Instantaneous Kinematics
Slide companion notes

Kinematics: refers to the motion of points or bodies. Here we deal with velocities only.

Instantaneous: Means at a precise instant of time. The velocities that we would see at a specific "snapshot", or configuration, of the global robot movement.

Goal: Our purpose is to provide geometric tools to establish the relationship between the actuated joint and end-effector velocities of a robot manipulator.

Slide 2 Let us remind the basic concepts of Module 2, Statics:

1. Line in the XY plane: $\hat{s} = \{L, M; R\}$.
2. Normalized (or unit) coordinates of previous line: $\hat{s} = \{c, s; p\}$.
3. Wrench: $\hat{w} = \{\boldsymbol{f}; \boldsymbol{c}_o\}$.
4. Pure couple: $\hat{w} = \{\boldsymbol{0}; \boldsymbol{c}_o\}$.
5. Translation and rotation: $\hat{w} = [\boldsymbol{e}]\hat{w}'$.
6. Statics of parallel robots: $\boldsymbol{f} = [\boldsymbol{j}]\boldsymbol{\tau}$.

We will discover that all of these concepts have kinematic analogous ones. The underlying mathematics is, again, Projective Geometry.

Slide 3 Consider a coordinate system $OXYZ$, and a line \$ parallel to its Z axis. The line intersects the XY plane at the point $(x_o, y_o, 0)$, with position vector $\boldsymbol{r}_o$, and has a direction vector $\boldsymbol{S} = N\boldsymbol{k}$. As we did in Statics, we want to provide a coordinate vector describing this line. What about this one?

$$\hat{s} = \{\boldsymbol{S}; \boldsymbol{S}_o\},$$

where $\boldsymbol{S}_o = \boldsymbol{r}_o \times \boldsymbol{S}$ is the moment of $\boldsymbol{S}$ about the origin O. Why not? Observe that

$$\boldsymbol{S}_o = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_o & y_o & 0 \\ 0 & 0 & N \end{vmatrix} = \begin{bmatrix} y_o N \\ -x_o N \\ 0 \end{bmatrix} = \begin{bmatrix} P \\ Q \\ 0 \end{bmatrix}.$$

Clearly, the line can be described by the coordinates N, P, and Q, because such values identify one and only one line. Although both $\boldsymbol{S}$ and $\boldsymbol{S}_o$ are spatial vectors, we accept a slight abuse of notation and we write

$$\hat{s} = \{N; P, Q\} = N\{1; y_o, -x_o\},$$

because N, P, and Q are the only non-null components of $\boldsymbol{S}$ and $\boldsymbol{S}_o$.

Remember that for a line in the XY plane, (L, M) determines a free vector on the line, and R the moment of such line about the Z axis. Now, for a line parallel to the Z axis, only N is needed to define a free vector on the line, and (P, Q) are moments about the X and Y axes, respectively. These two representations are consistent, and can in fact be written in the expanded form:

$$\{L, M, 0; 0, 0, R\} \quad \text{(Line in } XY \text{ plane)}$$

$$\{0, 0, N; P, Q, 0\} \quad \text{(Line parallel to } Z \text{ axis)}$$

In the course appendix we can see that these are particular cases of the way we represent general lines in 3-space. Indeed, the homogeneous coordinates for a general spatial line take the form

$$\{L, M, N; P, Q, R\} \quad \text{(General spatial line)}.$$

We add that, since in general $\hat{s} = N\{1; y_o, -x_o\}$, then for a unit line segment parallel to the Z axis it will be

$$\hat{s} = \{1; y_o, -x_o\}$$

because $N = 1$. These are the normalized (or unit) coordinates of a vertical line. The last two coordinates provide the point of intersection of the line with the XY plane, (x_o, y_o).

Slide 4 This provides the first two concepts in our analogy table. The line parallel to the Z axis, and the normalized coordinates of the line.

Slide 5 Left figure: An important point to recall from Mechanics is that the velocity field of a rigid lamina under planar motion can always be described as an instantaneous rotation about a point, called the *instant center of rotation*. This picture shows such a velocity field. We can always find an instant center, even in the case of pure translation. If we know the location of the instant center, and the angular velocity vector $\boldsymbol{\omega}$, how do we compute the linear velocity of any point of the lamina? This is explained in the right figure.

