Module 5
Introduction to Hybrid Control of Force and Position

Contents

Objective of this Module

A workpiece held by the gripper of a robot is in contact with the environment. This module presents hybrid control techniques to simultaneously control the position of the workpiece relative to the environment, and the force it exerts against such environment.

5.1 Introduction

Consider a robot that holds a workpiece with its gripper or end effector. Hybrid control techniques allow to simultaneously control:

- The displacements of the workpiece with respect to the environment.
- The forces that the workpiece applies to the environment.

To understand when hybrid control techniques are applicable, it is useful to start considering the extreme situations of complete restraint and complete freedom illustrated in Fig. 5.1. In Fig. 5.1(a), the workpiece is rigidly connected to the environment E, and it is possible to control any force that the robot may exert on E, as E will always provide the necessary reaction to guarantee equilibrium. However, the manipulator will be unable to control the displacements of the workpiece, because its movements are fully constrained. In Fig. 5.1(b), the workpiece is free to move in open space, and hence it is possible to control its three degrees of freedom. However, the manipulator cannot apply any wrench to the workpiece, as there is no environment available that can provide the necessary reaction. Whereas in the first situation force control is the strategy to follow, position control is the option of choice in the second case. The correcting actions to be taken in each case would be computed as follows:

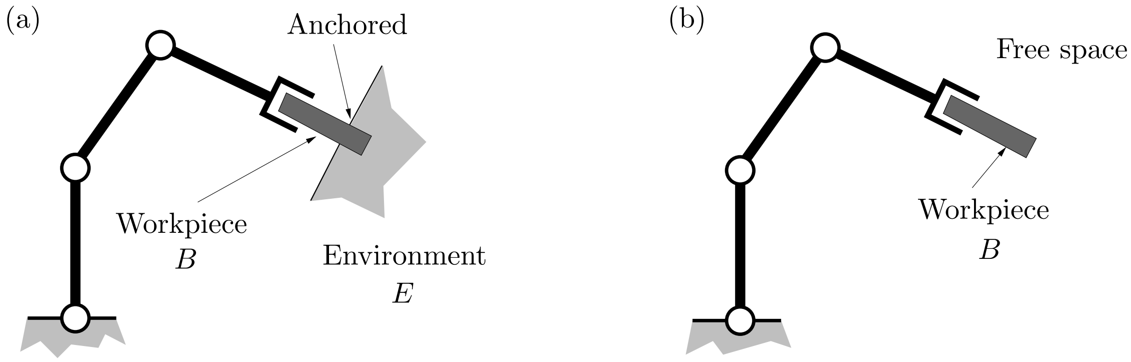


Figure 5.1: Two extreme scenarios: (a) The workpiece is anchored to the environment and there is complete restraint. (b) The workpiece is held by the robot in free space, and there is complete freedom.

Complete restraint Let us assume that in the situation of Fig. 5.1(a), the robot is in static equilibrium, and that the workpiece currently applies a wrench $\hat{w}_{cur}$ to the environment. The equilibrant torques that produce this wrench are

$$\boldsymbol{\tau}_{cur} = \mathbf{J}^{\mathsf{T}} \hat{w}_{cur}. \tag{5.1}$$

Thanks to this relation, and to the motor torque sensors that measure $\boldsymbol{\tau}_{cur}$, we can compute $\hat{w}_{cur}$ and thus know the force error with respect to the desired wrench $\hat{w}_{des}$:

$$\delta\hat{w} = \hat{w}_{des} - \hat{w}_{cur}.$$

The equilibrant torques that would be required to provide $\hat{w}_{des}$ are

$$\boldsymbol{\tau}_{des} = \mathbf{J}^{\mathsf{T}} \hat{w}_{des}, \tag{5.2}$$

and substracting Eq. (5.2) from Eq. (5.1) we find that the small variation $\delta\boldsymbol{\tau}$ required to correct the force error $\delta\hat{w}$ is

$$\delta\boldsymbol{\tau} = \boldsymbol{\tau}_{des} - \boldsymbol{\tau}_{cur} = \mathbf{J}^{\mathsf{T}} \delta\hat{w}.$$

Complete freedom Analogously, if in Fig. 5.1(b) we know that the position error is $\delta\hat{D}$, where $\delta\hat{D}$ is an infinitesimal displacement of the end effector (Appendix A), then we can compute the small joint angle variation $\delta\boldsymbol{\theta}$ that corrects this error as follows

$$\delta\boldsymbol{\theta} = \mathbf{J}^{-1} \delta\hat{D}, \tag{5.3}$$

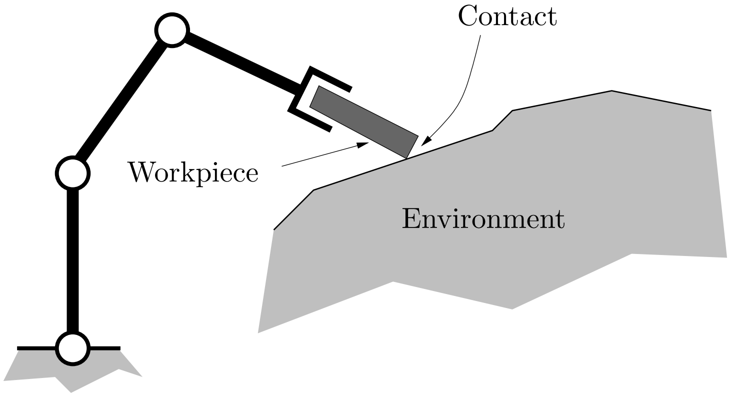


Figure 5.2: An intermediate scenario with a partial constraint on the motions of the workpiece.

where $\delta\boldsymbol{\theta} = [\delta\theta_1, \delta\theta_2, \delta\theta_3]^\mathsf{T}$. Note that Eq. (5.3) is obtained by multiplying $\boldsymbol{\gamma} = \mathbf{J}^{-1}\hat{T}$ by a small time increment (Appendix B).

Between the previous extreme scenarios, we may find situations in which the environment only partially constrains the motions of the workpiece, as in Fig. 5.2. In these situations one needs to simultaneously control:

- The force that the workpiece exerts on the environment.
- The position of the workpiece with respect to the environment.

This occurs for example in robotized welding. The robot has to lay a welding bead along a path, while maintaining the pressure against the welded part within acceptable limits, and controlling the position at every instant. The primary question that arises is: what variations of force and position can we make, and thus control, while maintaining the workpiece-environment contact?

5.2 Controllable position and force variations

Let us assume that in Fig. 5.2 the contact between the workpiece and the ground is punctual and frictionless. This means that the workpiece-ground contact can be modelled as a PR chain, as displayed in Fig. 5.3.

Note that in order to maintain the contact, it only makes sense to command workpiece velocities that correspond to the twists of freedom of the workpiece in the equivalent PR chain. Thus, the only contact-preserving displacements $\delta\hat{D}$ of the workpiece are those related to these twists of freedom.

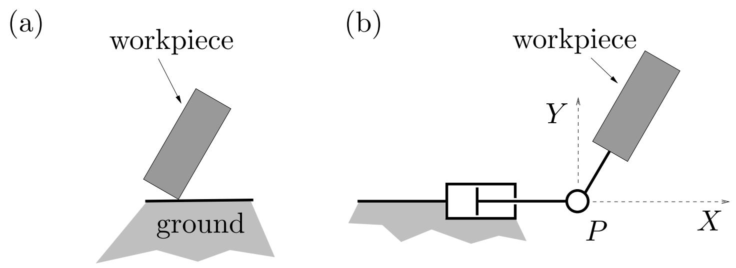


Figure 5.3: Workpiece-ground contact (a) and its kinematic model (b).

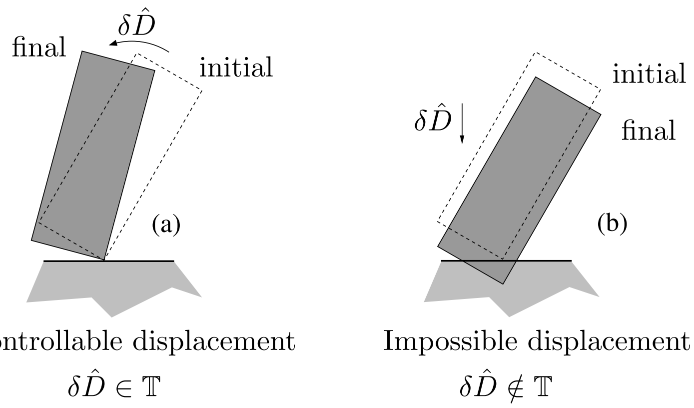


Figure 5.4: Controllable (a) and impossible (b) displacements.

