Exercises Module 4

Duality

Problems

In what follows, English Engineering units are assumed throughout.

Problem 1

1. Compute the Plücker coordinates and the mutual moment of the lines $\$_i$ in the XY-plane and $\$$ perpendicular to the plane passing through point $(1, 1)$ in the following coordinate systems (see Fig. 1).

   (a) The original coordinate system OXY.

   (b) A parallel coordinate system where O is located first at $(1, 0)$ and afterwards at $(0, -1)$.

   (c) A coordinate system $O'X'Y'$ where the origin is located at $(1, 2)$ and the system of coordinates has been rotated $60°$ anticlockwise.

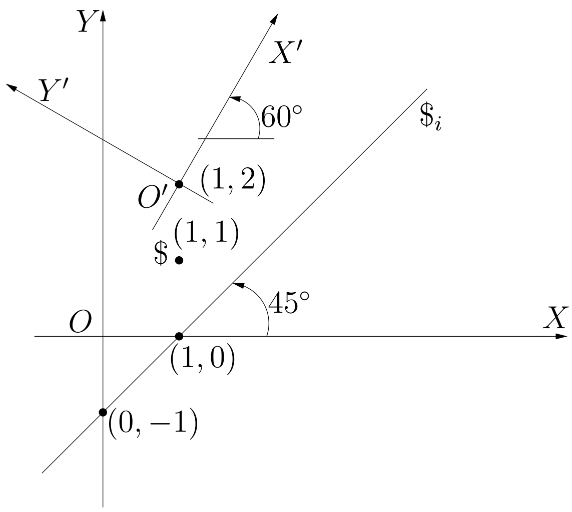


Figure 1: Mutual moments of two lines.

2. Consider a wheel that is rotating about point O, with angular speed ω (Fig. 2):

   (a) Suppose first that the wheel is rotating anticlockwise at $\omega = 10$ rad/s, and experiences an impulsive force of 5 lbf along the line shown in the figure. Compute the instant power gained by the system. Repeat the calculation for a clockwise rotation of $\omega = 10$ rad/s, and comment on your results.

   (b) The wrenches of constraint of the wheel form a two-dimensional vector space (also known as a "two-system") that can be characterized by the base forces with magnitudes f_1 and f_2 acting along the X and Y axes, as shown in the figure. Show that the impulsive force cannot be expressed as a linear combination of the base forces.

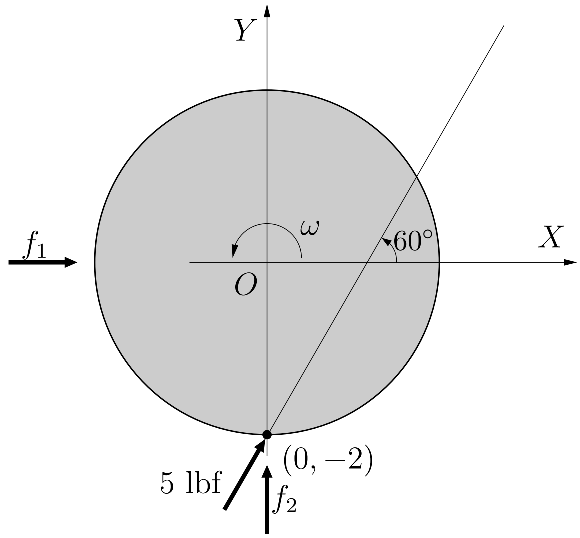


Figure 2: A rotating wheel and forces acting on it.

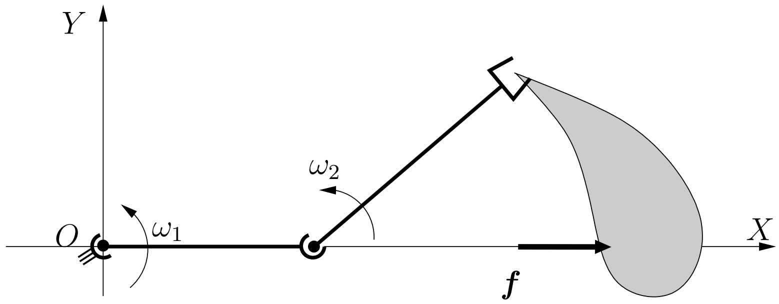


Figure 3: A $2R$ serial manipulator.

3. Compute the twist of the end effector of the $2R$ manipulator in Fig. 3 when $\omega_1 = 2$ rad/s, $\omega_2 = 8$ rad/s; $\omega_1 = -2$ rad/s, $\omega_2 = -8$ rad/s; $\omega_1 = -2$ rad/s, $\omega_2 = 8$ rad/s; and $\omega_1 = -8$ rad/s, $\omega_2 = 2$ rad/s. Show that, in general, the instant center must lie on the X axis and that any force along this axis is reciprocal to the end-effector twist.

4. Determine bases for the vector spaces of twists of freedom ($\mathbb{T}$) and wrenches of constraint ($\mathbb{W}$) for each one of the following bodies, relative to the indicated XY coordinate system in each case. Show that the bases you have chosen satisfy reciprocity.

   (a) The end effector of a $3R$ manipulator that maintains a point contact at P such that it is constrained to move along the X axis, as illustrated in Fig. 4.

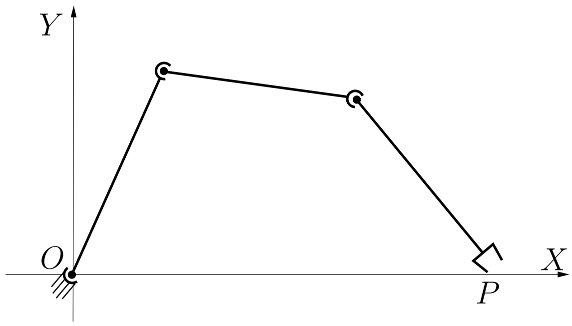


Figure 4: A $3R$ serial manipulator maintaining point contact at P.

