Consider the serial manipulator of the figure, where $\tau_1$, $\tau_2$, $\tau_3$ are the resultant joint torques. That is:

$$\tau_i = \text{resultant torque at joint } i$$

$$= \text{signed magnitude of the moment of } \mathbf{f} \text{ with respect to joint } i$$

The actuator in joint $i$ will exert an equilibrium joint torque $-\tau_i$ on link $i$ to keep the arm in equilibrium.

This is better understood with a 3D drawing of joint $i$. For example, for joint 3:

Consider the wrench $\mathbf{w}$ of force $\mathbf{f}$, i.e.

$$\mathbf{w} = \mathbf{f} \cdot \mathbf{a}$$

$\mathbf{a}$ unit coords. of line $\mathbf{f}$

signed magnitude of $\mathbf{f}$
and let \( \vec{\tau} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} \) be the vector of resultant joint torques. We wish to determine the relationship between \( \vec{\tau} \) and \( \vec{\omega} \).

To this end let us apply the principle of virtual power. For all \( \vec{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \neq 0 \) it must be

\[
-\tau_1 \omega_1 - \tau_2 \omega_2 - \tau_3 \omega_3 + \vec{\tau}^T \vec{\omega} = 0
\]

Power generated by the equivalent joint torques \( -\tau_i \) under the angular velocity \( \omega_i \).

\[\vec{\tau}^T \vec{\omega}\] Power generated by the wrench \( \vec{\tau} \) acting on the end effector, under a given rotor \( \vec{\omega} \) induced by \( \vec{\omega} \).

Note: This term is actually the simplification of the following sum

\[
-\tau_3 (\omega_3 + \omega_2 + \omega_1) + \tau_3 (\omega_2 + \omega_1) \\
-\tau_2 (\omega_2 + \omega_1) - \tau_2 (\omega_1) \\
-\tau_1 \omega_1
\]

in which each torque is multiplied by the absolute angular velocity of the body on which it is acting on
In vector form, the previous equation is

\[- \begin{bmatrix} \omega_1, \omega_2, \omega_3 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} + \hat{T} \hat{w} = 0\]

We now can write:

\[- \gamma^T \varepsilon + \hat{T} \hat{w} = 0\]

From the kinematic analysis of the arm, \( \hat{T} \) and \( \varepsilon \) are related in this way:

\[\hat{T} = J \varepsilon \iff \hat{T}^T = \varepsilon^T J^T\]

\[- \gamma^T \varepsilon + \frac{\gamma^T J^T}{J^T J} \hat{w} = 0\]

\[\forall \gamma (\varepsilon + J^T \hat{w}) = 0 \quad \forall \gamma \neq 0\]

Since the equality must hold \( \forall \gamma \neq 0 \), it must be:

\[- \varepsilon + J^T \hat{w} = 0\]

\[\varepsilon = J^T \hat{w}\]

which is the sought relationship \( \varepsilon \leftrightarrow \hat{w} \)

**Definition:** When \( \det J^T = 0 \) we say that the arm is in a static singularity.
IMPORTANT CONCLUSION: Since \( \det J = \det J^T \)
then a configuration is a **static singularity** if, and only if, it is a **kinematic singularity**.

From the point of view of statics, there are two important problems to be solved:

- **Inverse static problem**: given \( \hat{\omega} \), compute \( \hat{\varepsilon} \).
- **Forward static**: \( \varepsilon \rightarrow \hat{\varepsilon} \rightarrow \hat{\omega} \)

The inverse problem is always solvable, because \( \hat{\varepsilon} = J^T \hat{\omega} \).

The forward problem is not always solvable. Let us see this in some detail.

\( J^T \) is a linear transformation from the space of end-effector wrenches to the space of joint torques:

There are two situations:

- **When \( \det J^T \neq 0 \)**: The linear transformation is bijective.

  There is one, and only one, \( \hat{\omega} \) corresponding to a given \( \hat{\varepsilon} \), and vice versa.

  So \( \hat{\varepsilon} = J^T \hat{\omega} \) is solvable for all \( \hat{\varepsilon} \).
When $\det J^T = 0$ Then we have:

- $\det J^T = 0 \Rightarrow \text{rank } J^T < 3 \Rightarrow \text{Dim } [\text{Im } (J^T)] < 3$
  
  That is, there exist joint torques $\hat{z}$ that do not correspond to any $\hat{w}$. If we inject those torques no possible wrench exists to equilibrate the torques.

