Positional Inverse Kinematic Problems in $\mathcal{T}^n \times \Re^m$ Solved in $\mathcal{T}^{2(n+m)}$

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Abstract - Solving the inverse kinematic problem for a closed spatial mechanism with n translational and m rotational links is here reduced to the problem of navigating in the configuration space of the spherical orthogonal mechanism with, at most, $2(n+m)$ degrees of freedom.

A recursive algorithm to find the analytic solution of spherical orthogonal mechanisms is provided. This solution is thus amenable to differentiation, leading to a characterization of the tangent space of the self-motion manifold of such mechanisms. It is precisely this tangent space that provides the solution of the translational part of arbitrary spatial mechanisms.

The approach taken for navigating in the configuration space of the spherical orthogonal mechanism is numerical in nature, with the advantage of working in a space with a well-defined norm.

The problem of finding points on the self-motion manifold satisfying a set of extra constraints, such as joint limits, might be addressed through reasonable extensions of the algorithm presented.

I. Introduction

It is generally accepted that no satisfactory solution has been found for the general positional inverse kinematic problem. This is why the redundant manipulator literature has focused on the linearized first-order instantaneous kinematic relation between joint velocities (for recent advances in this area see [1] and [2]). Given the position and velocity states, the set of joint coordinates can be obtained either by directly solving positional equations (for a classical reference see [3]) or by solving the first-order differential equations derived from the linearization (see, for example, [4]). The latter alternative is relatively easier than the former. Nevertheless, it exhibits important difficulties to provide a description, at least a local one, of the self-motion manifold of the mechanism to be analyzed.

This paper deepens on the former alternative providing a way around these difficulties. To this end, we exploit the following two facts: (a) any kinematic loop equation can be modeled as the loop equation derived from the so-called n-bar mechanism by taking as many bars as needed and constraining some of the resulting degrees of freedom; and (b) the solution of the translational component of the loop equation of the n-bar mechanism is provided by the tangent bundle of the self-motion manifold of its spherical indicatrix. These two facts lead to a unified approach for the analysis of any closed loop containing independent revolute, prismatic and cylindrical pairs, which has been already published elsewhere [5] [6]. Herein, we concentrate ourselves on the description of a numerical algorithm derived from this analysis that permits converging,

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from an unfeasible point of the configuration space of the spherical indicatrix, to the *nearest* solution point. The relevance of this algorithm derives from the fact that it only requires knowledge of the self-motion manifold of the indicatrix, that is, it looks for points with certain characteristics in T^n, where n is not greater than twice the number of degrees of freedom (d.o.f.) of the mechanism.

The paper is structured as follows. Section II briefly describes the theory and notation used throughout the paper, showing the great relevance of the study of the n-bar mechanism and, in particular, of its spherical indicatrix: the orthogonal spherical mechanism. Section III is devoted to the analysis of this latter mechanism. In particular, a recursive analytic procedure for obtaining a local description of its self-motion manifold and its tangent space, at the lowest computational cost, is derived in this section. Section IV shows the application of the previous two sections to solving inverse kinematic problems of any closed loop containing independent revolute, prismatic and cylindrical pairs, through the definition of two error functions in T^n and, finally, Section V provides a summary of the main points in the paper, as well as the conclusions and prospects for future research.

II. Basics

A closed kinematic chain is determined by a sequence $X_1, \ldots, X_p$ of screws of the corresponding links through their Plücker coordinates. Its *geometry* is determined by the dual quantities [7] $\hat{\alpha}_1, \hat{\alpha}_3, \ldots, \hat{\alpha}_{2p-1}$, where $\hat{\alpha}_i = \alpha_i + \varepsilon a_i$, α_i being the angle from $X_{(i+1)/2}$ to $X_{(i+3)/2}$ and a_i, the distance from $X_{(i+1)/2}$ to $X_{(i+3)/2}$. Its *configuration* is determined by the dual quantities $\hat{\theta}_2, \hat{\theta}_4, \ldots, \hat{\theta}_{2p}$, where $\hat{\theta}_i = \theta_i + \varepsilon t_i$, θ_i being the angle around $X_{i/2}$ and t_i, the offset along $X_{i/2}$. Note that α_i, a_i, θ_i and t_i are the Denavit-Hantenberg parameters of the mechanism. Then, by assigning $\hat{\phi}_i = \hat{\alpha}_i$ when i is odd and $\hat{\phi}_i = \hat{\theta}_i$ when i is even, the loop equation of a closed kinematic chain can be expressed as:

