The Stability of Buckled Icosahedral Structures

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Résumé

On peut construire une famille d'icosaèdres concaves dont chaque icosaèdre serait composé de huit triangles équilatéraux d'une longueur d'arête e et de 12 triangles isocèles d'une longueur d'arête longue 2a. Dépendant de la valeur du ratio 2a/e, le polyèdre en résultant peut être rigide, infinitésimalement mobile ou bistable.

1. The buckled icosahedron. Among the unstable polyhedral structures described in (Goldberg 1978) is the buckled icosahedron shown in **Figures 1 and 2.** The short edges of length e are the edges of eight equilateral triangle faces. The long edges of length 2a are the long sides of twelve isoceles triangle faces. If the faces are considered as rigid triangular plates hinged at the edges, then this polyhedral structure may be rigid, infinitesimally movable or bistable, depending on the ratio 2a/e.

Let the variable distance between the two apices of two adjacent isoceles triangles be designated by 2x. Then, we have the equation

(1) $(a - x)^2 + x^2 + a^2 = e^2$.

From this equation, we obtain

(2) $2x = a \pm \sqrt{2e^2 - 3a^2}$.

We insist that $0 < x < a$ for some solutions to (2), which implies that $\sqrt{2} < 2a/e$. In order for there to be a solution to (2), $2e^2 - 3a^2$ must be non-negative. So $\sqrt{2} < 2a/e \leqslant 2\sqrt{6}/3$.

Hence, if $2e^2 - 3a^2$ is positive, there are two solutions for x. This signifies that there are two distinct closed polyhedral configurations that the twenty faces can assume. If the material of the plates is sufficiently flexible, as in a cardboard model, then either configuration can be deformed temporarily to snap into the other configuration. This is the bistable configuration mentioned in the foregoing paragraph.

2. The orthogonal icosahedron. If $2e^2 - 3a^2$ is zero, then there is only one solution. However, infinitesimal variations of x are possible. For this reason, it has been called a shaky configuration. It is the orthogonal icosahedron in which all the dihedral angles are right angles, and it is shown as Figure 1 in (Goldberg 1978). It was first described by (Jessen 1967). The ratio between the long edges and the short edges is $2a/e = 2\sqrt{6}/3 \approx 1.633$.

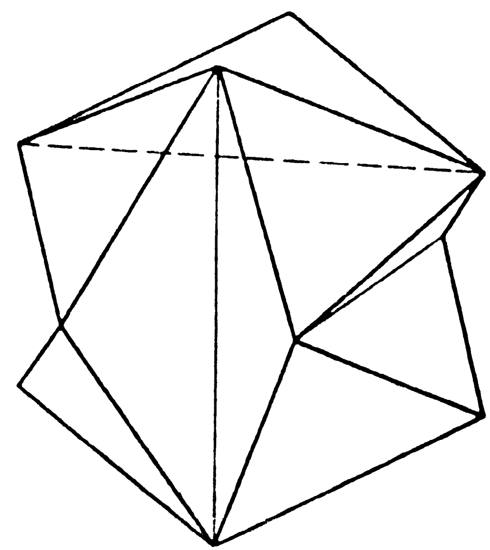

Figure 1.

3. The modified regular icosahedron. If six pairs of equilateral triangles of a regular icosahedron are replaced by isoceles triangles, without moving the vertices, then $2a/e = (\sqrt{5} + 1)/2 \approx 1.618$ (the golden ratio). Then,

$2x = ((\sqrt{5} + 1) \pm \sqrt{14 - 6\sqrt{5}})\, e/4 =$

$\sqrt{5} + 1 \pm (3 \cdot \sqrt{5})\, e/4 = e$ or $(\sqrt{5} - 1)\, e/2 =$

e or 0.618 e.

This is a special case of the bistable configurations. Other bistable configurations are produced fo $1.414 \approx \sqrt{2} < 2a/e < 2\sqrt{6}/3 \approx 1.633$.

4. The degenerate rigid polyhedra. When $2a/e = \sqrt{2}$, then either of two degenerate rigid polyhedra is produced. Each pair of isoceles triangles may become a single square face, and the icosahedron becomes the 14-faced cuboctahedron, one of the Archimedean solids. The other polyhedron is the regular octahedron which is produced when each pair of isoceles faces is collapsed inside of the octahedron.

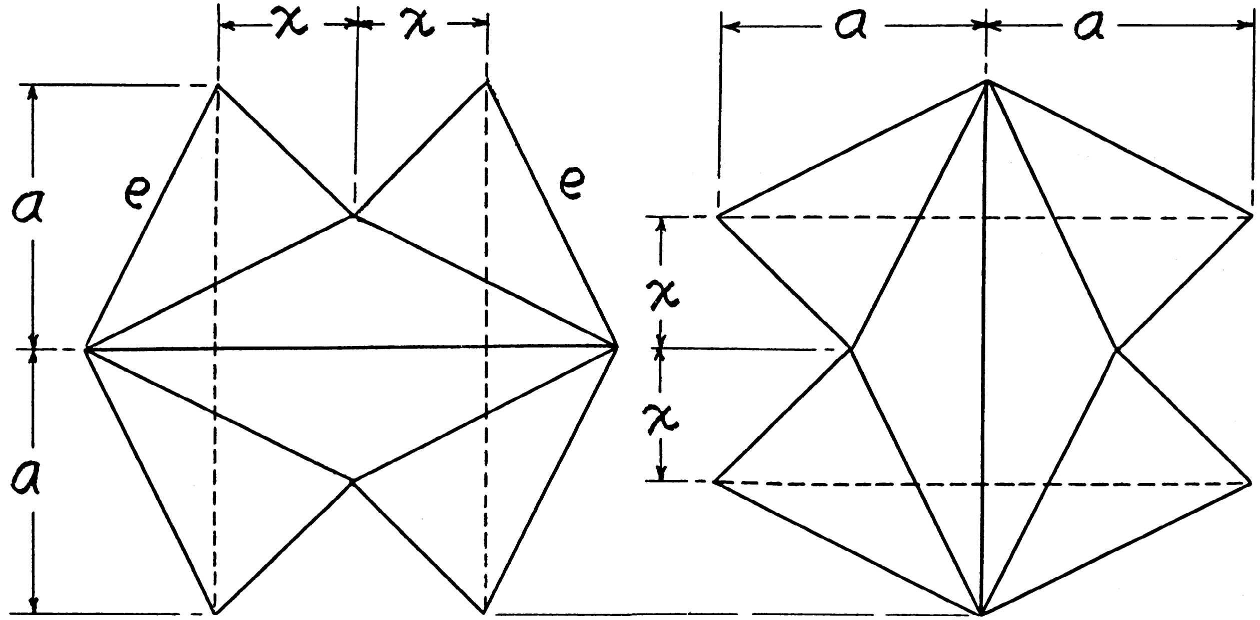

Figure 2. **Side view.** **Front view.**

Michael Goldberg, engineer-mathematician, retired from the Bureau of Naval Weapons in Washington D.C. in 1963. His work included the design of structures and mechanisms, particularly analog computers for weapon control. In addition to his papers listed in *Structural Topology #4*, his name has appeared frequently as a problem solver in the mathematical journals since 1920.

Goldberg 1978 Michael Goldberg A—M—P	**Unstable Polyhedral Structures,** Mathematics Magazine 51, 165-170 (1978).	
Jessen 1967 B. Jessen A—M—P	**Orthogonal icosahedron** Nordisk Mat. Tidskr. 15, 90-96 (1967).	