  ![Diagram](image)

- $\text{Null } (J^T) = \frac{\text{Dim } (\hat{w} \text{ space})}{\text{Rank } (J^T)} < 3$

- $\text{Null } (J^T) \geq 1$ (The kernel of $J^T$ is non-trivial)

  A feasible $\hat{z}$ (one in $\text{Im } J^T$) corresponds to infinitely many $\hat{w}$.

  **Proof:**

  Let $\hat{w}$ and $\hat{z}$ be such that $\hat{z} = J^T \hat{w}$
  Let $\hat{w}_0$ be such that $J^T \hat{w}_0 = 0$ (in the kernel)
  Then:

  $J^T (\hat{w} + \hat{w}_0) = J^T \hat{w} + J^T \hat{w}_0 = J^T \hat{w} = \hat{z}$

  To any $\hat{w}$, we can add a vector $\hat{w}_0$ from the kernel of $J^T$, and we obtain the same $\hat{z}$.

- The linear transformation is non-surjective (det $J^T = 0$) and non-injective (Null $J^T \geq 1$)
How do we solve the forward static problem in the 2 situations?

**When \( \det J^T \neq 0 \)**

Trivial, the solution is simply

\[
\hat{\mathbf{w}} = (J^T)^{-1} \tilde{\mathbf{w}}
\]

**When \( \det J^T = 0 \)**

In this case \( J^T \) is singular, we first check whether \( \tilde{\mathbf{w}} \in \text{Im } J^T \), using

\[
\tilde{\mathbf{w}} \in \text{Im } J^T \iff \text{rank } J^T = \text{rank } [J^T; \tilde{\mathbf{w}}]
\]

If \( \tilde{\mathbf{w}} \in \text{Im } J^T \), then \( J^T \hat{\mathbf{w}} = \tilde{\mathbf{w}} \) has a whole vector space of dimension \( \geq 1 \) of solutions. We can use the following result to obtain a parametric representation of this space.

**Proposition:** Consider the system \( A \mathbf{x} = \mathbf{b} \), where \( \mathbf{b} \in \text{Im } A \),

then

\[
\mathbf{x} = A^+ \mathbf{b} + (I - A^+A) \mathbf{y} + \mathbf{y} \in \mathbb{R}^n
\]

is a parametrization of the solution set of \( A \mathbf{x} = \mathbf{b} \), where \( A^+ \) is the Moore-Penrose pseudoinverse of \( A \).

\( A^+ \) always exists regardless of the rank of \( A \), and can be obtained from the singular value decomposition of \( A \).

The vector \( \mathbf{y} \in \mathbb{R}^n \) in the previous expression is the vector of free parameters of the parametrization. Varying \( \mathbf{y} \) we obtain all feasible \( \mathbf{x} \).
A COMPARISON OF THE STATIC AND KINEMATIC ANALYSES

The static analysis of a serial arm is analogous to its kinematic analysis. In the latter case we had

\[ \dot{T} = J \dot{\xi} \]

end-effector twist \[ [ \omega_x \omega_y \omega_z ] \]

which represents a linear transformation as well:

\[ \dot{\xi} \text{ space} \hspace{1cm} \dot{T} \text{ space} \]

(point velocities) \hspace{1cm} (end-effector twists)

We had:

- If \( \det J \neq 0 \) \( \Rightarrow \) \( \text{Im } J = \mathbb{R}^3 \) \( \Rightarrow \) \( \ker J = 0 \) \( \Rightarrow \)
  inverse kinematic problem always solvable. Bijective relationship between \( \xi \) and \( \dot{T} \).

- If \( \det J = 0 \) \( \Rightarrow \) \( \dim (\text{Im } J) < 3 \Rightarrow \text{Null } J > 1 \)
  inverse kinematic problem only solvable for a given \( \dot{T} \)
  if \( \text{rank } J = \text{rank } [J ; \dot{T}] \). There are unreachable twists \( \dot{T} \), and there is a many \( \xi \) to one \( \dot{T} \) relationship for the feasible \( \dot{T} \).
# Summary Table