(b) A beam that maintains contact with the X and Y axes at points A and B, as shown in Fig. 5(a).

(c) A beam that maintains contact at points A, B and D, as shown in Fig. 5(b).

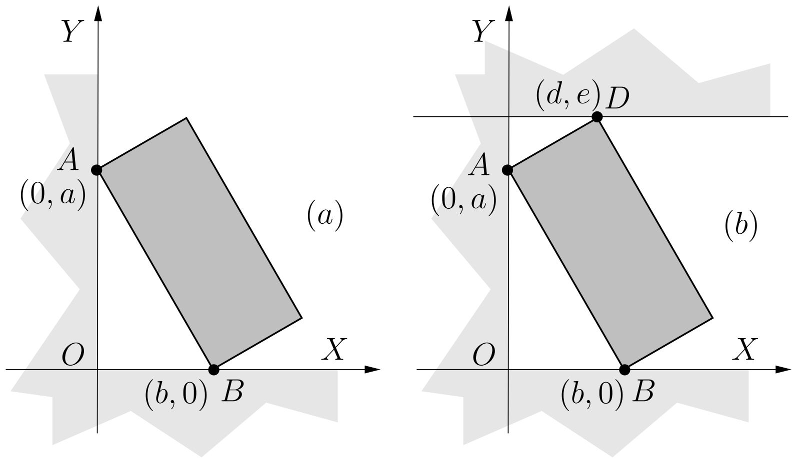


Figure 5: A beam maintaing (a) two and (b) three contact points.

Problem 2

1. Fig. 6 illustrates two positions of a $3R$ manipulator. All lengths are in feet.

   (a) A force of 5 lbf acts on the end effector as shown in the figure. Compute the corresponding resultant torques for the two positions and the equilibrating joint torques.

   (b) Assume that the end effector is in contact with some object and is in equilibrium with equilibrating joint torques $\tau_1 = 2$, $\tau_2 = 3$, and $\tau_3 = 4$ lbf·ft. Compute the external forces that act upon the end effector for the two positions.

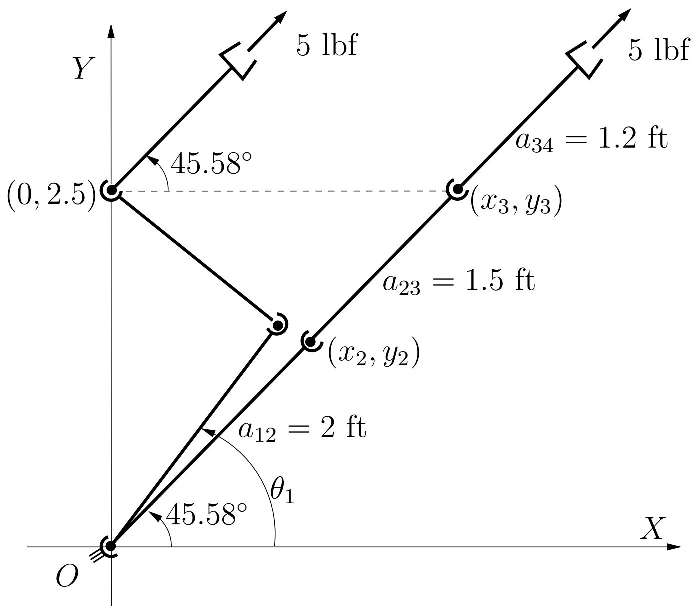


Figure 6: 3R serial manipulator.

2. The end effector of the $3R$ manipulator is instantaneously rotating about point C $(0, 2)$, as illustrated in Fig. 7. The velocity of point P is $v_P = 1$ in/s. Determine the twist of the end effector relative to the XY coordinate system. Compute the corresponding values for the joint speeds ω_1, ω_2, and ω_3 from the twist equation

$$\hat{T} = \omega_1 \cdot \hat{S}_1 + \omega_2 \cdot \hat{S}_2 + \omega_3 \cdot \hat{S}_3$$

   by forming reciprocal products with the coordinates of lines $\$_{23}$, $\$_{31}$, and $\$_{12}$, respectively. Compare your results with part 1 of Problem 1 of Module 3 (Instantaneous kinematics).

3. We require that the end effector of the RPR manipulator in Fig. 8 rotates on an instant center with coordinates $(2, 1)$ with a clockwise angular speed of 10 rad/s. Compute ω_1, v_2, and ω_3 by forming in turn reciprocal products of the twist equation with three lines; each line is reciprocal to pairs of joint lines, as shown in Fig. 8.

4. It has been shown that for a $3R$ manipulator, $\det J = a_{12}\hat{s}_{12}^T\hat{S}_3$, which is twice the area of the triangle 123 shown in Fig. 9.

   (a) Deduce that $\det J = a_{12}a_{23}\sin\theta_2$.

   (b) Deduce that the maximum value for $\det J$ is $\det J_{max} = a_{12}a_{23}$ and that $\lambda = \frac{\det J}{\det J_{max}} = \sin\theta_2$.

   (c) Sketch the graph $\lambda = \sin\theta_2$. for $0 \leq \theta_2 \leq 360°$. Draw the configurations of the manipulator for $\lambda = 0$ and $\lambda = \pm 1$.