Right figure: This is our lamina in $3D$, rotating about the red point—the instant center I. The velocity of the black point, with position vector $\boldsymbol{r}$ from I, is simply $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$.

Our goal now is to find a compact way to describe the whole velocity field. But ... there is an infinity of velocity vectors! How can we describe them with only a few numbers? This leads to the concept of *twist*.

Slides 6-7 Left figure: It turns out that the velocity field can be described by the twist[1] $\hat{t} = \{\boldsymbol{\omega}; \boldsymbol{v}_o\}$:

- ω encodes the angular velocity generating the velocity field.
- $\boldsymbol{v}_o$ is more tricky. It is the velocity of the point in the lamina that *instantaneously coincides* with the origin of the coordinate system. The lamina is moving on a plane, which is the ground, and the coordinate system is fixed to that ground. The lamina can be thought of as unlimited in all directions. At any instant of time, a point of the lamina will pass over the origin O with some velocity. That velocity is precisely $\boldsymbol{v}_o$. If $\boldsymbol{r}_o$ is the position vector of the instant center relative to O, then clearly:

$$\boldsymbol{v}_o = \boldsymbol{\omega} \times (-\boldsymbol{r}_o) = \boldsymbol{r}_o \times \boldsymbol{\omega}$$

Note that the structure of the twist

$$\hat{t} = \{\boldsymbol{\omega}; \boldsymbol{v}_o\}$$

is analogous to that of the wrench

$$\hat{w} = \{\boldsymbol{f}; \boldsymbol{c}_o\}$$

presented in Module 2. In both cases we have a vector, and the moment of such vector w.r.t. the orgin O.

Right figure: The twist $\hat{t} = \{\boldsymbol{\omega}; \boldsymbol{v}_o\}$ provides a complete representation of the velocity field because the velocity of any point P can be computed from $\boldsymbol{\omega}$ and $\boldsymbol{v}_o$ as follows:

$$\boldsymbol{v}_P = \boldsymbol{\omega} \times \boldsymbol{p} + \boldsymbol{v}_o$$

Certainly

$$\begin{aligned} \boldsymbol{v}_P &= \boldsymbol{\omega} \times (\boldsymbol{p} - \boldsymbol{r_o}) \\ &= \boldsymbol{\omega} \times \boldsymbol{p} + \boldsymbol{\omega} \times -\boldsymbol{r}_o \\ &= \boldsymbol{\omega} \times \boldsymbol{p} + \boldsymbol{r}_o \times \boldsymbol{\omega} \\ &= \boldsymbol{\omega} \times \boldsymbol{p} + \boldsymbol{v}_o. \end{aligned}$$

The origin O of the coordinate system can be chosen arbitrarily. We have drawn it below the lamina, but it could be located outside it, as in slide 7.

Slides 8-9 Consider again our body rotating with angular velocity $\boldsymbol{\omega} = [0, 0, \omega]^\mathsf{T}$ about a point with position vector $\boldsymbol{r}_o = [x_o, y_o, 0]^\mathsf{T}$ from the origin. The body twist is

$$\hat{t} = \{\boldsymbol{\omega}; \boldsymbol{v}_o\}$$

where $\boldsymbol{v}_o$ is the velocity of the lamina point at the origin. Since

$$\boldsymbol{v}_o = \boldsymbol{r}_o \times \boldsymbol{\omega} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_o & y_o & 0 \\ 0 & 0 & \omega \end{vmatrix} = \begin{bmatrix} y_o\omega \\ -x_o\omega \\ 0 \end{bmatrix}$$

we can write

$$\begin{aligned} \hat{t} &= \{\omega; y_o\,\omega, -x_o\,\omega\} \\ &= \omega \cdot \{1; y_o, -x_o\} \\ &= \omega \cdot \hat{s} \end{aligned}$$

In other words, the twist $\hat{t}$ can be seen as a multiple of the vertical unit line $\hat{s} = \{1; y_o, -x_o\}$, in which the pair (x_o, y_o) provides the location of the instant center.