## Kinematic and Static Analysis of the Serial 3R Manipulator

<table>
<thead>
<tr>
<th></th>
<th>Kinematic Analysis</th>
<th>Static Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input</strong></td>
<td>( \mathbf{\theta} = \begin{bmatrix} \theta_1 \ \theta_2 \ \theta_3 \end{bmatrix} ) Joint angular velocities</td>
<td>( \mathbf{\tau} = \begin{bmatrix} \tau_1 \ \tau_2 \ \tau_3 \end{bmatrix} ) Joint torques</td>
</tr>
<tr>
<td><strong>Output</strong></td>
<td>( \mathbf{\hat{T}} ) End-effector twist</td>
<td>( \mathbf{\hat{w}} ) End-effector wrench</td>
</tr>
</tbody>
</table>
| **Linear transformation**| \( \mathbf{\hat{T}} = \mathbf{J} \mathbf{\hat{y}} \)  \\
By columns, the line coods, of the joint axes | \( \mathbf{\hat{z}} = \mathbf{J}^{\mathbf{T}} \mathbf{\hat{w}} \)  \\
By rows, the line coods, of the joint axes |
| **Always solvable problem** | Forward kinematic  \\
\( \mathbf{\hat{y}} \to \mathbf{\hat{T}} \) | Inverse static  \\
\( \mathbf{\hat{w}} \to \mathbf{\hat{z}} \) |
| **Not always solvable problem** | Inverse kinematic  \\
\( \mathbf{\hat{T}} \to \mathbf{\hat{y}} \)  \\
If \( \det \mathbf{J} \neq 0 \Rightarrow \text{always solvable} \)  \\
If \( \det \mathbf{J} = 0 \Rightarrow \text{solvable when} \)  \\
\( \text{rank}[\mathbf{J}] = \text{rank}[\mathbf{J}^{\mathbf{T}} \mathbf{\hat{T}}] \) | Forward static  \\
\( \mathbf{\hat{z}} \to \mathbf{\hat{w}} \)  \\
If \( \det \mathbf{J}^{\mathbf{T}} \neq 0 \Rightarrow \text{always solvable} \)  \\
If \( \det \mathbf{J}^{\mathbf{T}} = 0 \Rightarrow \text{solvable when} \)  \\
\( \text{rank}[\mathbf{J}^{\mathbf{T}}] = \text{rank}[\mathbf{J}^{\mathbf{T}} \mathbf{\hat{z}}] \) |
| **Singularity when**      | \( \det \mathbf{J} = \det \mathbf{J}^{\mathbf{T}} = 0 \iff 7, 2, 3 \text{ aligned} \) |
Duality diagram of a serial manipulator

The kinetostatic performance of a serial manipulator is described by the linear maps

\[ \dot{T} = J \cdot \dot{\theta} \] (Joint velocities \( \to \) end-eff. twists)

\[ \dot{\varepsilon} = J^T \cdot \dot{\hat{W}} \] (end-eff. wrenches \( \to \) joint torques)

The two maps take values in \( \mathbb{R}^3 \) and return images in \( \mathbb{R}^3 \); we can summarize the connections of the two maps with the following **duality diagram**:

The following can be proved:

**Prop. (1)** \( \dim \mathbb{T} + \dim \mathbb{W} = 3 \)

**Prop. (2)** \( \dim \mathbb{T} + \dim \mathbb{\varepsilon} = 3 \)

**Prop. (3)** \( \mathbb{\hat{W}}^T \cdot \dot{T} = 0 \) \( \forall \mathbb{\hat{W}} \in \mathbb{W}, \forall \dot{T} \in \mathbb{T} \)

**Prop. (4)** \( \mathbb{\hat{T}}^T \cdot \dot{\varepsilon} = 0 \) \( \forall \mathbb{\hat{T}} \in \mathbb{T}, \forall \dot{\varepsilon} \in \mathbb{\varepsilon} \)
Proofs of Prop (1) and (2)

We know that

\[
\text{Dim } T = \text{Dim } (\text{Im } J) = \text{rank } J = \text{rank } J^T = \text{Dim } (\text{Im } J^T)
\]

and by the rank-nullity theorem of linear algebra it is

\[3 = \text{Dim } T + \text{Dim } \Pi \tag{1}\]
\[3 = \text{Dim } E + \text{Dim } W \tag{2}\]

From (1) we can write

\[3 = \frac{\text{Dim } T + \text{Dim } \Pi}{\text{Dim } E} = \text{Dim } \Pi + \text{Dim } E \leftarrow \text{Proved Prop (1)}
\]

From (2)

\[3 = \frac{\text{Dim } E + \text{Dim } W}{\text{Dim } T} = \text{Dim } T + \text{Dim } W \leftarrow \text{Proved Prop (2)}
\]