The space $\mathbb{T}$ of twists of freedom of the workpiece is, using the coordinate system displayed in Fig. 5.3,

$$\mathbb{T} = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rangle.$$

Thus, a displacement $\delta\hat{D}$ of the workpiece relative to the ground will be controllable if $\delta\hat{D} \in \mathbb{T}$, i.e., if

$$\delta\hat{D} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$$

for some $[\varepsilon_1, \varepsilon_2]^\mathsf{T} \in \mathbb{R}^2$. This is illustrated in Fig. 5.4.

In a similar way, note that the workpiece may only exert against the ground those forces that are structurally supportable by the workpiece, seen

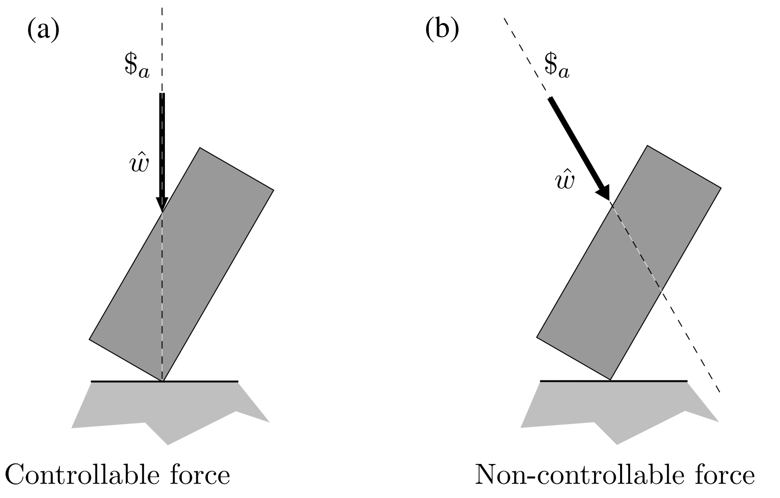


Figure 5.5: (a) Controllable force: the environment provides the necessary reaction to control it. (b) Non-controllable force: the environment cannot provide the necessary reaction.

as a component of the equivalent PR mechanism (Fig. 5.5). These forces are precisely the wrenches of constraint of the workpiece in the PR chain, i.e. the wrenches of the space

$$\mathbb{W} = \langle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rangle$$

whose action line is $\$_a$. Thus, we will be able to command force variations $\delta\hat{w}$ such that $\delta\hat{w} \in \mathbb{W}$, i.e., such that

$$\delta\hat{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mu$$

for some $\mu \in \mathbb{R}$. Table 1 summarizes the contact-preserving displacements $\delta\hat{D}$ and the contact-compliant force increments $\delta\hat{w}$ that it makes sense to control under this and other contact models.

One might think that to control the position of the workpiece and the force that it exerts against the environment it is only necessary to translate the position error and the force error into correcting variations of the joint positions and torques, according to $\delta\hat{w} = \hat{w}_{des} - \hat{w}_{cur}$ and $\delta\boldsymbol{\theta} = \mathbf{J}^{-1}\delta\hat{D}$, but this option is not possible as a motor cannot be controlled simultaneously in force and position. To control the force and position of the workpiece it

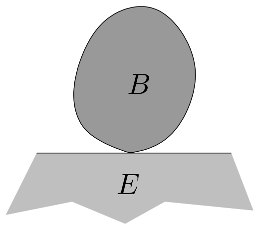
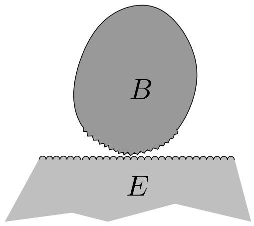
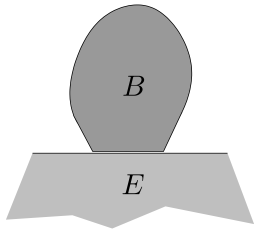
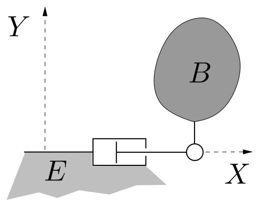
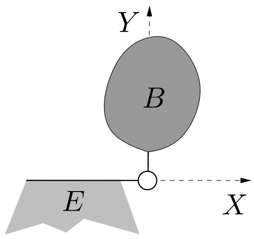
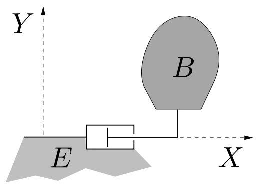
Contact model	Frictionless point contact	Point contact with friction	Frictionless planar contact
Representation			
Kinematic model			
Controllable displacements $\delta\hat{D}$ of B relative to E	$\langle \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \rangle$	$\langle \begin{bmatrix}0\\0\\1\end{bmatrix} \rangle$	$\langle \begin{bmatrix}1\\0\\0\end{bmatrix} \rangle$
Controllable force variations $\delta\hat{w}$ of B on E	$\langle \begin{bmatrix}0\\1\\0\end{bmatrix} \rangle$	$\langle \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \rangle$	$\langle \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \rangle$

Table 1: Controllable displacements and force variations under different contact models, in the indicated coordinate systems.

is necessary to adopt a *control strategy* that reduces the problem to provide only position commands, or only force commands, to the robot actuators. These are the so-called *hybrid control* strategies.

We note that the term “Hybrid Control” is used in other contexts as well, to name mixed analogic-digital control schemes. In our context, the meaning of “hybrid” is pretty different, and solely refers to the need of simultaneously controlling forces and positions.

In the remainder of this module we will derive a hybrid control strategy that reduces the correcting actions to position commands of the robot joints

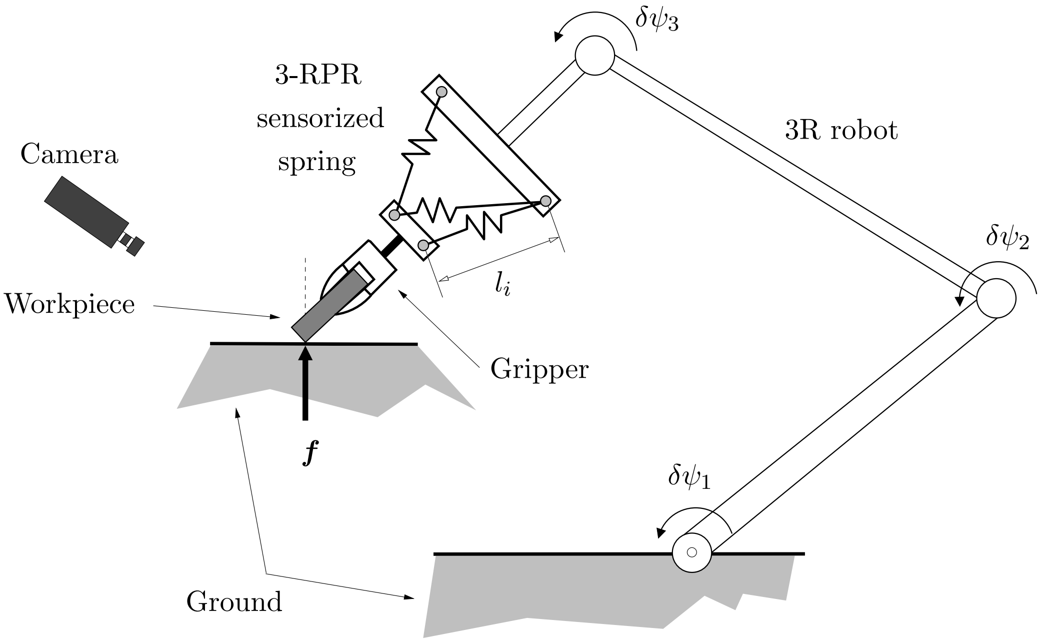


Figure 5.6: A serial 3R robot equipped with a force sensor in its wrist. This sensor has the structure of a 3-RPR parallel mechanism, where the linear actuators have been replaced by a linear spring, whose force can be measured.

exclusively. For concreteness, we will study the strategy in the particular system shown in Fig. 5.6, in which the gripper is pressing the workpiece against the ground. The goal is to control the force that the workpiece exerts on the ground, and also the relative position of the workpiece with respect to such ground. To measure the workpiece-ground contact force, we mount a 3-RPR spring in the wrist of the 3R robot, equipped with sensors that provide the lengths l_i of the individual springs. Moreover, the position of the workpiece is measured using a camera that provides visual feedback of the scene. From the force feedback provided by the sensorized 3-RPR spring, and from the position feedback of the workpiece given by the camera, we will design a control strategy that translates the force and position errors of the workpiece into position commands that correct these errors, for the 3R robot joints. To develop such a strategy it is first necessary to find the relationship between the force applied to a 3-RPR spring, and the linear displacements of its three legs.