[1] In planar kinematics, some texts also use the term *rotor* for a twist.

Slides 10-11 Just as couples are special cases of forces, instantaneous translations can also be seen as special cases of instantaneous rotations.

Suppose that a lamina is undergoing a pure translation. Its velocity field looks like the top-left figure. It is a constant field. Since $\boldsymbol{\omega} = \mathbf{0}$, all points move with the same velocity vector $\boldsymbol{v}_o$, and the lamina twist is:

$$\hat{t} = \{\mathbf{0}; \boldsymbol{v}_o\}.$$

We can see this field as the limit case of a family of fields in which the instant center is pushed to infinity. Suppose that initially the field is like in the top-right figure. The body is rotating with angular velocity $\boldsymbol{\omega}$ about the instant center in Q. We select two points on a line through Q, say O and P, and choose the X and Y axes depicted. Let v_o and v be the Y components of the velocities of O and P, respectively. Also let p and q be the X coordinates of P and Q. Using

$$\begin{aligned} v_o &= -\omega\, q \\ v &= -\omega\, (q-p) \end{aligned}$$

it is easy to see that the three quantities v, v_o, and p, completely determine the angular velocity ω and the position q of the instant center. In other words, v, v_o, and p can be seen as parameters determining the velocity field.

Certainly, dividing the first equation by the second we get q as a function of v, v_o, and p

$$q(v, v_o, p) = \frac{v_o\, p}{v_o - v}$$

and substituting this expression into $\omega = -v_o/q$ we get

$$\omega(v, v_o, p) = \frac{v - v_o}{p}$$

Now, if we try to convert the field to a constant field, e.g. by making v tend to v_o, we have

$$\begin{aligned} r &\to \infty \\ \omega &\to 0 \end{aligned}$$

so the instant center really tends to the point at infinity in the direction orthogonal to $\mathbf{v}_o$, and the angular velocity tends to zero.

Consider a general translational twist

$$\hat{t} = \{\mathbf{0}; \boldsymbol{v}_o\}$$

If we express $\boldsymbol{v}_o$ as $\boldsymbol{v}_o = v_o\, \boldsymbol{S}_o$, where $\boldsymbol{S}_o$ is a unit vector, we can write

$$\hat{t} = v_o\{\mathbf{0}; \boldsymbol{S}_o\}$$

The line $\{\mathbf{0}; \boldsymbol{S}_o\}$ is the unit vertical line through the point at infinity in the direction $\boldsymbol{S}_o$. Thus, a translational twist can be seen as a multiple of a vertical line at infinity. The instant center is the point at infinity in the direction orthogonal to $\boldsymbol{v}_o$.

Slide 12 Now try to find the twist of each ball at the moment shown in the figures, relative to the coordinate system OXY indicated. Assume that ω, L and v are all of them positive values.

Slide 13 We next introduce a somewhat artificial convention. It is difficult to grasp at this point, but it will be useful in Module 4 "Duality" to write the reciprocal product between twists and wrenches as a standard dot product.

Twists and wrenches are geometrically the same thing: a vector, and the moment of this vector with respect to the origin. We can write the vector and the moment in either of the following orders:

Order	**Order name**
{vector ; moment}	Ray coordinates
{moment ; vector}	Axis coordinates

From now on:

- Wrenches will be written in ray coordinates using lowercase letters:

  $$\hat{w} = \{\boldsymbol{f}; \boldsymbol{c}_o\} = f\hat{s}$$

  with $\hat{s} = \{\boldsymbol{S}; \boldsymbol{S}_o\}$.

- Twists will be written in axis coordinates using capital letters:

  $$\hat{T} = \{\boldsymbol{v}_o; \boldsymbol{\omega}\} = \omega\hat{S},$$

  with $\hat{S} = \{\boldsymbol{S}_o; \boldsymbol{S}\}$.

Note how the vector and the moment parts of the line coordinates are swapped between ray and axis coordinates. Since here we deal with the planar case, wrenches and their supporting lines $\hat{s}$ will be in the XY plane, and twists and their supporting lines $\hat{S}$ will be parallel to the Z axis.

