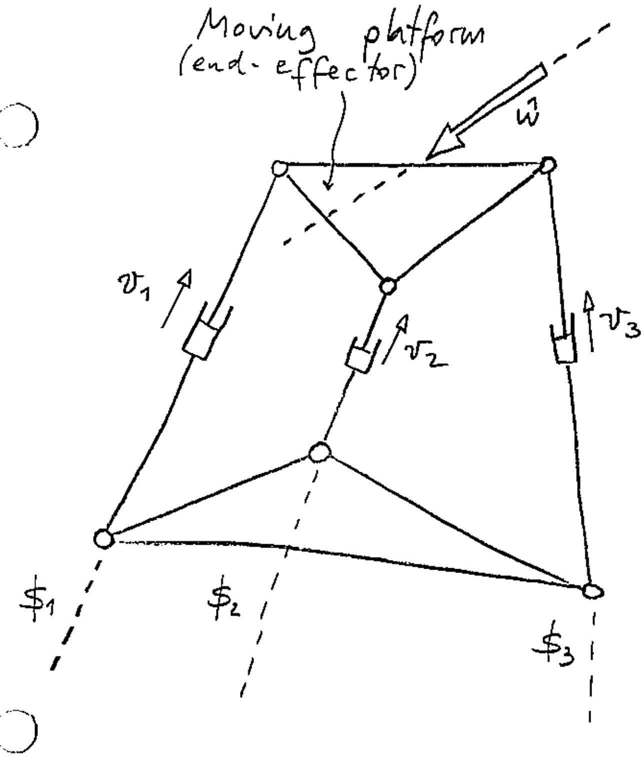
KINEMATIC ANALYSIS OF A PARALLEL MANIPULATOR

Consider the parallel 3-R$\underline{P}$R manipulator of the figure, where v_1, v_2, v_3 are the prismatic joint speeds of the three legs:

$v_i > 0 \Rightarrow$ the leg increases its length
$v_i < 0 \Rightarrow$ " " decreases " "



Let us assume that in the shown configuration the moving platform is moving under a twist

$$\hat{T} = \begin{bmatrix} v_{ox} \\ v_{oy} \\ \omega \end{bmatrix} \quad \text{(End-effector twist)}$$

we wish to find the relationship between $\hat{T}$ and the vector

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \text{(Vector of joint speeds)}$$

We can find the relationship using the principle of virtual power. We assume that a wrench $\hat{w} \neq 0$ is acting on the platform, so that, as a result, the platform exerts resultant forces of magnitudes f_1, f_2, and f_3 on the legs. That is, the wrench that the platform is exerting on the i-th leg is

$$\hat{w}_i = f_i \, \hat{s}_i$$

where $\hat{s}_i$ is the vector of unit coordinates of line $\$_i$ (in ray coordinates).

To be able to equilibrate $\hat{w}$, the actuators will exert <u>equilibrant forces</u> of magnitudes $-f_1, -f_2, -f_3$.

Let us apply the principle:

See note in the next page on how the $f_i v_i$ are obtained.

$$\underbrace{-f_1 v_1 - f_2 v_2 - f_3 v_3}_{\text{Power generated by the equilibrant forces}} + \underbrace{\hat{w}^T \cdot \hat{T}}_{\text{Power generated by } \hat{w}} = 0$$

By recalling that $\vec{\lambda} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$, we note that the previous equation must hold for any value $\vec{\lambda} \neq 0$. The equation can be compactly written as follows

$$\vec{\lambda}^T \vec{v} = \hat{w}^T \cdot \hat{T}$$

But we know from module 2 "statics" that

$$\hat{w} = J \vec{\lambda} \quad \Downarrow \quad \hat{w}^T = \vec{\lambda}^T \cdot J^T$$