5.3 Rigidity analysis of 3-RPR springs

Consider the 3-RPR mechanism in Fig. 5.7 where the P joints are not actuated, but coupled with springs of elastic constants k_1, k_2, and k_3. Asume that the platform is in equilibrium under the action of an external force $\boldsymbol{f}$, of wrench $\hat{w}$ and magnitude f, on a line \$. Then we have:

$$\hat{w} = f_1\hat{s}_1 + f_2\hat{s}_2 + f_3\hat{s}_3, \tag{5.4}$$

where each f_i is the magnitude of the resultant force on leg "i" and $\hat{s}_i$ is the unit coordinate vector of leg "i".

A small change $\delta\hat{w}$ in the applied force will cause a small displacement $\delta\hat{D}$ of the platform with respect to the ground. The quantities $\delta\hat{w}$ and $\delta\hat{D}$ are related by a 3×3 matrix $\mathbf{K}$, called the *rigidity matrix*.

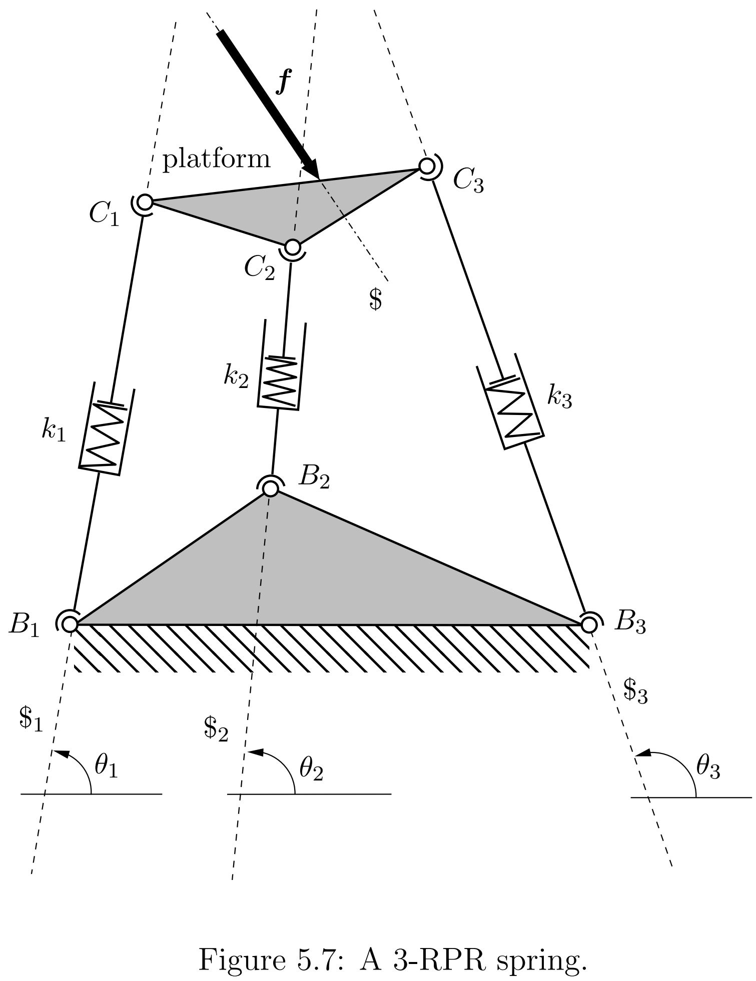


Figure 5.7: A 3-RPR spring.

To find **K** we will use the following notation:

- l_i = length of leg i.
- l_{0i} = length of leg i in rest position (i.e., when the spring is unloaded).
- k_i = elastic constant of the spring of leg i.
- $\hat{s}_i$ = normalized coordinates of the action line of leg i, with the unit direction vector pointing from the base towards the platform.

We will also assume that the resultant force acting on the spring of leg i follows the usual relation

$$f_i = k_i(l_i - l_{0i}). \tag{5.5}$$

Substituting Eq. (5.5) into Eq. (5.4) we obtain

$$\hat{w} = k_1(l_1 - l_{01})\hat{s}_1 + k_2(l_2 - l_{02})\hat{s}_2 + k_3(l_3 - l_{03})\hat{s}_3 \tag{5.6}$$

To determine the relation between $\delta\hat{w}$ and $\delta\hat{D}$ we now differentiate the previous Eq. (5.6) [1]. We must take into account that the quantities that change as a function of time are l_i and $\hat{s}_i$, whereas l_{0i} and k_i are fixed parameters. Furthermore, we assume that the coordinates of $\hat{s}_i$ are expressed as a function of the θ_i angles (Figure 5.7) and that θ_i varies as a function of time:

$$\begin{aligned}
\delta\hat{w} = {} & k_1\delta l_1\hat{s}_1 + k_1(l_1 - l_{01})\frac{\partial\hat{s}_1}{\partial\theta_1}\delta\theta_1 \\
& + k_2\delta l_2\hat{s}_2 + k_2(l_2 - l_{02})\frac{\partial\hat{s}_2}{\partial\theta_2}\delta\theta_2 \\
& + k_3\delta l_3\hat{s}_3 + k_3(l_3 - l_{03})\frac{\partial\hat{s}_3}{\partial\theta_3}\delta\theta_3
\end{aligned}$$

By noting that $\frac{\partial\hat{s}_i}{\partial\theta_i}$ are the coordinates of a line $\hat{s}_{iB}$ orthogonal to $\hat{s}_i$ through points B_i (Appendix D) and defining $\rho_i = \frac{l_{0i}}{l_i}$, we can write

$$\begin{aligned}
\delta\hat{w} = {} & \hat{s}_1 k_1\delta l_1 + \hat{s}_2 k_2\delta l_2 + \hat{s}_3 k_3\delta l_3 + \\
& + \hat{s}_{1B} k_1(1 - \rho_1) l_1\delta\theta_1 \\
& + \hat{s}_{2B} k_2(1 - \rho_2) l_2\delta\theta_2 \\
& + \hat{s}_{3B} k_3(1 - \rho_3) l_3\delta\theta_3,
\end{aligned}$$

[1] A reminder of the process of differentiation is given in Appendix C.