(QED)

Proof of Prop. (3)

If \( \hat{T} \in \text{Im } J \), \( \exists \ \hat{\gamma} \) such that \( \hat{T} = J \hat{\gamma} \), thus:

\[
\hat{\omega}^T \cdot \hat{T} = \hat{\omega}^T \cdot J \hat{\gamma} = (J^T \hat{\omega})^T \cdot \hat{\gamma} = 0 \tag{QED}
\]

\( \text{Ker } J^T \cap \text{Im } J = 0 \), because \( \hat{\omega} \in \text{Ker } J^T \)

Proof of Prop (4)

Analogous. If \( \hat{Z} \in \text{Im } J^T \), \( \exists \ \hat{\gamma} \) such that \( \hat{Z} = J^T \hat{\gamma} \), thus:

\[
\hat{\omega}^T \cdot \hat{Z} = \hat{\omega}^T \cdot J^T \hat{\gamma} = (J \cdot \hat{\gamma})^T \cdot \hat{\omega} = 0 \tag{QED}
\]

\( \text{Ker } J \cap \text{Im } J^T = 0 \), because \( \hat{\gamma} \in \text{Ker } J \)
Corollary: Propositions (4) - (9) prove that

\[ \Pi \text{ and } \Xi \text{ are orthogonal complements w.r.t. } \mathbb{R}^3 \]
\[ \Pi \text{ and } W \text{ " } \Pi \text{ " } \Xi \text{ " } \mathbb{R}^3 \]

**Note:** When we say "orthogonal complements" we mean in an abstract way, as vector spaces. However, the vectors in \( \Pi \) (resp. \( \Pi' \)) have different units than those of \( \Xi \) (resp. \( W \)). It is more correct to say that the two spaces are reciprocal complements.

**Physical interpretation of the diagram**

**A** if \( \det J \neq 0 \) then
\[
\begin{align*}
\text{Null } J &= \text{Null } J^T = 0 \\
\text{Dim } \text{Im } J &= \text{Dim } \text{Im } J^T = 3
\end{align*}
\]

The two kernels reduce to the trivial zero vector.
The two image spaces span the whole \( \mathbb{R}^3 \).
There are no internal velocities nor wrenches of constraint (except the trivial zero ones).

Both \( J \) and \( J^T \) are bijective maps.

**B** if \( \det J = 0 \) then
\[
\begin{align*}
\text{Null } J &= \text{Null } J^T = 1 \\
\text{Dim } \text{Im } J &= \text{Dim } \text{Im } J^T = 2
\end{align*}
\]

and we are in a singular configuration relative to **A**

\[ \Pi \text{ and } W \text{ " inflate" } \]
\[ \Xi \text{ and } \Pi \text{ " deflate" } \]

The following table summarizes all physical consequences.
<table>
<thead>
<tr>
<th>Kernel elements</th>
<th>Kinematic behaviour</th>
<th>Static behaviour</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\exists$ internal joint velocities $\dot{\theta}$ that produce $\dot{T} = 0$</td>
<td>$\exists$ wrenches of constraint $\dot{w}$ that produce $\dot{\tau} = 0$, also called &quot;structurally supportable&quot; forces</td>
</tr>
<tr>
<td>Many-to-one relationships</td>
<td>$\forall$ a same twist $\dot{T}$ can be produced by infinitely many joint velocities $\dot{\theta}$ (it is a consequence of $\mathbb{A}$)</td>
<td>$\forall$ a same joint torque $\dot{\tau}$ can be equilibrated by infinitely many wrenches $\dot{w}$ (a consequence of $\mathbb{B}$)</td>
</tr>
<tr>
<td>Impossible input or output</td>
<td>$\exists$ impossible $\dot{T}$ such that cannot be produced by any $\dot{\theta}$ (let's call them $\dot{T}_{\text{imp}}$)</td>
<td>$\exists$ impossible $\dot{\tau}$ such that can't be equilibrated by any $\dot{w}$ (let's call them $\dot{\tau}_{\text{neg}}$)</td>
</tr>
<tr>
<td>Near the Singularity</td>
<td>The $\dot{T}_{\text{imp}}$ require very large $\dot{\theta}$ → over speeding in resolved motion rate control</td>
<td>The $\dot{\tau}_{\text{neg}}$ may equilibrate very large end-effector forces ↓ Gain of mechanical advantage when transporting heavy loads</td>
</tr>
</tbody>
</table>