Thus:

$$\vec{\lambda}^T \vec{v} = \vec{\lambda}^T \cdot J^T \cdot \hat{T}$$

and since this relation must hold for any $\vec{\lambda} \neq 0$, we have

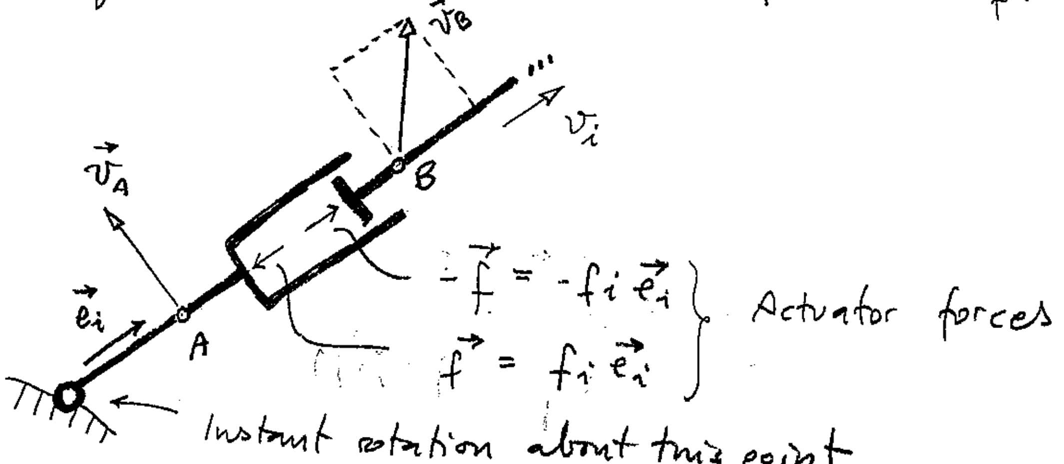
$$\boxed{\vec{v} = J^T \hat{T}}$$

which is the sought relationship between $\vec{v}$ and $\hat{T}$.

A configuration of the 3-R$\underline{\text{P}}$R manipulator is said to be in a <u>kinematic singularity</u> if $\det J^T = 0$.

> Important result: since $\det(J^T) = \det(J)$, a configuration of the 3-R$\underline{\text{P}}$R robot is in a static singularity if and only if it is in a kinematic singularity.

A note on how to obtain the power generated by the equilibrant actuator forces $-f_i v_i$

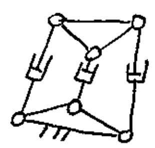
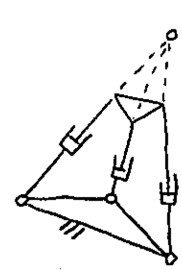


The power of the actuator forces is

$$\underbrace{\vec{f}^T \vec{v}_A}_{=0\ (\perp)} - \vec{f}^T \vec{v}_B = -\underbrace{f_i \vec{e}_i^{\,T}}_{\vec{f}} \cdot \underbrace{v_i \vec{e}_i}_{\text{The longitudinal component of } \vec{v}_B} = -f_i v_i$$

Summary table

Kinetostatic analysis of the 3-R$\underline{P}$R robot

	Static analysis	Kinematic analysis
Input	$\vec{\lambda} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$ Leg forces	$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ Prismatic joint speeds
Output	$\hat{w}$ End-effector wrench	$\hat{T}$ End-effector twist
Linear map	$\hat{w} = J \cdot \vec{\lambda}$ J: by columns the coords of the legs	$\vec{v} = J^T \hat{T}$ J^T: by rows the coords. of the legs
Always-solvable problem	Forward static $\vec{\lambda} \to \hat{w}$	Inverse kinematic $\hat{T} \to \vec{v}$
Not always a solvable problem	Inverse static $\hat{w} \to \vec{\lambda}$ If $\det J \neq 0 \Rightarrow$ always solvable $\forall \hat{w}$ If $\det J = 0 \Rightarrow$ only solvable when $\operatorname{rank}[J] = \operatorname{rank}[J \,\vdots\, \hat{w}]$	Forward kinematic $\vec{v} \to \hat{T}$ If $\det J^T \neq 0 \Rightarrow$ always solvable $\forall \vec{v}$ If $\det J^T = 0 \Rightarrow$ only solvable when $\operatorname{rank}[J^T] = \operatorname{rank}[J^T \,\vdots\, \vec{v}]$
Singularity when	$\det J = \det J^T = 0$  i.e., when the three leg lines are concurrent	