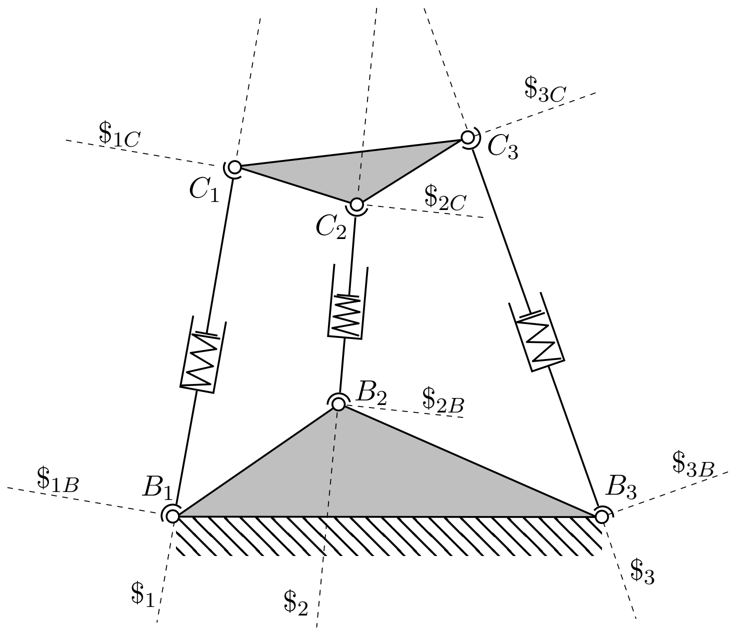


Figure 5.8: Significant lines in the 3-RPR spring.

which can be expressed in matrix form as:

$$\delta\hat{w} = \begin{bmatrix}\hat{s}_1 & \hat{s}_2 & \hat{s}_3\end{bmatrix} \mathbf{k} \begin{bmatrix}\delta l_1 \\ \delta l_2 \\ \delta l_3\end{bmatrix} + \begin{bmatrix}\hat{s}_{1B} & \hat{s}_{2B} & \hat{s}_{3B}\end{bmatrix} \mathbf{k}(1-\rho) \begin{bmatrix}l_1\delta\theta_1 \\ l_2\delta\theta_2 \\ l_3\delta\theta_3\end{bmatrix}, \qquad (5.7)$$

where

$$\mathbf{k} = \begin{bmatrix}k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3\end{bmatrix}, \quad \mathbf{k}(1-\rho) = \begin{bmatrix}k_1(1-\rho_1) & 0 & 0 \\ 0 & k_2(1-\rho_2) & 0 \\ 0 & 0 & k_3(1-\rho_3)\end{bmatrix}.$$

Now note from Appendix B that

$$\begin{bmatrix}\delta l_1 \\ \delta l_2 \\ \delta l_3\end{bmatrix} = \mathbf{j}^{\mathsf{T}}\delta\hat{D}, \qquad \begin{bmatrix}l_1\delta\theta_1 \\ l_2\delta\theta_2 \\ l_3\delta\theta_3\end{bmatrix} = \mathbf{C}^{\mathsf{T}}\delta\hat{D},$$

where

$$\mathbf{C} = \begin{bmatrix}\hat{s}_{1C} & \hat{s}_{2C} & \hat{s}_{3C}\end{bmatrix}$$

is the matrix of the lines $\$_{1C}$, $\$_{2C}$, $\$_{3C}$ orthogonal to $\$_1$, $\$_2$, $\$_3$ through the points C_1, C_2, and C_3 (Fig. 5.8). By substitution of these expressions in

Eq. (5.7), and using the notation

$$\mathbf{j} = \begin{bmatrix} \hat{s}_1 & \hat{s}_2 & \hat{s}_3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \hat{s}_{1B} & \hat{s}_{2B} & \hat{s}_{3B} \end{bmatrix},$$

we arrive at

$$\delta\hat{w} = \underbrace{\{\mathbf{j}\,\mathbf{k}\,\mathbf{j}^{\mathsf{T}} + \mathbf{B}\,\mathbf{k}\,(1-\rho)\,\mathbf{C}^{\mathsf{T}}\}}_{\mathbf{K}}\,\delta\hat{D},$$

This equation can be written as

$$\delta\hat{w} = \mathbf{K}\,\delta\hat{D}, \tag{5.8}$$

where $\mathbf{K}$ is the rigidity matrix anticipated at the beginning.

Eq. (5.8) can be interpreted in the following way. If the platform is in equilibrium under the effect of an external wrench $\hat{w}$, applied by a robot for example, and then we move the platform slightly by performing an infinitesimal displacement $\delta\hat{D}$, the new force that the robot will have to apply to keep the platform in equilibrium is (Fig. 5.9):

$$\hat{w} + \delta\hat{w} = \hat{w} + \mathbf{K}\,\delta\hat{D}.$$

Alternatively, we can also read Eq. (5.8) as follows: if the environment (the robot) varies the applied force from $\hat{w}$ to $\hat{w} + \delta\hat{w}$, then, to maintain equilibrium, the platform has to be displaced an amount $\delta\hat{D}$ with respect to the ground. The reasoning is similar to the one on a linear spring (Fig. 5.10).

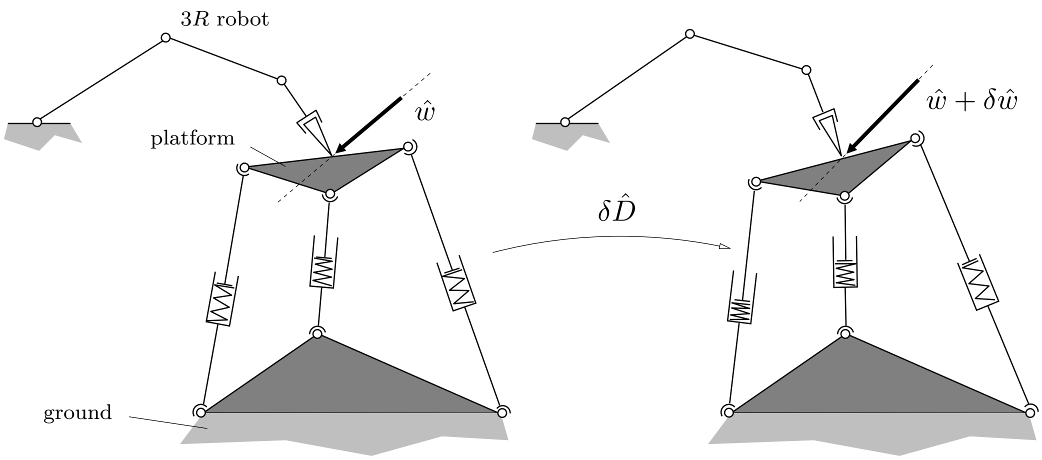


Figure 5.9: The meaning of $\delta\hat{w}$. $\delta\hat{D}$ is the displacement of the platform relative to the ground.

Figure 5.10: To maintain the equilibrium of forces when the spring is displaced from its rest position x_0 to a new position x, we need to apply an external force $F = k(x - x_0)$. Reversely, to compensate an externally applied force F, the spring has to deform until position x is reached.

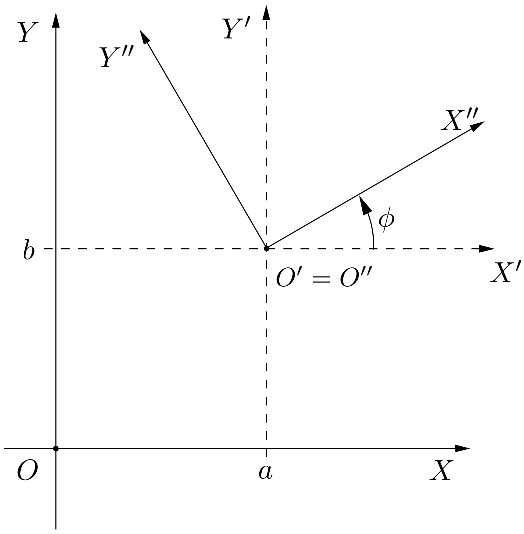


Figure 5.11: Translation and rotation of the coordinate system.