Slide 14 Try to find the correct answer/s.

Slides 15-16 The coordinates of a line \$ can be expressed in different coordinate systems. Let OXY and $O'X'Y'$ be two coordinate systems, such that the vector between O and O' is $[a, b]^\mathsf{T}$, and the angle between X and X' is ϕ. The system $O'X'Y'$ is translated and rotated with respect to OXY. If $\hat{S}$ and $\hat{S}'$ are the coordinates of \$ in OXY and $O'X'Y'$, respectively, then

$$\hat{S} = [\boldsymbol{E}]\hat{S}',$$

with

$$[\boldsymbol{E}] = \begin{bmatrix} c & -s & b \\ s & c & -a \\ 0 & 0 & 1 \end{bmatrix},$$

where $c = \cos\phi$ and $s = \sin\phi$.

Slide 17 It can be shown that the matrix $[E]$ is related to the matrix $[e]$ that appeared in Module 2 "Statics" through

$$[e]^{-1} = [E]^\mathsf{T}.$$

This connection stresses the duality between statics and kinematics, i.e. between the forces and moments applied on a body, and the rotations and translations that the body performs. This will be further developed in the following course modules.

Slide 18 We should make it clear that the velocity field of a rigid body—and hence its representative twist—is *relative* to the observational frame used. An observational frame is an observer and all points fixed to that observer. I.e., all points that keep their pairwise distances fixed along time. Think of it as a large rigid body on which the observer is firmly standing.

In the figure there is a wheel rotating about its center. The velocity of P seen by observer B is the black arrow. The velocity of P seen by the observer in A is the blue arrow. The two differ by the red vector, which is the translational velocity of the train car on which observer B is standing.

Slide 19 Once an observational frame is chosen, we usually attach a *coordinate system* to it: a point, and three direction vectors. Such a system allows us to specify the coordinates of points and velocity vectors in that frame.

There is an infinity of coordinate systems that we may choose. The choice affects the coordinates given to points and velocities in the frame, but not how they are perceived by the observer.

Since the twist coordinates depend on both the observational frame and coordinate system assumed, an orthodox sentence to refer to a twist is (e.g., in the setting of this slide):

> "Twist of the wheel relative to observational frame A, and expressed in coordinate system OXY"

Since this is rather long, and very often the observational frame is the body on which the coordinate system is drawn, or known by context, we usually omit it and say

> "Twist of the wheel relative to coordinate system OXY"

Slide 20 Let us now recall the law of composition of velocities that relates the velocity vectors observed from two different observational frames.

We have two observers, A and B, in their respective observational frames. They measure different velocities of point P. These velocities are related as follows, where *absolute* means "observed from frame A":

> **Law of composition of velocities:** The absolute velocity of P is equal to the velocity of P observed from frame B, plus the absolute velocity that P would have if it were a fixed point in frame B.

Slide 21 We now wish to perform the kinematic analysis of a 3R serial arm: establish the relationship between the articular and end-effector velocities. The manipulator consists of three bodies, labelled L_1 to L_3, articulated through revolute joints. The first joint is attached to the ground, labelled as L_0. Each joint J_i makes L_i rotate with angular velocity ω_i relative to L_{i-1}.
Now let

- $\boldsymbol{\gamma} = [\omega_1, \omega_2, \omega_3]^\mathsf{T}$ be the vector of joint angular velocities.

- $\hat{T} = [\boldsymbol{v}_o{}^{\mathsf{T}}, \omega]^{\mathsf{T}}$ be the twist describing the velocity field of the end effector relative to an observer on the ground, expressed in the coordinate system OXY.

The twist $\hat{T}$ is here written in square brackets $[\cdot]$, instead of curly braces $\{\cdot\}$, because we shall soon operate with it in vector form.

There are two problems we wish to solve:

- The forward instantaneous kinematic problem (FIKP): Given γ, compute $\hat{T}$.
- The inverse instantaneous kinematic problem (IIKP): Given $\hat{T}$ compute γ.