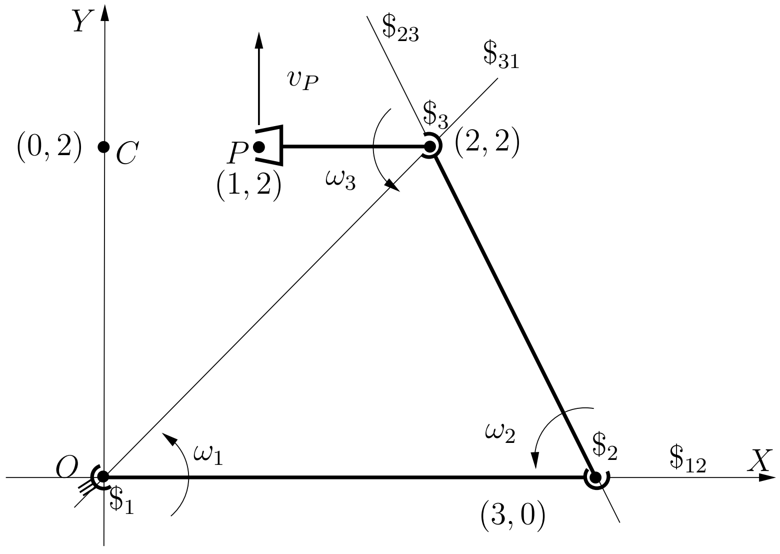


Figure 7: A 3R serial manipulator.

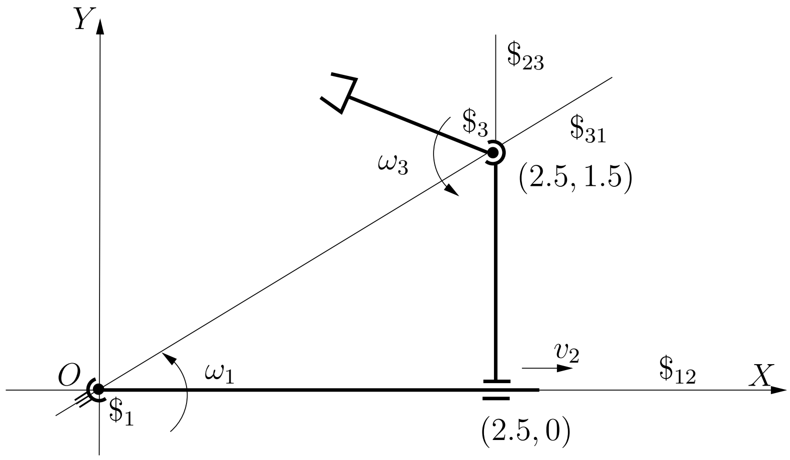


Figure 8: A RPR serial manipulator.

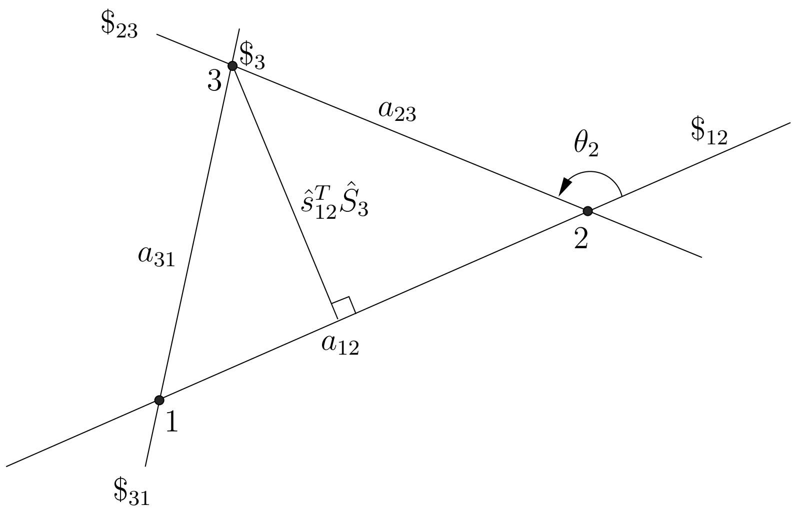


Figure 9: Triangle of lines of a $3R$ manipulator.

(Note: λ is independent of the manipulator dimensions.)

Problem 3

1. Recall the two $3RPR$ manipulators of Fig. 10, studied in Problem 3 of Module 2 "Statics". Recall that in that problem we computed the wrench $\hat{w}$ associated to each one of the forces indicated in Fig. 10, together with the unitized coordinates $\hat{s}_1$, $\hat{s}_2$, and $\hat{s}_3$ of the manipulators' legs. Solve questions 1b, 1c, 2b, and 2c of that problem, by isolating f_1, f_2, and f_3 from the force equation

$$\hat{w} = f_1 \cdot \hat{s}_1 + f_2 \cdot \hat{s}_2 + f_3 \cdot \hat{s}_3$$

by forming reciprocal products.

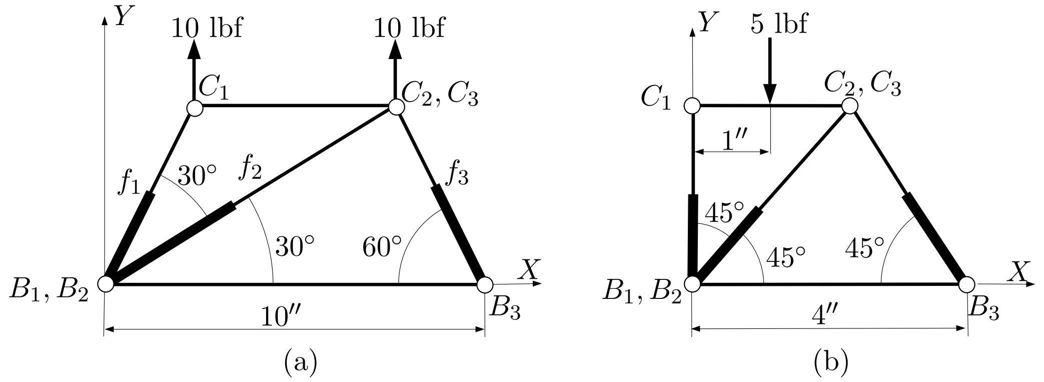


Figure 10: Two 3RPR manipulators.