5.4 The rigidity matrix in a new coordinate system

Note that **j**, **B** and **C**, and hence **K**, are not invariant under changes of coordinate system. How does **K** change when another coordinate system is chosen? For example, assume that **K** is expressed in coordinate system OXY and we want to express it in the new coordinate system $O''X''Y''$ shown in Fig. 5.11. On the one hand, we know from Module "Statics" that

$$\delta\hat{w} = \mathbf{e}\,\delta\hat{w}'', \tag{5.9}$$

where $\delta\hat{w}$ and $\delta\hat{w}''$ are the force increments expressed in coordinate systems OXY and $O''X''Y''$, respectively,

$$\mathbf{e} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & a & 1 \end{bmatrix} \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and c and s stand for $\cos\phi$ and $\sin\phi$. On the other hand, we know from Module "Kinematics" that

$$\delta\hat{D} = \mathbf{E}\,\delta\hat{D}'', \tag{5.10}$$

where $\delta\hat{D}$ and $\delta\hat{D}''$ are the infinitesimal displacements expressed in the OXY and $O''X''Y''$ coordinate systems, and

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, we can substitute Eqs. (5.9) and (5.10) in (5.8), obtaining:

$$\mathbf{e}\,\delta\hat{w}'' = \mathbf{K}\,\mathbf{E}\,\delta\hat{D}''.$$

If we multiply by $\mathbf{e}^{-1}$ we get

$$\delta\hat{w}'' = \mathbf{e}^{-1}\,\mathbf{K}\,\mathbf{E}\,\delta\hat{D}'',$$

and since $\mathbf{e}^{-1} = \mathbf{E}^{\mathsf{T}}$ we finally obtain

$$\delta\hat{w}'' = \mathbf{E}^{\mathsf{T}}\,\mathbf{K}\,\mathbf{E}\,\delta\hat{D}''.$$

Thus, the new rigidity matrix $\mathbf{K}''$ relates to the old one through

$$\mathbf{K}'' = \mathbf{E}^{\mathsf{T}}\,\mathbf{K}\,\mathbf{E}.$$

5.5 A hybrid control strategy

We now have the elements to develop the desired hybrid control strategy. Fig. 5.12 recalls the whole system under study. For concreteness we assume that the contact at P is frictionless and punctual, but other contact models could also be assumed (Table 1). Relative to ground, the workpiece is able to rotate and translate while keeping the shown contact constraint. The space of wrenches of constraint of the workpiece is formed by the vertical forces with action line $\$_a$ through point P. We first explain the hybrid control strategy in a simple limit case (Section 5.5.1), then describe it for frictionless point contacts (Section 5.5.2), and finally see how it extends to arbitrary contact models or robots (Section 5.5.3).

5.5.1 Control with an anchored workpiece

Consider the situation in Fig. 5.13, in which the workpiece is initially anchored to ground. Clearly, any force that the ground exerts on the workpiece can be controlled. The position of the workpiece cannot be changed, and thus it makes no sense to control it. Note that any displacement $\delta\hat{D}_F$ of the wrist base with respect to the workpiece (caused by the motion of the 3R robot, in this case) will cause the following variation of the force that the ground applies on the workpiece:

$$\delta\hat{w} = -\mathbf{K}\,\delta\hat{D}_F. \tag{5.11}$$

Here $\mathbf{K}$ is the rigidity matrix of the 3-RPR spring, which is a particular case of the one shown in Fig. 5.7. The negative sign in this equation is due to the fact that $\delta\hat{D}_F$ has now been defined as the displacement of the wrist base $B_{12}B_3$ relative to the workpiece C_1C_{23}, whereas in Eq. (5.8) it was defined considering the displacement of $C_1C_2C_3$ with respect to $B_1B_2B_3$.

From Eq. (5.11) we see that if we aim at causing a variation of the force of the ground on the workpiece of $\delta\hat{w}$, the necessary correcting displacement of the wrist base with respect to the ground will be

$$\delta\hat{D}_F = -\mathbf{K}^{-1}\delta\hat{w}$$

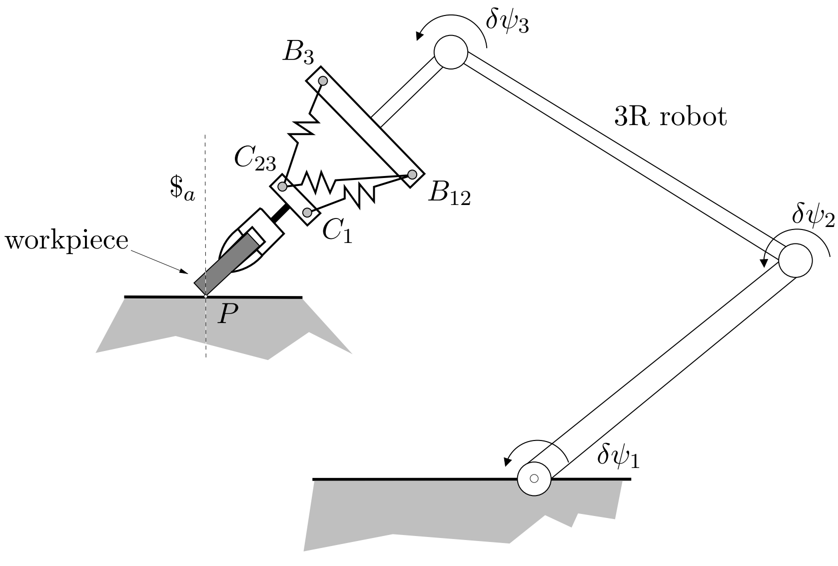


Figure 5.12: A serial 3R robot with a compliant 3-RPR wrist.

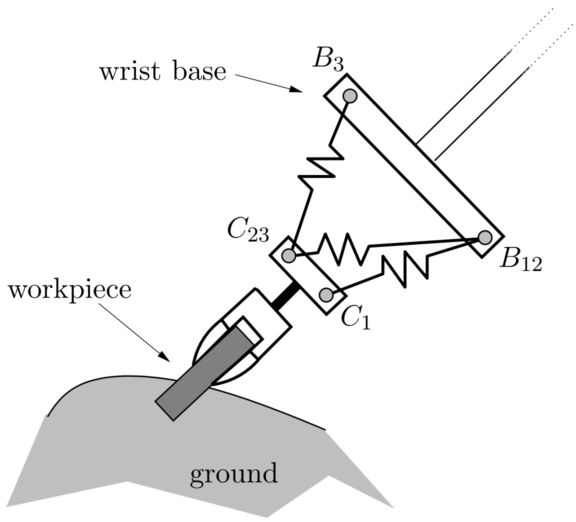


Figure 5.13: Simple case: the workpiece is anchored to the ground.

and the required displacements at the 3R robot joints will be:

$$\delta\boldsymbol{\psi} = \begin{bmatrix} \delta\psi_1 \\ \delta\psi_2 \\ \delta\psi_3 \end{bmatrix} = \mathbf{J}^{-1}\delta\hat{D}_F.$$

5.5.2 Control under frictionless point contact

In the general case both the force that the workpiece exerts on the ground and also its position can be controlled. To this end, note that any displacement $\delta\hat{D}_G$ of the wrist base with respect to the ground can be viewed as the composition of the displacement $\delta\hat{D}_F$ of the wrist base with respect to the workpiece, maintaining the workpiece fixed, plus a displacement $\delta\hat{D}_E$ of the whole set (workpiece and wrist) with respect to the ground, maintaining the spring lengths fixed (Fig. 5.14):

$$\delta\hat{D}_G = \delta\hat{D}_F + \delta\hat{D}_E \tag{5.12}$$

These displacements have the following effects:

- The displacement $\delta\hat{D}_F$ causes a change in the reaction force of the ground against the workpiece in an amount of $\delta\hat{w}_a = -\mathbf{K}\,\delta\hat{D}_F$, and thus one can see this displacement as correcting the reaction of the ground. It is important to notice that this displacement will not cause

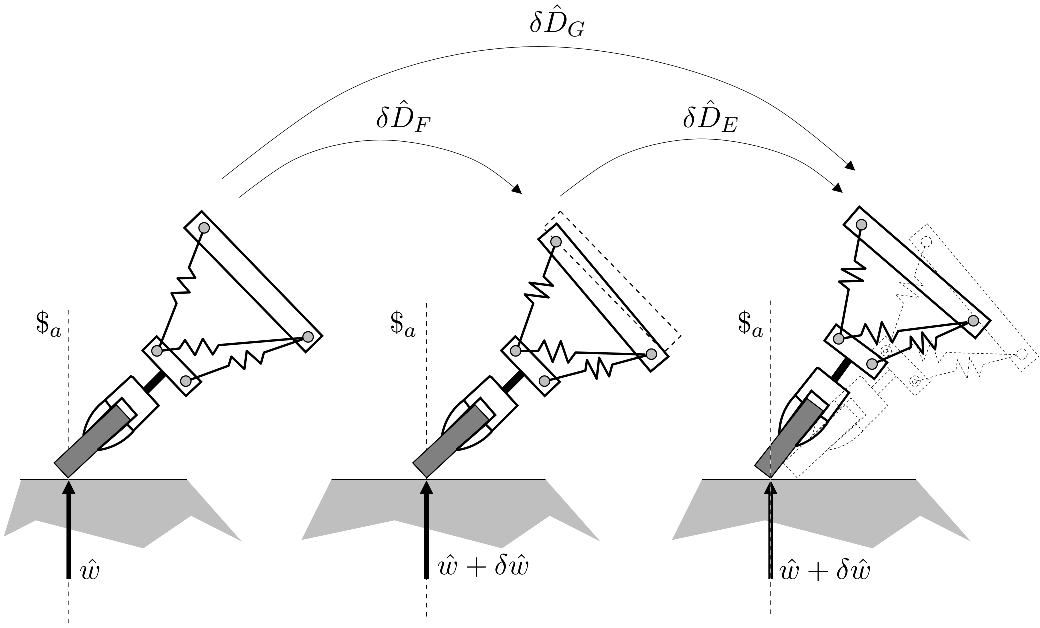


Figure 5.14: Decomposition of $\delta\hat{D}_G$ into the force- and position-correcting displacements $\delta\hat{D}_F$ and $\delta\hat{D}_E$.