To that end, we next prove that the twist $\hat{T}$ can be expressed as:

$$\hat{T} = \hat{T}_1 + \hat{T}_2 + \hat{T}_3,$$

where

$$\hat{T}_i = \begin{bmatrix} \boldsymbol{v}_{oi} \\ \omega_i \end{bmatrix},$$

in which $\boldsymbol{v}_{oi}$ is the velocity of the point of link L_i that instantaneously coincides with the origin O, assuming that all revolute joints, except joint i, are locked.

We first write the equation in expanded form

$$\begin{bmatrix} \boldsymbol{v}_o \\ \omega \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_{o1} \\ \omega_1 \end{bmatrix} + \begin{bmatrix} \boldsymbol{v}_{o2} \\ \omega_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{v}_{o3} \\ \omega_3 \end{bmatrix}$$

and note that if $\alpha = \theta_1 + \theta_2 + \theta_3$ is the orientation angle of the end effector, we have $\dot{\alpha} = \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3$. Since $\omega = \dot{\alpha}$ and $\omega_i = \dot{\theta}_i$, it must be $\omega = \omega_1 + \omega_2 + \omega_3$, which proves the angular part of the equation.

To prove that $\mathbf{v}_o = \mathbf{v}_{o1} + \mathbf{v}_{o2} + \mathbf{v}_{o3}$ we apply the law of composition of velocities.

Slide 22 For simplicity, we apply this law to the serial 2R manipulator depicted. Here link 1 is rotating about P at ω_1 with respect to the ground, and link 2 is rotating about Q at ω_2 with respect to link 1. Consider the following observational frames: a "frame 0" on the ground, and a "frame 1" on link 1. Frame 0 acts as the absolute frame. Then, the absolute velocity of any point R on link 2 is the sum of its velocity relative to frame 1 plus the absolute velocity that R would have if it were a fixed point in frame 1:

$$\boldsymbol{v}(R) = \boldsymbol{\omega}_2 \times \overrightarrow{QR} + \boldsymbol{\omega}_1 \times \overrightarrow{PR}$$

This equation is true for any point R of link 2. In particular, it is true when R is the origin point O of link 2 (the one that coincides with the origin). We thus can write

$$\boldsymbol{v}(O) = \boldsymbol{\omega}_2 \times \overrightarrow{QO} + \boldsymbol{\omega}_1 \times \overrightarrow{PO},$$

and, reordering and writing in "moment form" we have

$$\boldsymbol{v}(O) = \underbrace{\overrightarrow{OP} \times \boldsymbol{\omega}_1}_{\boldsymbol{v}_{o1}} + \underbrace{\overrightarrow{OQ} \times \boldsymbol{\omega}_2}_{\boldsymbol{v}_{o2}}$$

So, clearly,

$$\boldsymbol{v}_o = \boldsymbol{v}_{o1} + \boldsymbol{v}_{o2}$$

which proves the linear velocity part of the equation for the 2-link manipulator. The extension to a 3-link manipulator is straightforward.

Slide 23 Now we know that, for the 3R manipulator

$$\hat{T} = \hat{T}_1 + \hat{T}_2 + \hat{T}_3$$

where

$$\hat{T}_i = \begin{bmatrix} \boldsymbol{v}_{oi} \\ \omega_i \end{bmatrix}$$

Assuming that the coordinates of the i-th joint are (x_i, y_i), we can compute $\boldsymbol{v}_{oi}$ as:

$$\boldsymbol{v}_{oi} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_i & y_i & 0 \\ 0 & 0 & \omega_i \end{vmatrix} = \omega_i \begin{bmatrix} y_i\,\omega_i \\ -x_i\,\omega_i \\ 0 \end{bmatrix}$$

and hence

$$\hat{T}_i = \omega_i \begin{bmatrix} y_i \\ -x_i \\ 1 \end{bmatrix}$$

Thus we can write:

$$\hat{T} = \omega_1 \begin{bmatrix} y_1 \\ -x_1 \\ 1 \end{bmatrix} + \omega_2 \begin{bmatrix} y_2 \\ -x_2 \\ 1 \end{bmatrix} + \omega_3 \begin{bmatrix} y_3 \\ -x_3 \\ 1 \end{bmatrix}.$$