2. The upper platform of the robotic mechanism shown in Fig. 11 is in contact equilibrium with another body. Anticlockwise joint torques of 1 lbf inch are applied first at the three base joints, then at the three intermediate joints, and finally at the three upper platform joints. Compute the corresponding external forces acting upon the upper platform for each of these three cases. Assume that the point $(1)_1$ is the origin of the coordinate system.

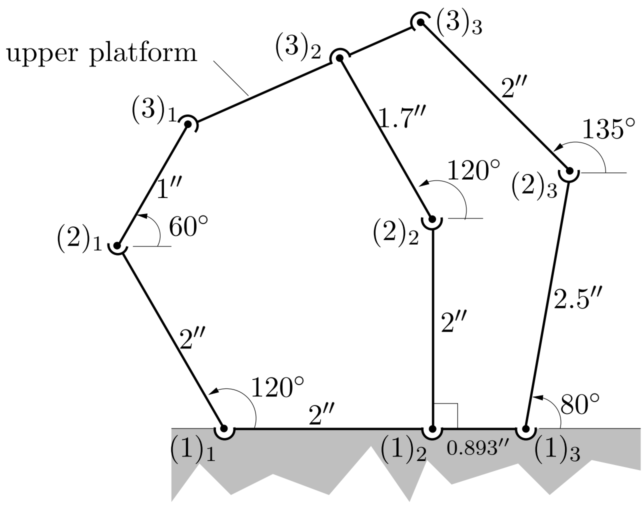


Figure 11: A 3RRR parallel manipulator.

Problem 4

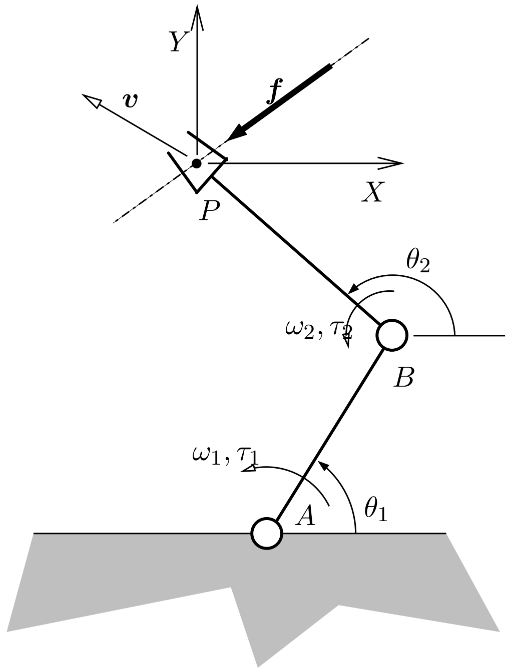


Figure 12: A 2R serial manipulator. While $\boldsymbol{v}$ is the velocity of point P, $\boldsymbol{f}$ is the force applied on P by the environment.

The goal of this exercise is to check that all the phenomena mentioned in the last table of "The duality diagram of a serial manipulator" arise in practice. To easily visualize the phenomena we consider the simple 2R robot of Fig. 12, whose link lengths AB and BP are equal to one. Since it is a 2-revolute manipulator we speak of end-effector velocities $\boldsymbol{v}$ and forces $\boldsymbol{f}$, instead of twists $\hat{T}$ and wrenches $\hat{w}$, and use the kinetostatic relationships

$$\boldsymbol{v} = \boldsymbol{J}_r \boldsymbol{\gamma} \qquad \boldsymbol{\tau} = {\boldsymbol{J}_r}^{\mathsf{T}} \boldsymbol{f}$$

given in the bottom note, where $\boldsymbol{J}_r$ is the so-called reduced jacobian. Using $\boldsymbol{J}_r$, determine the singular configurations of the robot and

(a) Find the internal velocities $\boldsymbol{\gamma}$.

(b) Find the forces of constraint $\boldsymbol{f}$.

(c) Prove that a same velocity $\boldsymbol{v}$ can be produced by infinitely-many joint angular velocities.

(d) Prove that a same joint torque $\boldsymbol{\tau}$ can be equilibrated by infinitely-many end-effector forces $\boldsymbol{f}$.

(e) Find the impossible end-effector velocities.

(f) Find the non-equilibrable joint torques.

(g) Select an impossible end-effector velocity $\boldsymbol{v}$ and show that it requires very high joint speeds $\boldsymbol{\gamma}$ as we approach the singularity.

(h) Select a non-equilibrable joint torque $\boldsymbol{\tau}$ and show that it requires very large end-effector forces $\boldsymbol{f}$ as we approach the singularity.