the workpiece to move, because the change in the reaction force $\delta\hat{w}_a$ acts on the line $\$_a$ and does not generate virtual power under any twist of freedom of the workpiece relative to ground. Therefore, it cannot cause a position error of the workpiece.

- The displacement $\delta\hat{D}_E$ changes the pose of the whole set with respect to the ground. Thus, one can view $\delta\hat{D}_E$ as a correcting displacement of the position of the workpiece. It is important to notice that since the relative position between the workpiece and the wrist base does not change under $\delta\hat{D}_E$, this displacement cannot cause a force error. The reaction force of the ground will remain constant while performing this displacement.

Therefore, to simultaneously attain

- a force variation $\delta\hat{w}_a$ (of the ground on the workpiece)
- a position change $\delta\hat{D}_E$ (of the workpiece relative to ground)

the following displacement of the wrist base relative to ground has to be performed:

$$\delta\hat{D}_G = \delta\hat{D}_F + \delta\hat{D}_E = -\mathbf{K}^{-1}\delta\hat{w}_a + \delta\hat{D}_E \tag{5.13}$$

Vector equations	$\sharp$ equations	$\sharp$ variables
$\delta\hat{D}_G = \delta\hat{D}_F + \delta\hat{D}_E$	3	6
$\delta\hat{D}_F = -\mathbf{K}^{-1}\delta\hat{w}_a$	3	3
$\delta\hat{w}_a = \mathbf{W}\,\mu$	3	1
$\delta\hat{D}_E = \mathbf{T}\,\boldsymbol{\varepsilon}$	3	2
Total	12	12

Table 2: The system of equations that determines $\delta\hat{D}_F$ and $\delta\hat{D}_E$ (left column) together with its number of scalar equations and variables (middle and right columns). In the right column, only the new unknowns not appearing in the previous equations are counted.

To this end, it is only necessary to command the following angular increments $\delta\boldsymbol{\psi}$ at the joints of the 3R robot:

$$\delta\boldsymbol{\psi} = \begin{bmatrix}\delta\psi_1\\ \delta\psi_2\\ \delta\psi_3\end{bmatrix} = \mathbf{J}^{-1}\delta\hat{D}_G. \tag{5.14}$$

We can guarantee that the desired displacement $\delta\hat{D}_G$ will always decompose into the required displacements $\delta\hat{D}_F$ and $\delta\hat{D}_E$ because the decomposition $\delta\hat{D}_G = \delta\hat{D}_F + \delta\hat{D}_E$ is, for a given $\delta\hat{D}_G$, unique. Note that the values $\delta\hat{D}_F$ and $\delta\hat{D}_E$ verify the system of equations in Table 2, where the columns of $\mathbf{W}$ constitute a basis of the space of wrenches of constraint, the columns of $\mathbf{T}$ contain a basis of the space of twists of freedom (of the workpiece relative to the ground), $\mu \in \mathbb{R}$ is a scalar, and $\boldsymbol{\varepsilon}$ is a vector of $\mathbb{R}^2$. The system has as many equations as variables, and, thus, in general it will have a unique solution.

5.5.3 Control under arbitrary contact models or robots

Note that, the methodology described in the previous section is equally applicable under arbitrary contact models, in particular those of Table 1. Eq. (5.12) always applies irrespectively of the contact model assumed, and the correcting displacement $\delta\hat{D}_G$ and associated joint motions $\delta\boldsymbol{\psi}$ are always computed using Eqs. (5.13) and (5.14). We only need to check initially that the force variation $\delta\hat{w}_a$ and position change $\delta\hat{D}_E$ demanded lie within the spaces $\mathbb{W}$ and $\mathbb{T}$ of the contact model assumed.

Finally, we have assumed that the compliant wrist is connected to a serial robot, but we could connect it to a different robot as well, e.g., a fully parallel robot. We would equally compute the correcting displacement $\delta\hat{D}_G$ using Eq. (5.13), but we would then replace Eq. (5.14) by an analogous relationship derived from the velocity equation of the considered robot.

A Infinitesimal displacements

Consider a lamina in the plane instantaneously rotating about a vertical axis \$ through point $Q = (x_Q, y_Q)$, with angular velocity $\boldsymbol{\omega} = \omega \boldsymbol{k}$ (Fig. 5.15). The twist of the lamina is

$$\hat{T} = \omega \begin{bmatrix} y_Q \\ -x_Q \\ 1 \end{bmatrix} = \begin{bmatrix} v_{ox} \\ v_{oy} \\ \omega \end{bmatrix}$$

where $\boldsymbol{v}_o = v_{ox}\, \boldsymbol{i} + v_{oy}\, \boldsymbol{j}$ is the velocity of the origin point of the lamina. Assume now that the lamina rotates a certain infinitesimal angle $\delta\phi$ during a very small time increment δt. Then, in analogy to the twist, the *infinitesimal displacement* of the lamina is defined as the vector

$$\delta\hat{D} = \delta\phi \begin{bmatrix} y_Q \\ -x_Q \\ 1 \end{bmatrix} = \begin{bmatrix} \delta r_{ox} \\ \delta r_{oy} \\ \delta\phi \end{bmatrix}$$

where $\delta r_{ox} = \delta\phi\, y_Q$ and $\delta r_{oy} = -\delta\phi\, x_Q$. Note that, since $\omega = \delta\phi/\delta t$, the displacement $\delta\hat{D}$ can be obtained by multiplying the twist $\hat{T}$ by δt.

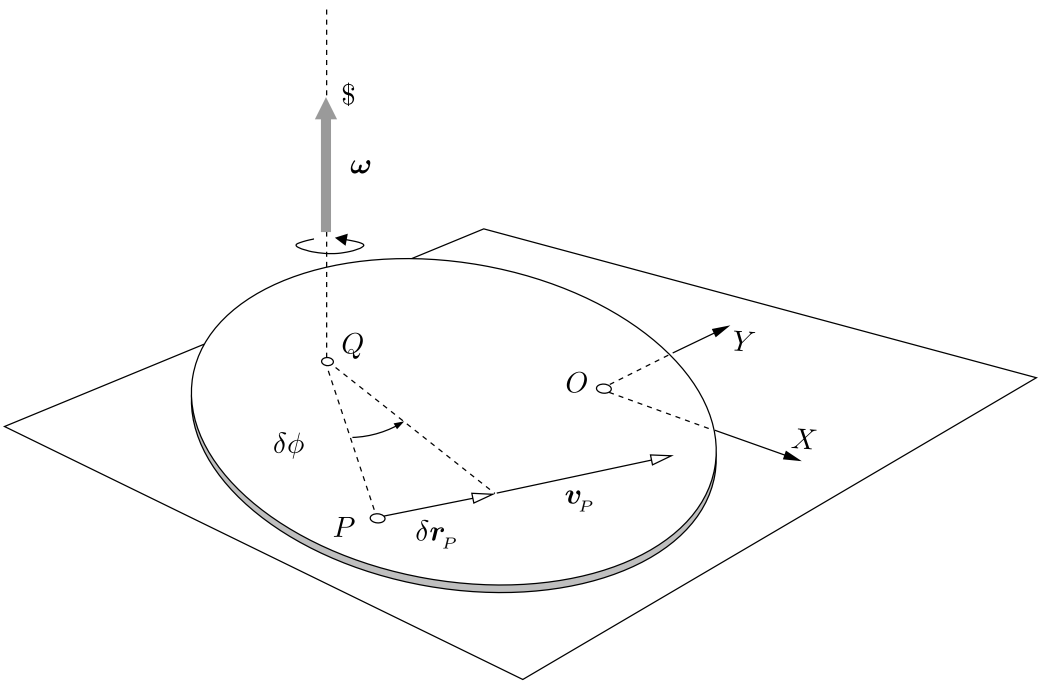


Figure 5.15: Infinitesimal displacements of a lamina.