The duality diagram of a parallel manipulator

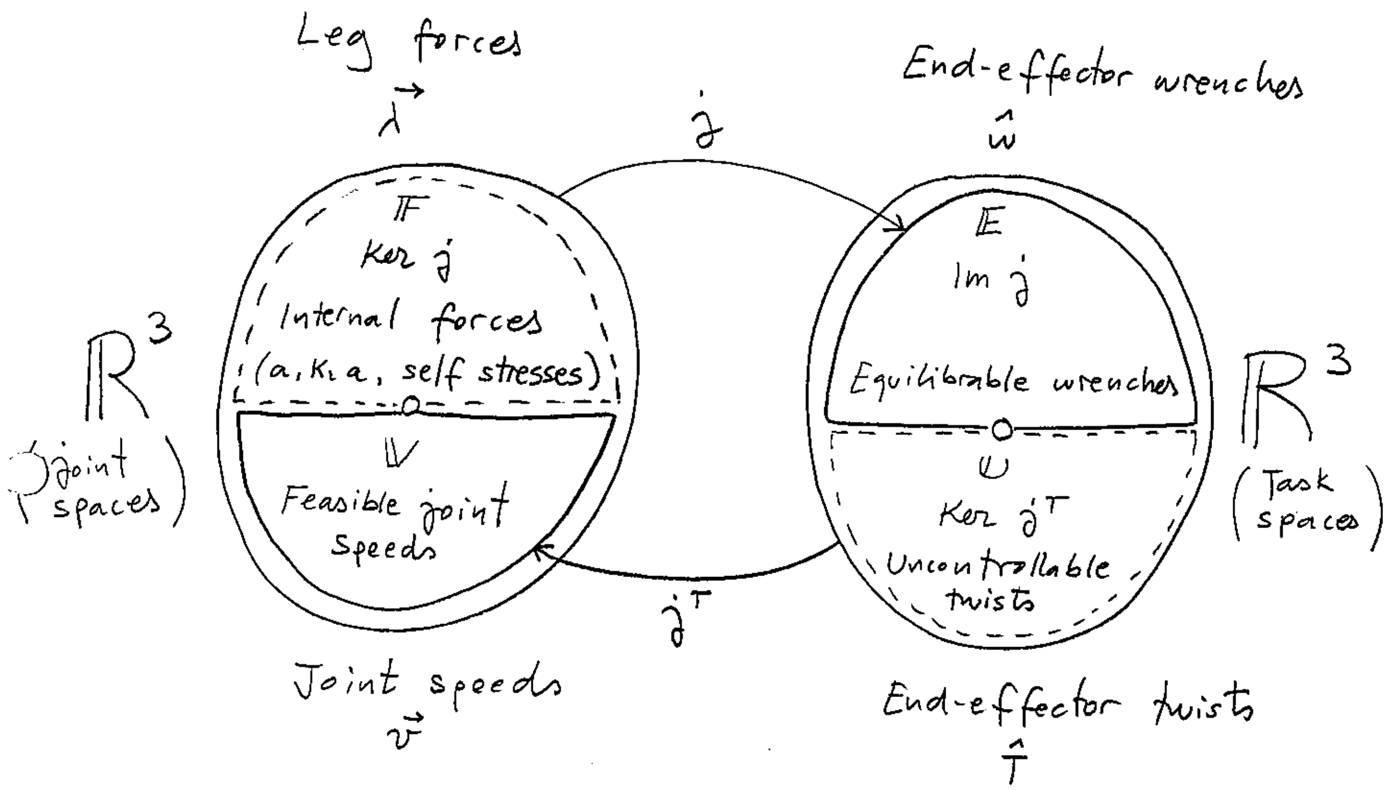
The kinetostatic performance of a parallel manipulator is described by the linear maps

$$\hat{w} = J \vec{\lambda} \qquad \text{leg forces} \rightarrow \text{end-effector wrenches}$$

$$\vec{v} = J^T \hat{T} \qquad \text{end-eff twists} \rightarrow \text{prismatic joint speeds}$$

