As in cell [b] this will actually happen for any $\vec{\tau} \in \text{Im}\, J^T$.

[g] To understand this cell, note that $\text{Im}\, J^T$ is a subspace of $\mathbb{R}^3$ when J is singular



$\vec{\tau}$ here is equilibrable by some $\hat{w}$

$\text{Im}\, J^T = \mathcal{T}$ is of dimension lower than 3

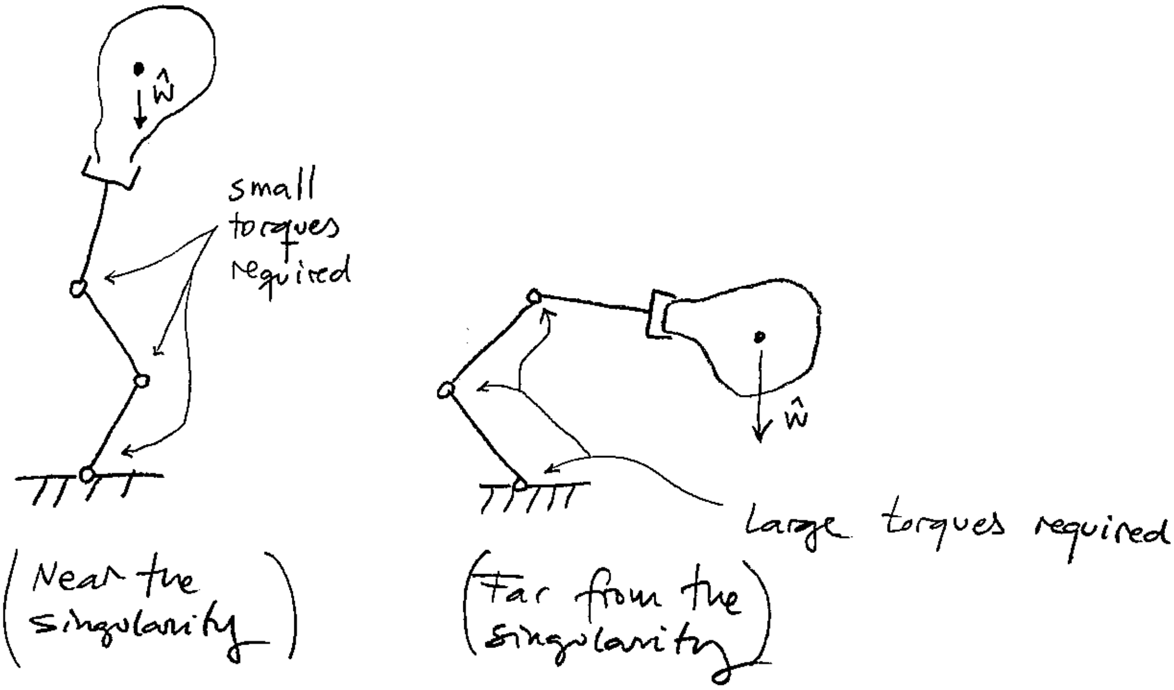
$\vec{\tau}$ here is <u>NOT</u> equilibrable by any $\hat{w}$

$\Rightarrow$ There exists a linear dependence on the feasible τ_i. They must fulfill, e.g. an equation of the form

$$0 = a_1 \tau_1 + a_2 \tau_2 + a_3 \tau_3$$

These $\vec{\tau}$, if produced by the actuators, can't be equilibrated by any external wrench $\Rightarrow$ we say there is a <u>LOSS OF EQUILIBRABILITY</u> (of the environment against the robot).

[h] Note that in the vicinity of a singular configuration the arm can equilibrate (withstand) the wrench $\hat{w}$ [e] by injecting small torques on the actuators



We obtain a gain of mechanical advantage near the singularity, which is beneficial in heavy-load transportation.

But we can also read things the other way around. Small actuator torques produce very large wrenches of the end-effector against the environment (in contact situations e.g.), which is dangerous in human-robot interaction scenarios. A small error in the injected joint torque might result in body penetration of the tool tip in a sugery operation.