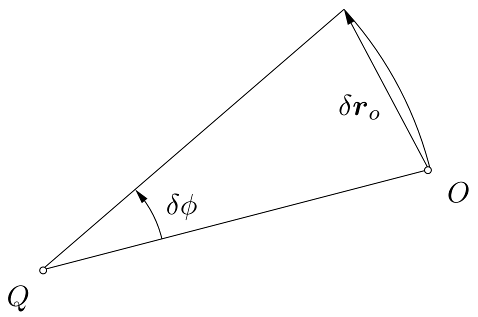


Figure 5.16: Small rotation about Q.

Remember that the twist $\hat{T}$ completely encodes the velocity $\boldsymbol{v}_P$ of any point P of the lamina (Fig. 5.15), because

$$\boldsymbol{v}_P = \boldsymbol{\omega} \times \boldsymbol{r}_P + \boldsymbol{v}_o,$$

where $\boldsymbol{r}_P = \overrightarrow{OP}$. In a similar way, we next prove that $\delta\hat{D}$ completely encodes the small displacement $\boldsymbol{r}_P$ undergone by P due to the small rotation $\delta\phi$, because it will turn out that

$$\delta \boldsymbol{r}_P = \delta \boldsymbol{\phi} \times \boldsymbol{r}_P + \delta \boldsymbol{r}_o, \tag{5.15}$$

where $\delta\boldsymbol{\phi} = \delta\phi\, \boldsymbol{k}$ and $\delta\boldsymbol{r}_o = \delta r_{ox}\, \boldsymbol{i} + \delta r_{oy}\, \boldsymbol{j}$.

To obtain Eq. (5.15), we first see that $\delta\boldsymbol{r}_o$ is precisely the displacement of the origin point of the lamina. Note that due to the rotation $\delta\phi$ about \$, the origin point follows a circular path centered in Q, of radius $\|\overrightarrow{QO}\|$, and since $\delta\phi$ is quite small, we can approximate the norm of $\delta\boldsymbol{r}_o$ by the length of the arc of circumference described (Fig. 5.16). Thus we can say that $\delta\boldsymbol{r}_o$ is a vector with the same direction than $\delta\phi\, \boldsymbol{k} \times \overrightarrow{QO}$, with norm $\delta\phi \cdot |\overrightarrow{QO}|$. Therefore, $\delta\boldsymbol{r}_o = \delta\phi\, \boldsymbol{k} \times \overrightarrow{QO}$, i.e.

$$\delta\boldsymbol{r}_o = \begin{bmatrix} 0 \\ 0 \\ \delta\phi \end{bmatrix} \times \begin{bmatrix} -x_Q \\ -y_Q \\ 0 \end{bmatrix} = \begin{bmatrix} \delta\phi y_Q \\ -\delta\phi x_Q \\ 0 \end{bmatrix} = \begin{bmatrix} \delta r_{ox} \\ \delta r_{oy} \\ 0 \end{bmatrix}.$$

And now the displacement $\delta\boldsymbol{r}_P$ of an arbitrary point P of the lamina is

$$\begin{aligned} \delta\boldsymbol{r}_P = \delta\boldsymbol{\phi} \times \overrightarrow{QP} = \delta\boldsymbol{\phi} \times (\overrightarrow{QO} + \overrightarrow{OP}) = \delta\boldsymbol{\phi} \times \overrightarrow{OP} + \delta\boldsymbol{\phi} \times \overrightarrow{QO} \\ = \delta\boldsymbol{\phi} \times \boldsymbol{r}_P + \delta\boldsymbol{r}_o, \end{aligned}$$

which produces the desired Eq. (5.15).

It is important to note that since the infinitesimal displacement $\delta\hat{D}$ is a multiple of the twist $\hat{T}$, the vector space of twists of freedom of the lamina coincides with the vector space of feasible infinitesimal displacements of the lamina.

B The displacement equations

Recall that the relations that describe the kinematic behavior of serial and parallel robots are

$$\begin{aligned} \hat{T} &= \mathbf{J}\,\boldsymbol{\gamma} \quad \text{(Serial robot)} \\ \boldsymbol{v} &= \mathbf{j}^{\mathsf{T}}\,\hat{T} \quad \text{(Parallel robot)} \end{aligned}$$

Multiplying these relations by δt, analogous expressions are obtained that relate the infinitesimal displacement of the end effector $\delta\hat{D}$ with the infinitesimal rotations $\delta\boldsymbol{\theta}$ and linear displacements $\delta\boldsymbol{l}$ undergone by the joints:

$$\begin{aligned} \delta\hat{D} &= \mathbf{J}\,\delta\boldsymbol{\theta} \quad \text{(Serial robot)} \\ \delta\boldsymbol{l} &= \mathbf{j}^{\mathsf{T}}\,\delta\hat{D} \quad \text{(Parallel robot)} \end{aligned}$$

where

$$\delta\boldsymbol{\theta} = \begin{bmatrix} \delta\theta_1 \\ \delta\theta_2 \\ \delta\theta_3 \end{bmatrix} \quad \delta\boldsymbol{l} = \begin{bmatrix} \delta l_1 \\ \delta l_2 \\ \delta l_3 \end{bmatrix}.$$

In the case of the parallel robot, we also recall from Module 4 the expression that relates the transversal velocities of C_i, $i = 1, \ldots, 3$, (Fig. 5.17), and the end-effector twist $\hat{T}$

$$\boldsymbol{v}_t = \mathbf{C}^{\mathsf{T}}\,\hat{T}, \tag{5.16}$$

where the columns of $\mathbf{C}$ are the normalized coordinates of the lines $\$_{iC}$ through C_i, $i = 1, \ldots, 3$. Again, through multiplication by δt, we can can convert Eq. (5.16) into a displacement equation, obtaining

$$\begin{bmatrix} l_1\,\delta\theta_1 \\ l_2\,\delta\theta_2 \\ l_3\,\delta\theta_3 \end{bmatrix} = \mathbf{C}^{\mathsf{T}}\delta\hat{D},$$

where $l_i\,\delta\theta_i$ is the infinitesimal displacement of C_i. Note that this point traces an arc of circumference of length $l_i\,\delta\theta_i$, with $\delta\theta_i$ being the small angle rotated by the leg.

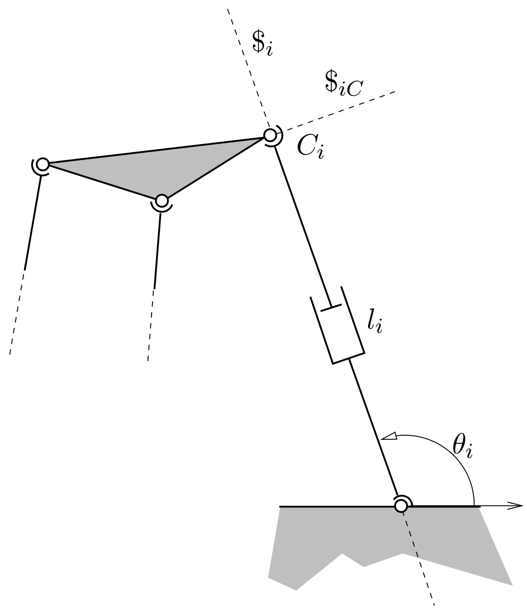


Figure 5.17: Lines at points C_i.