The two maps take values in $\mathbb{R}^3$ and return images in $\mathbb{R}^3$

We can summarize their connections with the following <u>duality diagram</u>:



In analogy to the properties of the duality diagram of a serial manipulator it can be proved that:

PROP (1) $\quad \text{Dim } \mathbb{E} + \text{Dim } \mathbb{U} = 3$

" (2) $\quad \text{Dim } \mathbb{F} + \text{Dim } \mathbb{V} = 3$

" (3) $\quad \hat{w}^T \cdot \hat{T} = 0 \quad \forall \hat{w} \in \mathbb{E}, \ \forall \hat{T} \in \mathbb{U}$

" (4) $\quad \vec{\lambda}^T \cdot \vec{v} = 0 \quad \forall \vec{\lambda} \in \mathbb{F}, \ \forall \vec{v} \in \mathbb{V}$

Physical interpretation of the diagram

$\boxed{A}$ If $\det J \neq 0$ then
$$\begin{cases} \text{Null } J = \text{Null } J^T = 0 \\ \text{Dim Im } J = \text{Dim Im } J^T = 3 \end{cases}$$

The two kernels reduce to the trivial zero vector
The two image spaces span the whole $\mathbb{R}^3$
All end-effector wrenches are equilibrable.
All joint speeds are feasible.
We have the following bijections:

Leg forces		Joint speeds
$\updownarrow$	bijection	$\updownarrow$
End-eff. wrenches		End-eff. twists

This bijection allows solving the inverse static problem using:
$$\vec{\lambda} = J^{-1} \hat{w}$$

This bijection allows solving the forward instantaneous kinematic problem using:
$$\hat{T} = J^{-1} \vec{v}$$

$\boxed{B}$ If $\det J = 0$ then
$$\begin{cases} \text{Null } J = \text{Null } J^T \geqslant 1 \\ \text{Dim Im } J = \text{Dim Im } J^T \leqslant 2 \end{cases}$$

and we are in a <u>singular configuration</u>

Relative to the situation in $\boxed{A}$:

$\mathbb{F}$ and $\mathbb{U}$ "inflate"
$\mathbb{V}$ " $\mathbb{E}$ "deflate"

The following table summarizes all physical consequences

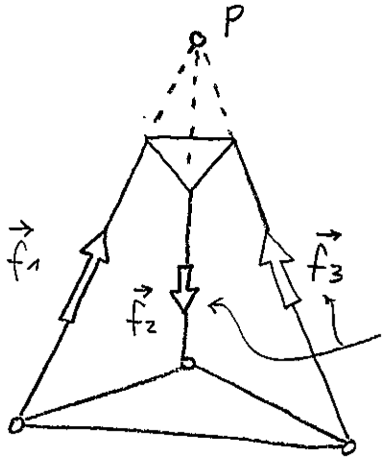
Parallel manipulators
Physical behaviour at a singularity

a.k.a. "self stresses"

	Static behaviour	Kinematic behaviour
Kernel elements	a) $\exists$ internal forces (infinitely-many $\vec{\lambda}$ that produce $\hat{w}=0$)	e) $\exists$ uncontrollable twists (infinitely-many $\hat{T}$ that correspond to $\vec{v}=0$)
Many-to-one relationships	b) An equilibrable wrench $\hat{w}\neq 0$ on the effector can be withstanded by infinitely-many leg forces $\vec{\lambda}$ (this is a consequence of $\boxed{a}$)	f) A feasible joint speed vector $\vec{v}\neq 0$ is compatible with infinitely-many end-effector twists (this is a consequence of $\boxed{b}$)
Impossible input or output	c) $\exists$ impossible $\hat{w}$. There exist wrenches $\hat{w}_{imp}$ that cannot be equilibrated by the robot. Loss of equilibrability	g) $\exists$ impossible $\vec{v}$. There exist joint speed vectors $\vec{v}_{imp}$ that are not compatible with any end-effector twist. There is a linear dependence between the values v_1, v_2, v_3. The manipulator is stressed if we don't satisfy such dependence
Near the singularity	d) Near the singularity the $\hat{w}_{imp}$ require very large leg forces. The legs or the actuators may break	h) Near the singularity the velocities $\vec{v}_{imp}$ produce large end-eff. twists. Small position errors at the joints produce large end-effector position errors

As in the serial arm, let us illustrate the meaning of the table cells. We only work out the main ones. First the column on the static behaviour.