$$\mathbf{F}(\hat{\Phi}) = \prod_{i=1}^{2p} \mathbf{Rx}(\hat{\phi}_i)\mathbf{Rz}(\pi/2) = \prod_{i=1}^{n} \mathbf{B}(\hat{\phi}_i) = \mathbf{I}, \tag{1}$$

where $\hat{\Phi} = (\hat{\phi}_1, \hat{\phi}_2, \ldots, \hat{\phi}_n) = (\phi_1 + \varepsilon t_1, \phi_2 + \varepsilon t_2, \ldots, \phi_n + \varepsilon t_n)$ is called the *vector of displacements*; $\Phi = (\phi_1, \phi_2, \ldots \phi_n)$, the *vector of rotations*; and $D = (d_1, d_2, \ldots d_n)$, the *vector of translations*. This equation corresponds to the loop equation of what in [5] and [6] is called the n-bar mechanism.

A. Fundamental Theorems

Theorem I. The solution of the non-dual part of (1)

$$\prod_{i=1}^{n} \mathbf{B}(\phi_i) = \mathbf{I}, \tag{2}$$

is a connected $(n-3)$-dimensional pseudomanifold that can be characterized, outside of its singular points and at least locally, using $r = n-3$ parameters, $\Psi = (\psi_1, \psi_2, \ldots, \psi_r)$.

Proof. The fact that the solution is a connected pseudomanifold is proved in subsection B and local parameterizations are discussed in subsection C. □

Theorem II. *Spatial to spherical transference.* The solution of the dual part of (1), outside of the singular points of the non-dual part, can be expressed as:

$$D = \mathbf{K}\Lambda, \qquad \forall \Lambda = (\lambda_1, \ldots, \lambda_r)^T \in \Re^r, \tag{3}$$

where

$$\mathbf{K} = \begin{bmatrix} \frac{\partial \phi_1}{\partial \psi_1} & \cdots & \frac{\partial \phi_1}{\partial \psi_r} \\ \vdots & & \vdots \\ \frac{\partial \phi_n}{\partial \psi_1} & \cdots & \frac{\partial \phi_n}{\partial \psi_r} \end{bmatrix}. \tag{4}$$

Proof. For a full proof of this theorem see [5]. □

Equation (2) corresponds to the loop equation of an orthogonal spherical mechanism. This theorem shows the great relevance of deepening on the structure of the self-motion manifold of the orthogonal spherical mechanisms, and how a thorough understanding of them is very helpful in the study of spatial mechanisms.

B. The Self-Motion Set of the Orthogonal Spherical Mechanism as a Punched Manifold

The configuration space, C, of a spherical mechanism is a product space formed by the n-fold product of the individual variables of rotation, that is, $C = S^1 \times S^1 \times \ldots \times S^1 = T^n$, where T^n is an n-torus, which is a compact n-dimensional manifold.

An orthogonal spherical mechanism becomes redundant for $n > 3$. Then, let the redundant inverse kinematic solution of equation (2) be expressed as a $(n-3)$-dimensional algebraic set or *self-motion set*, M, embedded in T^n. This self-motion set, however, is not a $(n-3)$-manifold but rather a pseudomanifold or *punched manifold* because of the presence of singular points. Singular points correspond to those situations in which the mechanism becomes planar, that is when all axes of rotation lie on the same plane (see [5]). Then, from a topological point of view, splitting all these singular points yields a $(n-3)$-manifold M'. Thus, M can be obtained from M' by pinching M' at certain pairs of points.

In order to prove these facts, observe that equation (2) has a straightforward geometric interpretation as an n-sided spherical polygon. Consider a unit sphere centered at the coordinate origin. As a result of applying successive rotations, the z-axis will describe on the surface of the sphere a spherical polygon with sides of length ϕ_i and exterior angles equal to $\pi/2$. Alternatively, the y-axis will describe a spherical polygon with sides of length $\pi/2$ and exterior angles equal to ϕ_i.

Lemma I. The self-motion set of an orthogonal spherical mechanism with $n > 3$ is connected.