By defining

$$\boldsymbol{J} = \begin{bmatrix} y_1 & y_2 & y_3 \\ -x_1 & -x_2 & -x_3 \\ 1 & 1 & 1 \end{bmatrix}$$

we finally obtain the equation we were looking for:

$$\hat{T} = \boldsymbol{J} \cdot \boldsymbol{\gamma}.$$

This equation provides the mapping between the angular velocities at the actuated joints, $\boldsymbol{\gamma}$, and the resulting end-effector twist, $\hat{T}$. If we know the robot configuration, we know $\boldsymbol{J}$, and the equation solves the forward instantaneous kinematic problem. It is easy to compute $\hat{T}$ for a given $\boldsymbol{\gamma}$, and there will be one and only one such twist. To solve the inverse instantaneous kinematic problem for a given $\hat{T}$, we solve the associated system of linear equations. Depending on the situation, this system will have one solution, infinitely-many solutions, or no solution at all.

Slide 24 Notice that each column of $\boldsymbol{J}$ contains the Plücker coordinates $\hat{S}_1$, $\hat{S}_2$, and $\hat{S}_3$ of the lines parallel to the Z axis passing through the center of the corresponding joint. As we shall see soon, this interpretation is quite useful as it will allow us to rapidly detect singular configurations visually, those in which the serial robot loses dexterity.

The interpretation also works when a given joint is prismatic. A prismatic joint can be interpreted as a revolute joint at a point at infinity (the one in the direction orthogonal to the sliding direction of the joint). In that case, the column of $\boldsymbol{J}$ certainly gives the Plücker coordinates of the line through that point and parallel to the Z axis.

It can be shown also that the inverse of the Jacobian matrix can be computed as

$$\boldsymbol{J}^{-1} = \frac{1}{det\,\boldsymbol{J}} \begin{bmatrix} a_{23}\ \hat{s}_{23}^T \\ a_{31}\ \hat{s}_{31}^T \\ a_{13}\ \hat{s}_{12}^T \end{bmatrix},$$

where a_{ij} is the distance between J_i and J_j, and $\hat{s}_{ij}$ are the coordinates of the line in the XY plane passing through joints J_i and J_j. Moreover,

$$det\,\boldsymbol{J} = a_{23}\ \hat{s}_{23}{}^\mathsf{T}\ \hat{S}_1 = a_{31}\ \hat{s}_{31}{}^\mathsf{T}\ \hat{S}_2 = a_{12}\ \hat{s}_{12}{}^\mathsf{T}\ \hat{S}_3.$$

Slide 25 This completes our analogy table. The matrix $\boldsymbol{J}$ plays a role similar to the jacobian matrix $\boldsymbol{j}$ of parallel manipulators, but instead of encoding a mapping of forces, it encodes a mapping of velocities.

Slides 26 to 29 If we have a general chain of n links connected in series, the end-effector twist will be the sum of the intermediate link twists. This follows directly from applying the law of composition of velocities recursively to the whole chain.

The links are numbered from 1 to n. Link 1 is the one connected to ground. Link n is the end effector, and its twist relative to some observer in the ground, written in OXY, can be expressed as

$$\hat{T} = \hat{T}_1 + \ldots + \hat{T}_n$$

We can interpret $\hat{T}_i$ in either of two equivalent ways:

- As the twist of the i-th link in the observational frame 0 attached to ground, assuming all joints are locked except joint i.

- As the twist of link i in the observational frame of link $i-1$; i.e., as the twist of link i **relative to** link $i-1$.

In both cases, $\hat{T}_i$ is written in the same coordinate system used to describe the end-effector twist (e.g., OXY in the figure).