Semiai manipulators

Physical behaviour at a singularity
Let us illustrate the meaning of each table cell with examples and further insights. We proceed column-wise. First the column on the kinematic behaviour:

**a.** Note for example that in the following configuration

\[
\begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{bmatrix} = \begin{bmatrix}
\omega \\
0 \\
-\omega
\end{bmatrix}
\]

is an internal velocity for any \( \omega \)

The first two links can rotate freely, keeping the end effector fixed.

meant to be coincident in the origin of \( OXY \)

**b.** The statement is clearly true for \( \tau = 0 \) and \( \tau \) as in \( a \), in the shown configuration. But it is also true for any twist of freedom \( \tau \) because if for some \( \tau \), we have

\[
J \dot{\tau} = \tau
\]

then

\[
J (\dot{\tau} + \dot{\tau}_{int}) = \tau
\]

as well, for any \( \dot{\tau}_{int} \in \text{ker } J \).
Let us show one such $\overset{\wedge}{T}_{\text{imp}}$ in the configuration of $\bar{a}$. In oxy, the Jacobian $J$ is

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow \text{Im } J = \left\langle \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\rangle,$$

and one $\overset{\wedge}{T}_{\text{imp}}$ is, e.g.,

$$\overset{\wedge}{T}_{\text{imp}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Under such a twist, all end effector points should have velocities parallel to the ox axis, but note that point A can only have horizontal velocities (parallel to ox).

We say there is a loss of dexterity of the end effector.

We don't prove this rigorously, but note that in a non-singular configuration

$$\overset{\wedge}{Y} = J^{-1} \overset{\wedge}{T} = \frac{\text{Adj}(J)}{\text{det}(J)}, \overset{\wedge}{T}$$

Adjugate or adjoint of $J$ (the transpose of the cofactor matrix).

If $\text{det} \to 0$, then $\overset{\wedge}{Y} \to \infty$.

The overspeeding effect can be seen visually on a Geogebra animation posted on the course web. Such overspeeding may be dangerous in human-robot interactions (e.g., robot surgery). We don't want the arm to suddenly speed up for even small displacements of the end effector.
Now the column on the static behaviour:

2. The following wrench is a wrench of constraint \( \mathbf{w} \):

\[
\mathbf{w} = \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix}
\]

i.e. a "structurally supportable" one

Produced a resultant torque \( \mathbf{\tau} = 0 \),

can be equilibrated without the help of the actuators.

From cell 2, we clearly see, e.g., that \( \mathbf{\tau} = 0 \) corresponds to infinitely many wrenches \( \mathbf{w} = \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} \). As in cell 2, this will actually happen for any \( \mathbf{\tau} \in \text{Im } J^T \).

3. To understand this cell, note that \( \text{Im } J^T \) is a subspace of \( \mathbb{R}^3 \) when \( J \) is singular.

\[ \begin{array}{c}
\mathbb{R}^3 \\
\text{Im } J^T = \mathcal{Z} \\
is of dimension lower than 3 \\
\end{array} \]

\( \mathcal{Z} \) here is not equilibrable by any \( \mathbf{w} \)

\( \mathcal{Z} \) here is equilibrable by some \( \mathbf{w} \)

\[ \mathcal{Z} \text{ here is not equilibrable by any } \mathbf{w} \]

\[ \mathcal{Z} \text{ here is equilibrable by some } \mathbf{w} \]

There exist a linear dependence on the feasible \( \mathcal{Z} \).

They must fulfill, e.g., an equation of the form

\[ 0 = a_1 \mathcal{Z}_1 + a_2 \mathcal{Z}_2 + a_3 \mathcal{Z}_3 \]

These \( \mathcal{Z} \), if produced by the actuators, can't be equilibrated by any external wrench.

\[ \text{Thus there is a loss of equilibrability of the environment against the robot.} \]
Note that in the vicinity of a singular configuration the arm can equilibrate (withstand) the wrench if by injecting small torques on the actuators.

![Diagram](image)

(Near the Singularity)

(Large torques required)

(Far from the Singularity)

We obtain a gain of mechanical advantage near the singularity, which is beneficial in heavy-load transportation.

But we can also read things the other way around. Small actuator torques produce very large wrenches of the end-effector against the environment (in contact situations e.g.), which is dangerous in human-robot interaction scenarios. A small error in the injected joint torque might result in body penetration of the tool tip in a surgery operation.