Note: To obtain the relationships $\boldsymbol{v} = \boldsymbol{J}_r \boldsymbol{\gamma}$ and $\boldsymbol{\tau} = {\boldsymbol{J}_r}^{\mathsf{T}} \boldsymbol{f}$ we place the origin O of the coordinate system at point P of the end effector. In this way, the end-effector twist will be $\hat{T} = [v_x, v_y, \omega]^{\mathsf{T}}$, where v_x and v_y

are the X and Y components of the velocity $\boldsymbol{v}$ of point P, and ω is the angular velocity of the effector. The components of A and B in the coordinate system PXY are:

$$\begin{aligned} x_A &= -(\cos\theta_1 + \cos\theta_2) \qquad & x_B &= -\cos\theta_2 \\ y_A &= -(\sin\theta_1 + \sin\theta_2) \qquad & y_B &= -\sin\theta_2 \end{aligned}$$

and thus the standard relation $\hat{T} = \boldsymbol{J}\boldsymbol{\gamma}$ given in the course is

$$\begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix} = \begin{bmatrix} -(\sin\theta_1 + \sin\theta_2) & -\sin\theta_2 \\ \cos\theta_1 + \cos\theta_2 & \cos\theta_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \tag{1}$$

However, since $\operatorname{rank}(\boldsymbol{J}) = 2$ at most, the robot cannot generate all possible twists $\hat{T} \in \mathbb{R}^3$. Thus, it is better to view the 2R robot as a system that allows to control $\boldsymbol{v} = [v_x, v_y]^\mathsf{T}$ through commanding $[\omega_1, \omega_2]^\mathsf{T}$. Therefore, the relevant part of Eq. 1 is:

$$\boldsymbol{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} -(\sin\theta_1 + \sin\theta_2) & -\sin\theta_2 \\ \cos\theta_1 + \cos\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \boldsymbol{J}_r\boldsymbol{\gamma} \tag{2}$$

where $\boldsymbol{J}_r$ is called the reduced Jacobian of the manipulator.

From a statics viewpoint, the 2R robot should also be viewed as a system that allows to equilibrate the force $\boldsymbol{f} = [f_x, f_y]^\mathsf{T}$ applied on P, by commanding the joint torques $\boldsymbol{\tau} = [\tau_1, \tau_2]^\mathsf{T}$. The relation between $\boldsymbol{\tau}$ and $\boldsymbol{f}$ is given by $\boldsymbol{\tau} = \boldsymbol{J}^\mathsf{T}\hat{w}$, where $\boldsymbol{J}^\mathsf{T}$ is the standard static Jacobian explained in the course. In the coordinate system of the figure, the relation is

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -(\sin\theta_1 + \sin\theta_2) & \cos\theta_1 + \cos\theta_2 & 1 \\ -\sin\theta_2 & \cos\theta_2 & 1 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} \tag{3}$$

which is equivalent to:

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -(\sin\theta_1 + \sin\theta_2) & \cos\theta_1 + \cos\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix} = {\boldsymbol{J}_r}^\mathsf{T}\boldsymbol{f} \tag{4}$$

Note that the singularity condition of the 2R robot is thus $\det \boldsymbol{J}_r = \det {\boldsymbol{J}_r}^\mathsf{T} = 0$.

Problem 5

Fig. 13 shows a CNC drilling machine based on a parallel 3-$\underline{P}$RR robot. The prismatic joints are actuated and the revolute joints are passive.

1. Using the coordinate system OXY, compute the $\boldsymbol{A}$ and $\boldsymbol{B}$ matrices of the velocity equation

$$\boldsymbol{A}\,\hat{T} = \boldsymbol{B}\,\boldsymbol{\gamma}, \tag{5}$$

where $\hat{T} = [v_{ox}, v_{oy}, \omega]^\mathsf{T}$ is the end-effector twist, and $\boldsymbol{\gamma} = [\dot{l}_1, \dot{l}_2, \dot{l}_3]^\mathsf{T}$ is the vector of prismatic joint velocities.

2. In the shown configuration the drill bit is in contact with a planar surface. This surface applies a reaction wrench $\hat{w}$ on the drill. The manipulator is equipped with load sensors providing the