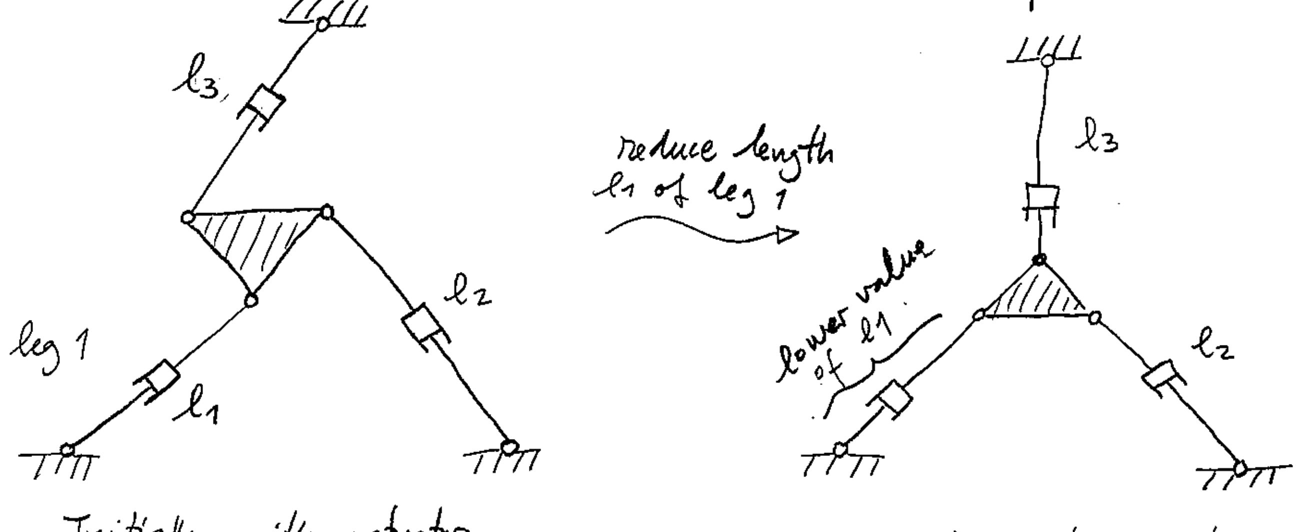
$\boxed{a}$ An internal force, or self-stress, is a set of leg forces (acting on the platform) that produce a zero net wrench. Here's one such self-stress



The concurrence of the three leg lines on P allows the existence of such self-stresses.

We can always choose $\vec{f}_2$ and $\vec{f}_3$ that equilibrate $\vec{f}_1$, without any $\hat{w}$ on platform.

Such a self-stress may produce the locking of the mechanism. This is shown in a video of the course web entitled "singularity effects in the 3-RPR robot". The video shows this sequence:



Initially, with actuators locked → mechanism is rigid. Leg 1 is unlocked then, and its length is reduced to bring the robot to a singularity

Once here, the mechanism is shaky with the actuators locked (effect of cell $\boxed{e}$). If we then push leg 1 to increase its length, a self stress locks the mechanism.

$\boxed{b}$ We don't explain this cell. Similar to $\boxed{b}$ in the serial robot, but using the statics interpretation.