If the i-th joint is a revolute joint, then we already know:

$$\hat{T}_i = \omega_i \begin{bmatrix} y_i \\ -x_i \\ 1 \end{bmatrix}$$

If, instead, it is a prismatic joint, then

$$\hat{T}_i = v_i \begin{bmatrix} a_i \\ b_i \\ 0 \end{bmatrix}$$

where v_i is the signed magnitude of the linear velocity of the joint, and $[a_i, b_i]^\mathsf{T}$ is a unit vector providing the direction of such velocity. Thus, in general the velocity equation will take the form:

$$\hat{T} = \begin{bmatrix} & y_i & & a_j & \\ \ldots & -x_i & \ldots & b_j & \ldots \\ & 1 & & 0 & \end{bmatrix} \cdot \begin{bmatrix} \vdots \\ \omega_i \\ \vdots \\ v_j \\ \vdots \end{bmatrix}$$

asuming that joints i and j are of revolute and prismatic type, respectively.

Slide 30 Match each manipulator with its corresponding Jacobian matrix.

Slide 31 In a general robotic mechanism (one with either open or closed kinematic chains), we define the forward and inverse instantaneous kinematic problems (FIKP and IIKP) as follows:

- The FIKP consists in, given the vector of input velocities $\boldsymbol{\gamma}$ (those of the actuated joints), compute the whole velocity state of the mechanism, obtaining the end-effector twist $\hat{T}$ in particular.

- The IIKP consists in, given the end effector twist $\hat{T}$, compute the whole velocity state of the mechanism, obtaining the vector of input velocities $\boldsymbol{\gamma}$ in particular.

Here, the "whole velocity state" means "enough velocity coordinates to compute the velocity of any point on the mechanism". For example, in a $3R$ arm, it is enough to know ω_1, ω_2 and ω_3 to determine the velocity of any point of the arm, and in particular

$$\hat{T} = \omega_1 \hat{S}_1 + \omega_2 \hat{S}_2 + \omega_3 \hat{S}_3$$

which solves the FIKP.

We say that a configuration $\boldsymbol{q}$ of our robot is *nonsingular* if in such a configuration both the FIKP and the IIKP have one and only one solution for any value of $\boldsymbol{\gamma}$ and $\hat{T}$. Otherwise the configuration is said to be *singular*. Thus, in nonsingular configurations we have a one-to-one correspondence between the vectors $\boldsymbol{\gamma}$ and their compatible velocity states, and between the twists $\hat{T}$ and their compatible velocity states. At least one of the bijections is lost at a singular configuration: we may find that one of the two problems is undetermined (infinite solutions) or unsolvable (no solution), for some value of $\boldsymbol{\gamma}$ or $\hat{T}$.

Slide 32 To see this on an example, consider again the 3R manipulator. The IIKP will be unsolvable for some $\hat{T}$ when $det \boldsymbol{J} = 0$, because in such a case $\boldsymbol{J}^{-1}$ does not exist. Assume for convenience that we put the origin of the coordinate system in joint 1. Then

$$det\ \boldsymbol{J} = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & -x_2 & -x_3 \\ 1 & 1 & 1 \end{vmatrix} = x_2 y_3 - x_3 y_2,$$

and $det\ \boldsymbol{J} = 0$ whenever

$$x_2 y_3 - x_3 y_2 = \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} = 0,$$

so that the singular configurations are those in which the three revolute joints are aligned.

Slide 33 We here see some examples of singular configurations. Do you see that in them there are twists $\hat{T}$ that cannot be produced by any combination of the angular velocities $\omega_1, \omega_2, \omega_3$? And that some other twists $\hat{T}$ can be produced by infinitely-many combinations of $\omega_1, \omega_2, \omega_3$? Try to find them for yourself.

Intuitively we see that in these configurations the velocity of P cannot be parallel to the line of the joints. Velocities of P orthogonal to that line, on the other hand, can be produced by infinitely-many combinations of the angular joint velocities.

Slide 34 In this configuration P can move under any velocity! Is this a singular configuration? Why?

Slide 35 Compute $det\ \boldsymbol{J}$ in each case and determine which of these configurations is singular. What happens with the instant centers relative to the joint twists $\hat{T}_1, \hat{T}_2, \hat{T}_3$ in each manipulator?

Slides 36-38 On a serial manipulator, only the IIKP may become unsolvable. To show a robotic mechanism where both the FIKP and IIKP may become unsolvable, we next analyze the 4-bar mechanism.