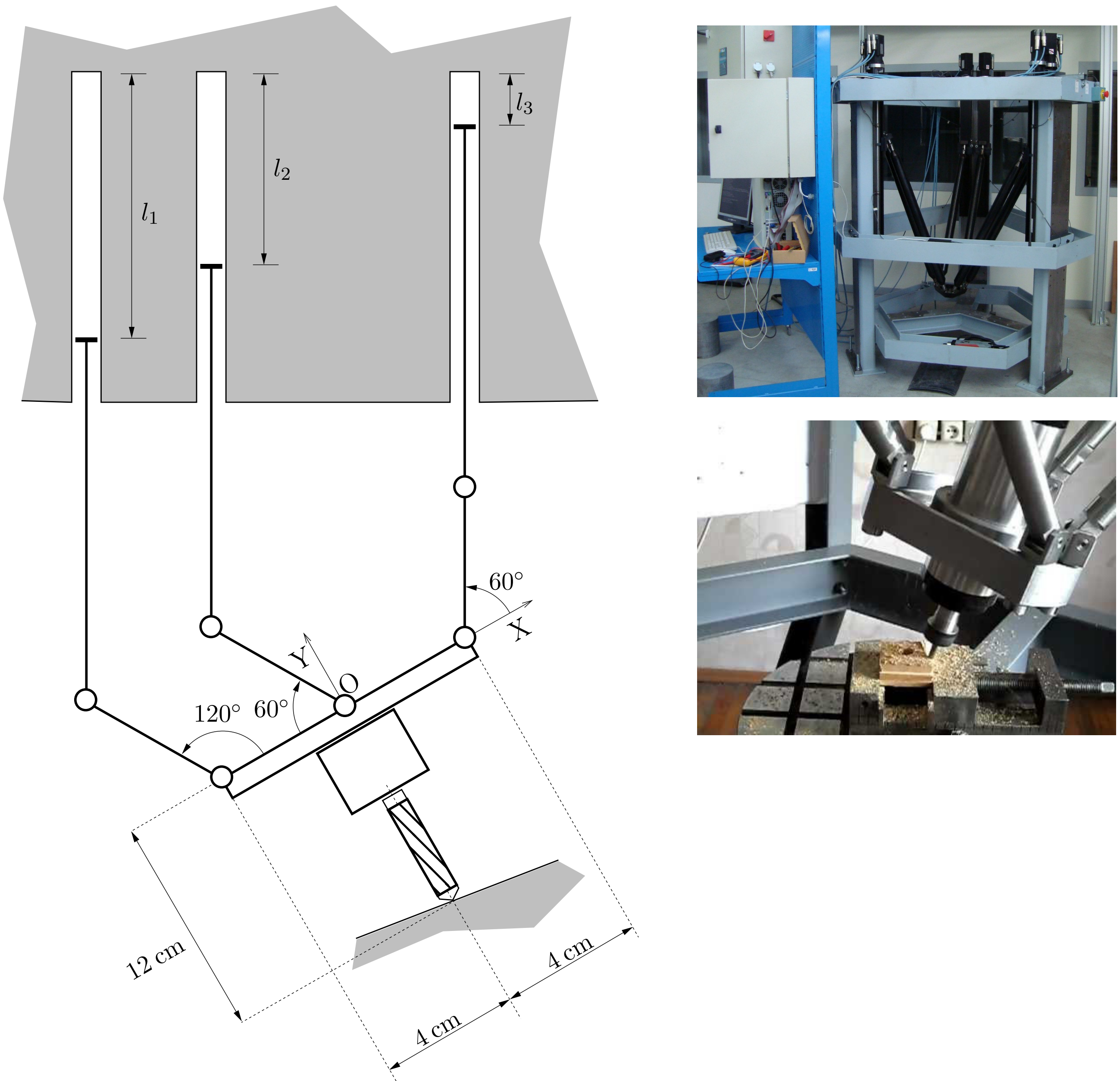


Figure 13: Left: A CNC drilling machine based on a fully parallel 3-$\underline{P}$RR robot. Right-top: A spatial version of this machine build at Universidad Pública de Navarra: `http://www.imac.unavarra.es/ cnc/`. Right-bottom: Detail of a drill mounted on a similar platform.

magnitudes of the resultant forces on the prismatic joints f_1, f_2, and f_3. Determine the value of $\hat{w}$ assuming that the current sensor readings are

$$\boldsymbol{\tau} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 0.627081 \\ -0.102857 \\ 1.257790 \end{bmatrix} \text{ N.}$$

Verify that the action line of $\hat{w}$ meets the tip point of the drill bit.

3. We want to drill a hole orthogonal to the surface. The contact bewteen the drill bit and the surface is punctual and frictionless. Use the geometric information encoded in $\hat{w}$ to determine the infinitesimal displacement $\delta\hat{D}$ that places the drill bit orthogonally to the surface.

4. By multiplying Eq. (5) by a small time increment δt, we obtain the relationship between the small displacements $\delta\hat{D}$ of the end effector, and the small displacements $\delta\boldsymbol{l} = [\delta l_1, \delta l_3, \delta l_3]^\mathsf{T}$ of the actuators. Compute the displacements $\delta\boldsymbol{l}$ required to perform the small correction $\delta\hat{D}$ obtained in point 3.

5. Suppose that we rotate the prismatic joint axes 90°, as shown in Fig. 14, keeping the rest of the configuration as in Fig. 13. Show that the drill loses dexterity in this new design. In particular, show that the robot will be unable to perform the small correction $\delta\hat{D}$ obtained in point 3.

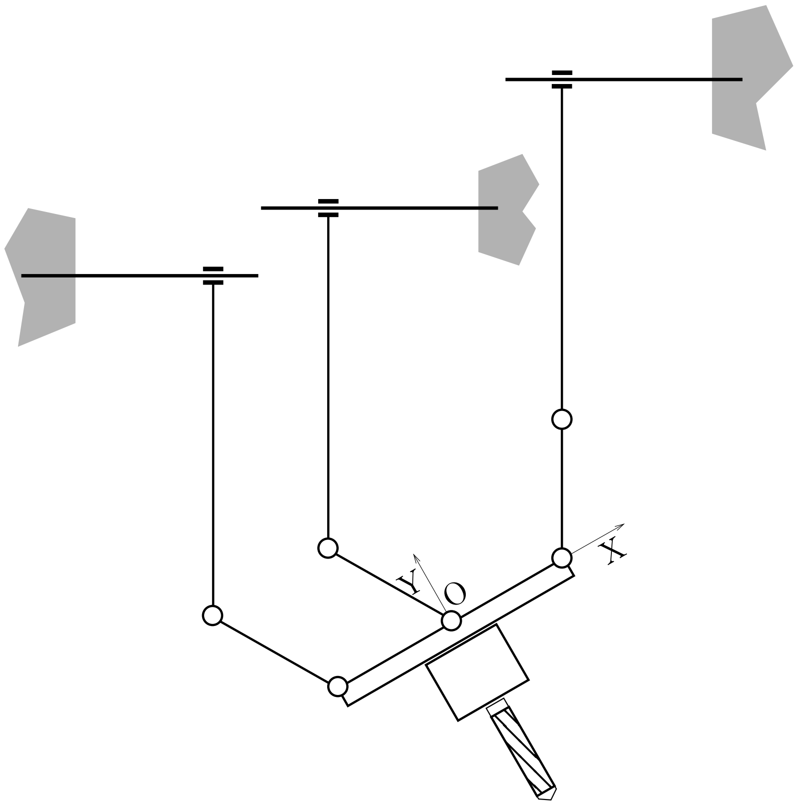
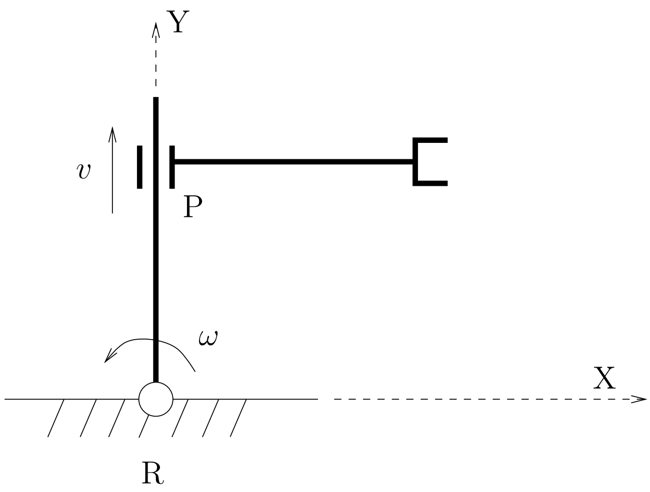