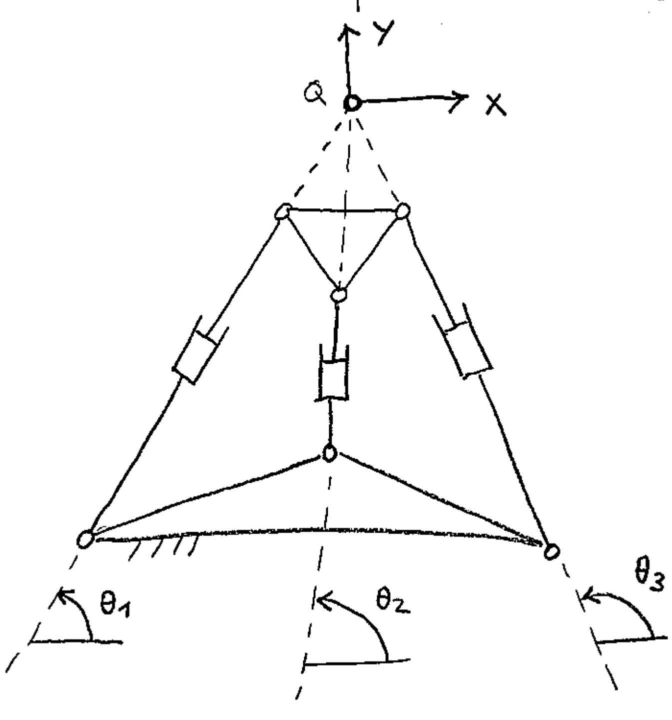
$\boxed{c}$ This was proved in module 2 "Statics". Only the wrenches through P are equilibrable.

$\boxed{d}$ See exercise 4 in Module 2 "Statics".

Now let us see the 2nd column on the kinematic behaviour. We only examine cell $\boxed{e}$, which is perhaps the most important.

$\boxed{e}$ There exist uncontrollable twists because infinitely-many $\hat{T}$ correspond to $\vec{v}=0$ (In practice we observe that the platform is "shaky" when we lock the actuators. See the previously-mentioned video.)

Let's find such $\hat{T}$. They are the $\hat{T}$ in Ker J^T, or equivalently, those that are orthogonal (reciprocal) to Im J.



The jacobian J in QXY is:

$$J = \begin{bmatrix} c_1 & c_2 & c_3 \\ s_1 & s_2 & s_3 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus:

$$\operatorname{Im} J = \overbrace{\left\langle \begin{bmatrix} c_1 \\ s_1 \\ 0 \end{bmatrix}, \begin{bmatrix} c_2 \\ s_2 \\ 0 \end{bmatrix} \right\rangle}^{\text{We assume here that the three legs are concurrent, and not parallel.}} = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\rangle$$

Therefore:

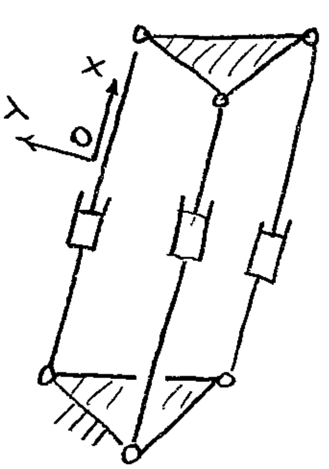
$$\operatorname{Ker} J^T = \left\langle \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$

And hence the twists of the form

$$\hat{T} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}$$

are uncontrollable. They clearly correspond to an instantaneous rotation about Q. This is the rotation we observe in class with a model of the 3-R$\underline{P}$R robot.

<u>EXERCISE</u>: Find the uncontrollable twists that arise when all legs are parallel. Hint: use the following coordinate system OXY:



<u>OBS</u>: In reality, <u>any</u> twist for the end-effector is uncontrollable (corresponds to infinitely-many $\vec{\tau}$). This is because if for a $\hat{T}$ we have

$$J^T \hat{T} = \vec{\tau}$$

then

$$J^T(\hat{T} + \underset{\operatorname{Ker} J^T}{\underset{\cap}{\hat{T}_{ker}}}) = \vec{\tau}$$

as well.

<u>OBS</u>: The shakiness of the platform (with actuators locked) also arises "near" the singularity, due to mechanical backlash of the robot parts. It is the cause of <u>increased position error</u> of the end effector.