C Derivatives and differentiation

Let w be a function $\mathbb{R}^n \to \mathbb{R}$, of n parameters p_i

$$w = F(p_1, \ldots, p_n). \tag{5.17}$$

Eq. (5.17) defines the graph of w in the augmented space of $p_1, \ldots, p_n, w$, a hypersurface in this space, as shown schematically in Fig. 5.18. Assume that the parameters $p_1, \ldots, p_n$ vary with time, according to a given trajectory

$$\boldsymbol{p}(t) = (p_1(t), \ldots, p_n(t)),$$

so that w can be viewed as $w(\boldsymbol{p}(t))$. By applying the chain rule, the derivative of w with respect to time is,

$$\frac{\delta w}{\delta t} = \frac{\partial F}{\partial p_1}\frac{\delta p_1}{\delta t} + \cdots + \frac{\partial F}{\partial p_n}\frac{\delta p_n}{\delta t}$$

or, equivalently,

$$\underbrace{\frac{\delta w}{\delta t}}_{w'} = \underbrace{\left[\tfrac{\partial F}{\partial p_1}, \cdots, \tfrac{\partial F}{\partial p_n}\right]}_{\mathbf{J}} \underbrace{\begin{bmatrix} \frac{\delta p_1}{\delta t} \\ \vdots \\ \frac{\delta p_n}{\delta t} \end{bmatrix}}_{\boldsymbol{p}'}. \tag{5.18}$$

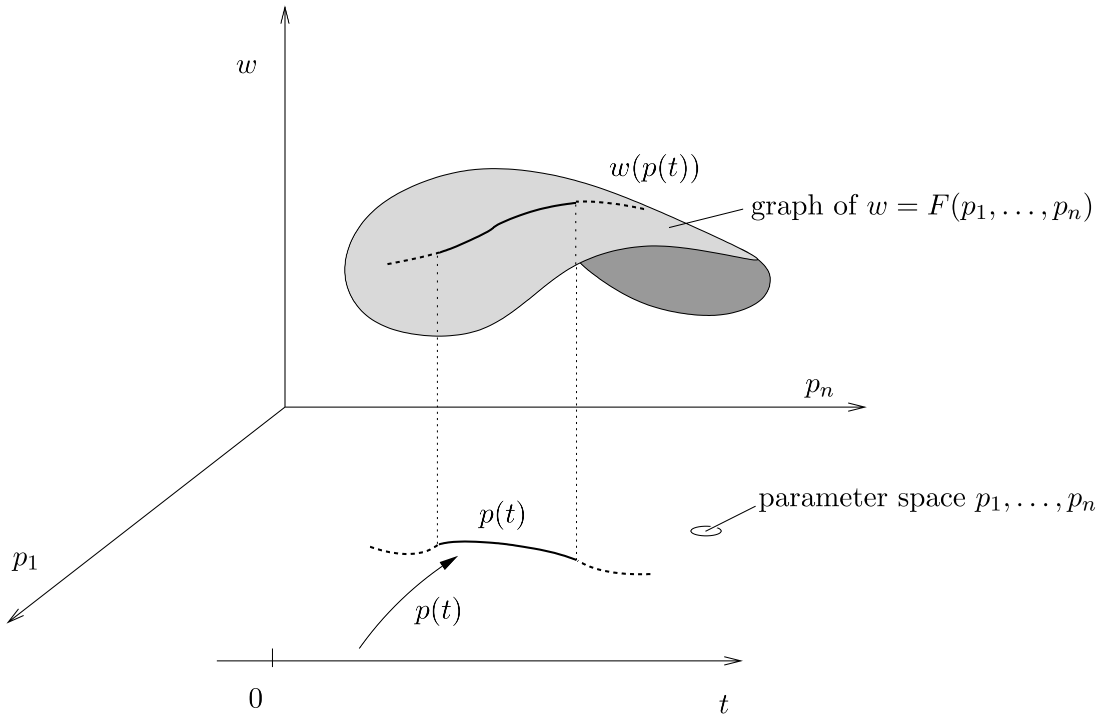


Figure 5.18: The graph of function w.

Eq. (5.18) is called the *total derivative* of w. It provides the relationship between the rate of change of $\boldsymbol{p}$, and the rate of change of w. By multiplying the equation by an infinitesimal time increment, we obtain the following expression

$$\delta w = \left[\tfrac{\partial F}{\partial p_1}, \cdots, \tfrac{\partial F}{\partial p_n}\right] \underbrace{\begin{bmatrix} \delta p_1 \\ \vdots \\ \delta p_n \end{bmatrix}}_{\delta \boldsymbol{p}}, \tag{5.19}$$

or, in matrix form,

$$\delta w = \mathbf{J}\, \delta \boldsymbol{p},$$

which relates the small variations in the parameters, $\delta\boldsymbol{p}$, with the small variations of the function, δw.

Eq. (5.19) is called the *total differential* of w. The process of obtaining Eq. (5.19) is analogous to the one followed to obtain Eq. (5.18), and it is called *differentiation*. The rules are also analogous to those used when computing derivatives. The total differential, or simply *differential*, of w is, thus,

$$\delta w = \frac{\partial F}{\partial p_1}\delta p_1 + \cdots + \frac{\partial F}{\partial p_n}\delta p_n.$$

Eqs. (5.18) and (5.19) can be quickly generalized to functions w from $\mathbb{R}^n$ to $\mathbb{R}^m$:

$$\begin{aligned} w_1 &= f_1(p_1, \ldots, p_n) \\ \vdots & \quad \vdots \\ w_m &= f_m(p_1, \ldots, p_n) \end{aligned}$$

In particular, Eq. (5.18) generalizes to:

$$\begin{bmatrix} \frac{\delta w_1}{\delta t} \\ \vdots \\ \frac{\delta w_m}{\delta t} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial p_1} & \cdots & \frac{\partial f_1}{\partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial p_1} & \cdots & \frac{\partial f_m}{\partial p_n} \end{bmatrix} \begin{bmatrix} \frac{\delta p_1}{\delta t} \\ \vdots \\ \frac{\delta p_n}{\delta t} \end{bmatrix}, \tag{5.20}$$

and the generalization of Eq. (5.19) is obtained by multiplying Eq. (5.20) by a small increment of time δt.

D The derivative of a line

Consider the leg of a parallel manipulator, which rotates about point B_i anchored to ground (Fig. 5.19). The leg line $\$_i$ forms an angle θ_i with respect to the X axis, as shown in the figure. The unit coordinates of $\$_i$ are

$$\hat{s}_i = \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \\ OB_i \ \sin(\theta_i - \alpha) \end{bmatrix}$$

The unit coordinates of $\frac{\delta \hat{s}_i}{\delta \theta_i}$ are

$$\hat{s}_{iB} = \frac{\delta \hat{s}_i}{\delta \theta_i} = \begin{bmatrix} -\sin\theta_i \\ \cos\theta_i \\ OB_i \ \cos(\theta_i - \alpha) \end{bmatrix}$$

which are clearly the coordinates of a line $\$_{iB}$ perpendicular to $\$_i$ through point B_i.

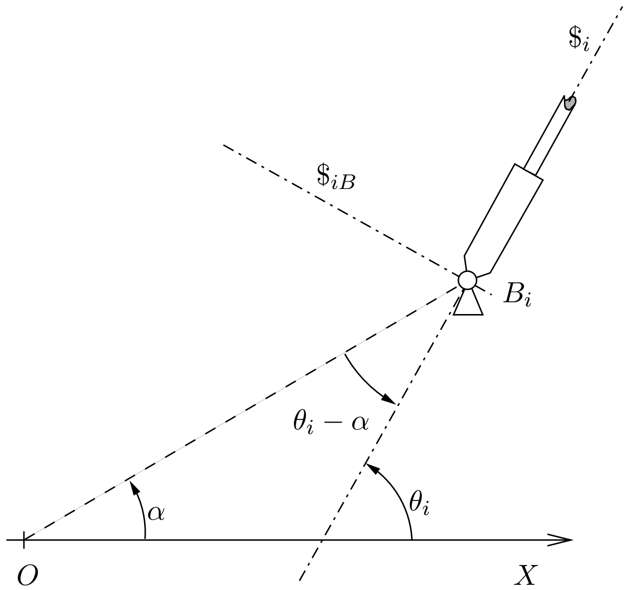


Figure 5.19: The derivative of line $\$_i$.