Figure 14: A 3-PRR CNC drilling machine with horizontal prismatic joints.

Short questions

Short question 1
Consider the following RP manipulator, in the drawn configuration, where joint R rotates with angular velocity ω, and joint P translates with linear velocity v in the direction of Y.



Which one of the following statements is false?

1. The twists of freedom of the end effector have the form $[0, v; \omega]^T$.
2. The wrenches of constraint of the end effector have the form $[F, 0; 0]^T$.
3. The twists of freedom of the end effector form a vector space of dimension two.
4. The wrenches of constraint of the end effector form a vector space of dimension one.
5. The end effector can support any couple without having to actuate the joints.

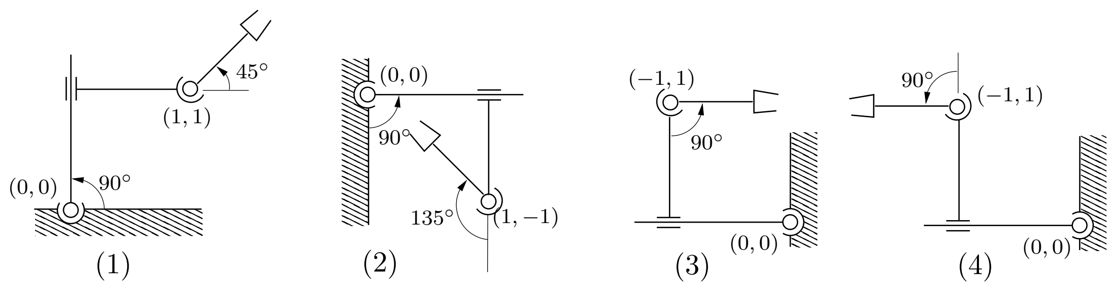
Short question 2

Remember that the static and kinematic behaviour of a serial manipulator is given by the relations $\boldsymbol{\tau} = J^T \cdot \hat{w}$, and $\hat{T} = J \cdot \boldsymbol{\gamma}$, respectively. If the manipulator is in a singularity, which one of the following statements is false?

1. There are non-zero joint velocities $\boldsymbol{\gamma}$ that produce a zero twist $\hat{T}$ at the end effector.
2. Some twists $\hat{T}$ can be produced by many joint velocities $\boldsymbol{\gamma}$.
3. As we are approaching a singularity, there are small twists $\hat{T}$ that require very large joint velocities.
4. At a singularity, the end effector can support certain wrenches $\hat{w}$, without actuating the joints.
5. There are non-zero joint torques $\boldsymbol{\tau}$ that produce a zero resultant wrench $\hat{w}$ at the end effector.

Short question 3

Consider the following four configurations of the same RPR manipulator. All of the links are one meter long.



If d_i is the value of $det(J)$ for configuration (i), it holds that

1. $d_1 > d_2 > d_3 > d_4$.
2. $d_4 > d_3 > d_2 > d_1$.
3. $d_1 = d_2 = d_3 = d_4$.
4. $d_1 = d_2$, and $d_3 = -d_4$.
5. No one of the previous answers is right.

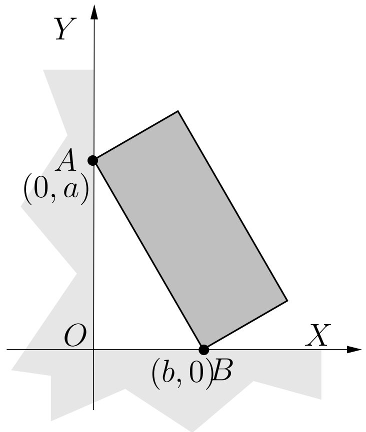
Short question 4

Consider a 3R serial robot. If J is the Jacobian that transforms joint velocities $\boldsymbol{\gamma}$ into end effector twists $\hat{T}$, which one of the following statements is true when the robot moves from a non-singular to a singular configuration?

1. Dim(Ker J) increases, Dim(Im J) decreases, Dim(Ker ${}^{\mathsf{T}}J$) increases, Dim (Im ${}^{\mathsf{T}}J$) decreases.
2. Dim(Ker J) decreases, Dim(Im J) increases, Dim(Ker ${}^{\mathsf{T}}J$) decreases, Dim (Im ${}^{\mathsf{T}}J$) increases.
3. Dim(Ker J) increases, Dim(Im J) increases, Dim(Ker ${}^{\mathsf{T}}J$) decreases, Dim (Im ${}^{\mathsf{T}}J$) decreases.
4. Dim(Ker J) decreases, Dim(Im J) decreases, Dim(Ker ${}^{\mathsf{T}}J$) increases, Dim (Im ${}^{\mathsf{T}}J$) increases.
5. The four spaces Ker J, Im J, Ker ${}^{\mathsf{T}}J$, and Im ${}^{\mathsf{T}}J$ decrease their dimension.