Proof. Consider the spherical polygon described by the y-axis, i.e. a polygon with all its sides of length $\pi/2$. Note that, by varying the angle between two sides of length

$\pi/2$, one can form a triangle whose third side may attain every value between 0 and π. Therefore, one can fix arbitrarily $n-3$ consecutive variables of an orthogonal spherical mechanism, this leading to a spherical chain with $n-2$ sides that can always be closed with the remaining two sides.

Moreover, if the resulting angle between these two sides is different from 0 and π, then there are two alternative solutions corresponding to the two possible symmetric placements of the two sides on the sphere. As the angle approaches 0 or π, the two solution branches fuse into one precisely at these two values.

To prove that the solution set is connected, it suffices to show that a reference configuration can be reached from every other configuration. Let us choose the reference configuration as $\phi_i = \pi, \forall i = 1, \ldots n$, when n is even, and as $\phi_1 = \phi_2 = \phi_3 = \pi/2, \phi_i = \pi, \forall i = 4, \ldots n$, when n is odd.

Now, from any initial configuration, one can make the exterior angles approach sequentially their values at the reference configuration. Thanks to the last two sides, the chain will remain closed throughout the process. □

Lemma II. $\Phi = (\phi_1, \ldots, \phi_n)$ is a singular point iff (i) $\phi_i (\bmod\ \pi) = 0$, $i = 1, \ldots, n$; and (ii) $(\sum_{i=1}^{n} \phi_i) (\bmod\ 2\pi) = 0$.

Proof. The former condition ensures that the mechanism lies on the plane defined by the first and the last axis, and the latter is a simplified version of the closure equation (2) when the former holds. Both provide a necessary and sufficient condition for the mechanism to be in a planar configuration and, hence, for the configuration point to be singular. □

Corollary I. When n is odd, the orthogonal spherical mechanism has no singularities.

Corollary II. When n is even, the number of singularities is 2^{n-2}.

Corollary III. When $n > 4$ the self-motion set remains connected after removing its singular points. (See [5] for a complete analysis of the 4-bar mechanism.)

C. Parameterizations and Symmetries of the Self-Motion Manifold

After removing the singular points, the self-motion set becomes an r-dimensional smooth manifold, M_c, of class C^∞, which will be called *self-motion manifold*. Then, r coordinates of the surrounding space T^n can be taken as local coordinates in the neighborhood of each point $\Phi_0 \in M_c$. This is, in fact, the implicit function theorem formulated in convenient terms, whose proof can be found in any textbook on differential geometry. In what follows we will study this simple parameterization.

Let us take r consecutive variables in the chain as parameters. Without loss of generality, let $\{\phi_1, \phi_2, \ldots, \phi_r\}$ be the set of parameters. Hence, the equation of rotations can be expressed as:

$$\mathbf{Rx}(\phi_{r+1})\ \mathbf{Rz}(\pi/2)\ \mathbf{Rx}(\phi_{r+2})\ \mathbf{Rz}(\pi/2)\ \mathbf{Rx}(\phi_{r+3})\ =\ \mathbf{A}, \tag{5}$$

which has always solution for any proper orthogonal matrix $\mathbf{A}$ encompassing all the parameters. In general, this equation has the following two discrete solutions:

$$\begin{aligned} \phi_{r+1} &= \text{atan2}(\pm a_{21}, \mp a_{31}) \\ \phi_{r+2} &= \mp\text{acos}(-a_{11}) \\ \phi_{r+3} &= \text{atan2}(\mp a_{12}, \mp a_{13}) \end{aligned} \tag{6}$$

where a_{ij} denotes the element (i,j) of **A**. One solution is obtained by taking the upper row of signs, and the other, by taking the lower one.

When $a_{11} = \pm 1$, there appear infinite solutions. The points of the self-motion manifold where this happens are called singularities of the parameterization, and it can be easily shown that they correspond to those situations in which the last three rotation axes are coplanar.

As it is shown in the next section, the above formulation, although correct, can be greatly improved using geometric arguments to reduce computational overhead during the computation of variables in terms of parameters, and partial derivatives of variables with respect to parameters.

Lemma III. *Symmetries.* Given a point $\Phi_0 = (\phi_1, \ldots, \phi_n)$ on the self-motion manifold, points $\Phi_1 = (\phi_1 + \pi, -\phi_2, \phi_3 + \pi, \phi_4, \ldots, \phi_n)$ and $\Phi_2 = (\phi_n, \phi_1, \ldots, \phi_{n-1})$ are also on it.