Consider the shown 4R manipulator. We have

$$\hat{T} = \hat{T}_1 + \hat{T}_2 + \hat{T}_3 + \hat{T}_4.$$

where $\hat{T}$ is the twist of the end effector relative to the ground, and $\hat{T}_i$ is the twist of link i relative to link $i-1$. The joints are numbered from 1 to 4, with joint i being the one between links i and $i-1$.

If we connect the end-effector to ground, then the effector is fixed and $\hat{T} = 0$. Therefore, for such a closed kinematic chain

$$\hat{T}_1 + \hat{T}_2 + \hat{T}_3 + \hat{T}_4 = \mathbf{0}.$$

Now, suppose that we actuate ω_4 and that we consider ω_1 as the output velocity. This corresponds to viewing $\hat{T}_4$ and $\hat{T}_1$ as input and output twists respectively. Note that the whole velocity state is determined once we know $\omega_1, \ldots, \omega_4$. With the choices just made, in this mechanism:

- The FIKP consists in, given ω_4, compute $\omega_1, \omega_2, \omega_3$ (from which it is immediate to obtain $\hat{T}_1$).
- The IIKP consists in, given ω_1 (and hence $\hat{T}_1$), compute $\omega_2, \omega_3, \omega_4$.

Slide 39 Let us solve the FIKP for this 4 bar mechanism. Since

$$\hat{T}_1 + \hat{T}_2 + \hat{T}_3 + \hat{T}_4 = \mathbf{0},$$

we can write

$$\omega_4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = - \underbrace{\begin{bmatrix} 0 & y_2 & y_3 \\ -x_1 & -x_2 & -x_3 \\ 1 & 1 & 1 \end{bmatrix}}_{\boldsymbol{J}_f} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

where ω_i is the relative angular velocity of joint i. The output velocity is ω_1, and ω_2 and ω_3 are passive joint velocities. Clearly, the FIKP will have one and only one solution for any value of ω_4 whenever $\boldsymbol{J}_f$ is a full rank matrix.

The configurations where *det* $\boldsymbol{J}_f = 0$ are called *forward* singularities of the 4-bar mechanism, because in them the FIKP is unsolvable or undetermined for some input ω_4. Since $\boldsymbol{J}_f$ is equivalent to the Jacobian of a 3R manipulator, such singularities arise when joints 1, 2, and 3 become aligned. In such a situation, the only *feasible* input velocity is $\omega_4 = 0$. Moreover, the input velocity ω_4 does not determine the global velocity state of the manipulator. Intuitively, observe that when we lock joint 4, joint 2 is still able to move infinitesimally in the direction orthogonal to the line through joints 1, 2, and 3.

Slide 40 To solve the IIKP on the same manipulator, we write

$$\hat{T}_1 = -(\hat{T}_2 + \hat{T}_3 + \hat{T}_4),$$

or, in expanded form,

$$\omega_1 \begin{bmatrix} 0 \\ -x_1 \\ 1 \end{bmatrix} = - \underbrace{\begin{bmatrix} y_2 & y_3 & 0 \\ -x_2 & -x_3 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{\boldsymbol{J}_i} \begin{bmatrix} \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix}.$$

In this case we want to find the values of ω_2, ω_3, and ω_4 that produce the desired output velocity ω_1. Similarly as before, the IIKP will be unsolvable or undetermined for some ω_1 whenever *det* $\boldsymbol{J}_i = 0$, which occurs if and only if joints 2, 3, and 4 are aligned. In such a configuration $\omega_1 = 0$ and $\hat{T}_1 = \mathbf{0}$ necessarily. Any value of $\hat{T}_4$ is in principle possible, and hence the velocity state of the manipulator is undetermined. Likewise, any $\hat{T}_1 \neq \mathbf{0}$ is unfeasible.

The configurations in which *det* $\boldsymbol{J}_i = 0$ are called *inverse* singularities of the 4-bar mechanism, because in them the FIKP is unsolvable or undetermined for some value of ω_4.

Following an analogous procedure, one can define the forward and inverse singularities of a general mechanism. In module 4 we shall illustrate the two kinds of singularities on a large class of useful parallel robots.