Short question 5

The beam maintains contact with the wall and the ground at points A and B shown in the figure ($a = 3$m y $b = 2$m), and is falling towards the ground, with angular velocity $\omega = 1$ rad/s. It can be assumed that there is no friction at the contacts.



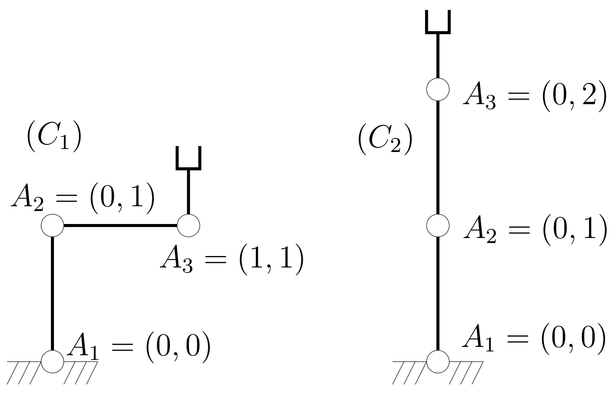
What happens if we apply with a robot one of these two wrenches on the beam?

$$\hat{w}_1 = [-2\ N, -3\ N; 0\ Nm]^T$$
$$\hat{w}_2 = [-1\ N, 0\ N; 0\ Nm]^T$$

1. We can not determine what will happen, as the application points of the forces are unknown.
2. Applying $\hat{w}_2$ we tend to stop the fall.
3. Applying $\hat{w}_1$ we tend to accelerate the fall.
4. Applying any one of the two forces we tend to stop the fall.
5. No one of the previous answers is right.

Short question 6

Consider the following two configurations of a 3R manipulator, with the indicated coordinates for the joints (anywhere SI units are used).



Which one of the following statements is false?

1. At configuration C_1, the space of twists of freedom of the end effector has dimension 3, and the space of wrenches of constraint has dimension 0.

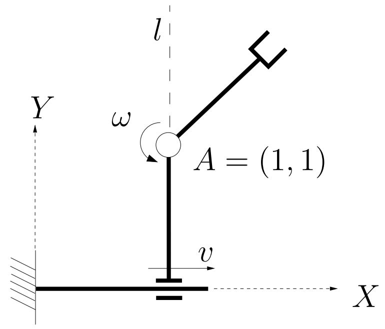
2. At configuration C_2, the space of twists of freedom of the end effector has dimension 2, and the space of wrenches of constraint has dimension 1.

3. At configuration C_2, the manipulator can support structurally the wrench $\hat{w} = {}^{\mathsf{T}}[-1, 0, 2]$ applied at the end effector.

4. At configuration C_1, a wrench $\hat{w} = {}^{\mathsf{T}}[-1, 0, 1]$, applied at the end effector, produces a vector of resultant couples at the joints of $\vec{\tau} = {}^{\mathsf{T}}[1, 0, 0]$.

5. At configuration C_2, there are joint velocities that produce a zero twist at the end effector.

Short question 7
Consider the following PR manipulator:



(Coordinates in meters)

Be $\mathbb{T}$ and $\mathbb{W}$, respectively, the space of twists of freedom and the space of wrenches of constraint of the end effector, given in the XY coordinate system displayed in the figure. Indicate which of the following statements are true:

A1: In “axis” coordinates,

$$\mathbb{T} = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \rangle$$

A2: In “ray” coordinates,

$$\mathbb{W} = \langle \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix} \rangle$$

A3: Any force applied at the end effector actuating on line l is structurally supportable.

A4: If $v = 1$ m/s, and $\omega = 2$ rad/s, then the instant center of rotation of the end effector will be at point $(1, \frac{3}{2})$.

A5: For every $\hat{w} \in \mathbb{W}$ and every $\hat{T} \in \mathbb{T}$, it will be ${}^{\mathsf{T}}\hat{w} \cdot \hat{T} = 0$.

The correct statements are:

1. All of the statements.

2. None of the statements.

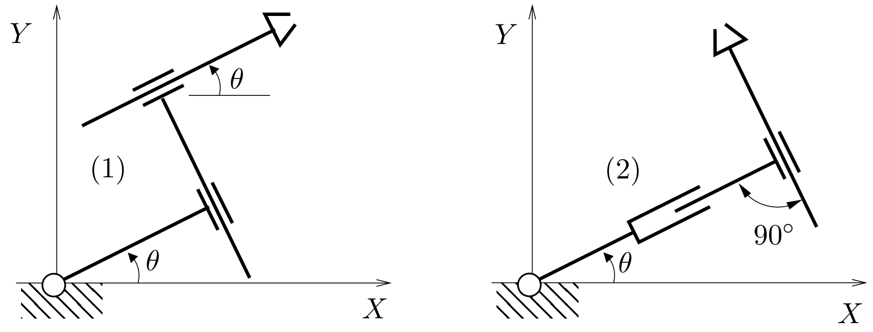
3. Only A1, A4, and A5.

4. Only A2 and A4.

5. Only A1, A3, A4, and A5.

Short question 8

Consider the two serial robots RPP in the following figure:



Mark the false statement:

1. Robot 2 can support structurally a wrench $\hat{w} = {}^{\mathsf{T}}[-\cos\theta, -\sin\theta, 0]$ applied on the effector (that is, without having to actuate the joints).
2. The determinant of the Jacobian of robot 1 has always an opposite sign to that of robot 2.
3. Robot 1 can never be in a singular configuration.
4. Robot 2 can never be in a singular configuration.
5. Neither robot 1 nor robot 2 can have internal velocities.