Proof. The first symmetry can be derived by analyzing (6). The second one is obvious. □

Corollary IV. The iterative computation of the symmetries in *Lemma III* leads at most to $n \cdot 2^n$ symmetric points for any point on the self-motion manifold.

III. Inverse Kinematics of Orthogonal Spherical Mechanisms

We have already seen that equation (2) has a geometric interpretation as an n-sided spherical polygon. In this section, after introducing some basic relations from Spherical Trigonometry, we will derive a recursive algorithm to find the inverse kinematics of any spherical orthogonal mechanism. The recursive nature of the solution allows us to compute the derivatives needed to find the solution for the translational component of the n-bar mechanism from the inverse kinematics of its spherical indicatrix, by applying equation (3).

A. Spherical Trigonometry Preliminaries

Let us denote ϕ_1, ϕ_2 and ϕ_3 the sides of an spherical triangle and α_{12}, α_{23} and α_{31} its exterior angles, the ***cosine***, ***sine-cosine*** and ***sine*** laws are [8]:

$$\begin{aligned} &\cos\alpha_{12} = \cos\alpha_{31}\cos\alpha_{23} - \sin\alpha_{31}\sin\alpha_{23}\cos\phi_3 \\ &-\sin\alpha_{12}\cos\phi_2 = \cos\alpha_{31}\sin\alpha_{23} + \sin\alpha_{31}\cos\alpha_{23}\cos\phi_3 \\ &\sin\alpha_{12}\sin\phi_2 = \sin\phi_3\sin\alpha_{31}. \end{aligned} \tag{7}$$

From this, one can derive the following relation for a triangle having two exterior angles equal to $\pi/2$. Taking $\alpha_{31} = \alpha_{23} = \pi/2$, we find that $\alpha_{12} = \pi - \phi_3$. Moreover, $\phi_1 = \phi_2 = \pi/2$.

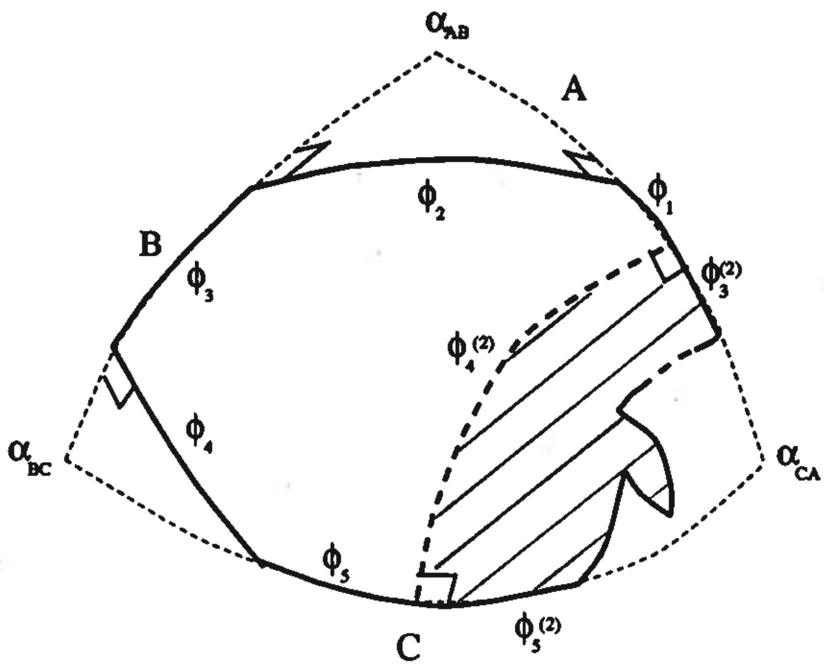

Fig. 1: Triangle ABC constructed by prolonging the sides ϕ_1, ϕ_3 and ϕ_5 of the original n-gon. Using the fact that the three small triangles have 2 exterior angles equal to $\pi/2$, the solution of the n-gon reduces to that of the $(n-2)$-gon, which is shown shaded.

Note that a sequence of $n > 2$ points on a unit sphere defines in general a unique spherical n-gon with sides $\leq \pi$, but a total of 2^n n-gons if we take into account sides of length $> \pi$. In the triangle relation above, we have taken the determination with sides $\leq \pi$.

B. A Recursive Algorithm for the Solution of Spherical Orthogonal Mechanisms

The spherical polygon corresponding to a spherical orthogonal loop has all its exterior angles equal to $\pi/2$. The crucial idea of the algorithm is to reduce the solution of one such n-gon to that of an $(n-2)$-gon. To this end, we construct a triangle by prolonging three alternate sides of the n-gon, as shown in *fig. 1.*

Note that two exterior angles in each of the two small triangles formed in this way have value $\pi/2$, and we can apply the relation derived above for spherical triangles to express two exterior angles and one side of the big spherical triangle as functions of ϕ_2, ϕ_3 and ϕ_4. Moreover, we can construct the third small triangle by drawing a side perpendicular to both ϕ_1 and ϕ_5. In this way, the three new sides $\phi_3^{(2)}$, $\phi_4^{(2)}$ and $\phi_5^{(2)}$ are originated. In sum, we have the following expressions for the sides of the big triangle:

$$\begin{aligned} A &= \phi_1 - \phi_3^{(2)} + \pi \\ B &= \phi_3 + \pi \\ C &= \phi_5 - \phi_5^{(2)} + \pi, \end{aligned} \tag{8}$$

and the following expressions for its exterior angles:

$$\begin{aligned} \alpha_{AB} &= \pi - \phi_2 \\ \alpha_{BC} &= \pi - \phi_4 \\ \alpha_{CA} &= \pi - \phi_4^{(2)}. \end{aligned} \tag{9}$$

Now, we can use the three laws of Spherical Trigonometry included in the preceding subsection to relate the new variables $\phi_3^{(2)}$, $\phi_4^{(2)}$ and $\phi_5^{(2)}$ to the old ones ϕ_1, ϕ_2, ϕ_3, ϕ_4 and ϕ_5 as follows:

$$\begin{aligned} \phi_3^{(2)} &= \phi_1 - \text{atan2}(\sin\phi_4\sin\phi_3, -\cos\phi_4\sin\phi_2 + \sin\phi_4\cos\phi_3\cos\phi_2) \\ \phi_4^{(2)} &= \text{acos}(-\cos\phi_2\cos\phi_4 - \sin\phi_2\sin\phi_4\cos\phi_3) \\ \phi_5^{(2)} &= \phi_5 - \text{atan2}(\sin\phi_2\sin\phi_3, -\cos\phi_2\sin\phi_4 + \sin\phi_2\cos\phi_3\cos\phi_4) \end{aligned} \tag{10}$$

As a result of this process, we have reduced the solution of the original n-gon to the solution of the $(n-2)$-gon with sides $\phi_3^{(2)}$, $\phi_4^{(2)}$, $\phi_5^{(2)}$, $\phi_6, \ldots \phi_n$. Note that the resulting $(n-2)$-gon has all its exterior angles also equal to $\pi/2$.

The following recurrence equations can thus be easily derived:

$$\begin{aligned} \phi_{2i-1}^{(i)} &= \phi_{2i-3}^{(i-1)} - \text{atan2}(f(\phi_{2i-1}^{(i-1)}, \phi_{2i}^{(1)}), g(\phi_{2i-2}^{(i-1)}, \phi_{2i-1}^{(i-1)}, \phi_{2i}^{(1)})), \\ \phi_{2i}^{(i)} &= \text{acos}(h(\phi_{2i-2}^{(i-1)}, \phi_{2i-1}^{(i-1)}, \phi_{2i}^{(1)})), \\ \phi_{2i+1}^{(i)} &= \phi_{2i+1}^{(1)} - \text{atan2}(f(\phi_{2i-2}^{(i-1)}, \phi_{2i-1}^{(i-1)}), g(\phi_{2i}^{(1)}, \phi_{2i-1}^{(i-1)}, \phi_{2i-2}^{(i-1)})), \end{aligned} \tag{11}$$

where, to ease the notation, we have introduced the following functions:

$$\begin{aligned} f(\alpha, \beta) &= \sin\alpha \ \sin\beta, \\ g(\alpha, \beta, \gamma) &= -\sin\alpha \ \cos\beta + \cos\alpha \ \sin\gamma \ \cos\beta, \\ h(\alpha, \beta, \gamma) &= -\cos\alpha \ \cos\beta - \sin\alpha \ \sin\gamma \ \cos\beta \end{aligned} \tag{12}$$

and this is valid $\forall i = 2, \ldots \lceil (n-4)/2 \rceil$.

The initial conditions can be derived in the same way depending on whether n is even or odd. In the former case, one has to consider the equations for an hexagon, which are essentially the same as the recurrence equations. Taking $n = 2i + 4$, these equations are:

$$\begin{aligned} \phi_n &= \text{atan2}(f(\phi_{2i}^{(i)}, \phi_{2i+1}^{(i)}), g(\phi_{2i-1}^{(i)}, \phi_{2i}^{(i)}, \phi_{2i+1}^{(i)})), \\ \phi_{n-1} &= \text{acos}(h(\phi_{2i-1}^{(i)}, \phi_{2i+1}^{(i)}, \phi_{2i}^{(1)})), \\ \phi_{n-2} &= \text{atan2}(f(\phi_{2i-1}^{(i)}, \phi_{2i}^{(i)}), g(\phi_{2i+1}^{(i)}, \phi_{2i}^{(i)}, \phi_{2i-1}^{(i)})). \end{aligned} \tag{13}$$

While, when n is odd, one has to consider the equations for a pentagon, which follow from those of the hexagon by forming a small triangle (with two exterior angles equal to $\pi/2$) limited by two consecutive sides of the pentagon. This amounts to making the following substitutions in the equations for the hexagon: $\phi_{2i-1}^{(i)}, \phi_{2i}^{(i)}$ and $\phi_{2i+1}^{(i)}$ have to be replaced by $\phi_{2i-1}^{(i)} - \pi/2$, $\pi/2$ and $\phi_{2i}^{(i)} - \pi/2$, respectively. Moreover, $n = 2i + 3$ in this case.

The above recurrence equations, together with the initial conditions, provide a general analytic expression of the inverse kinematics of spherical orthogonal mechanisms. The functions involved can be easily differentiated, leading to the general solution of spatial orthogonal mechanisms, as explained below.

C. Tangent Space

We begin by noting that most partial derivatives of the functions introduced in the preceding subsection can be written in terms of the functions themselves. The complete listing of these derivatives follows:

$$
\begin{aligned}
&\frac{\partial f}{\partial \alpha} = \cos\alpha \ \sin\beta, \quad \frac{\partial g}{\partial \alpha} = h(\alpha,\beta,\gamma), \quad \frac{\partial h}{\partial \alpha} = -g(\alpha,\beta,\gamma), \\
&\frac{\partial f}{\partial \beta} = \sin\alpha \ \cos\beta, \quad \frac{\partial g}{\partial \beta} = -\cos\alpha \ f(\alpha,\beta), \quad \frac{\partial h}{\partial \beta} = \sin\gamma \ f(\alpha,\beta), \\
&\frac{\partial g}{\partial \gamma} = f(\alpha,\beta) + \cos\alpha \ \cos\gamma \ \cos\beta, \quad \frac{\partial h}{\partial \gamma} = -g(\alpha,\beta,\gamma).
\end{aligned}
\tag{14}
$$

With these derivatives and those of the functions atan2 and acos, the partial derivatives of the variables $\phi_{n-2}, \phi_{n-1}, \phi_n$ with respect to the parameters $\phi_1, \dots \phi_{n-3}$ can be found using the same recursive structure of the algorithm described above. Space limitations prevent us from giving an exhaustive listing of all these derivatives, but a sample of three:

$$
\begin{aligned}
&\frac{\partial \phi_{2i-1}^{(i)}}{\partial \phi_{2i-2}^{(i-1)}} = \frac{f(\phi_{2i-1}^{(i-1)}, \phi_{2i}^{(1)}) \ h(\phi_{2i-2}^{(i-1)}, \phi_{2i-1}^{(i-1)}, \phi_{2i}^{(1)})}{f^2(\phi_{2i-1}^{(i-1)}, \phi_{2i}^{(1)}) + g^2(\phi_{2i-2}^{(i)}, \phi_{2i-1}^{(i)}, \phi_{2i}^{(1)})}, \\
&\frac{\partial \phi_{2i}^{(i)}}{\partial \phi_{2i-2}^{(i-1)}} = \frac{-g(\phi_{2i-2}^{(i-1)}, \phi_{2i-1}^{(i-1)}, \phi_{2i}^{(1)})}{\sin\phi_{2i}^{(i)}}, \\
&\frac{\partial \phi_{2i+1}^{(i)}}{\partial \phi_{2i+1}^{(1)}} = 1.
\end{aligned}
\tag{15}
$$

Now note that it is not necessary to derive the full analytic expression of these derivatives, since only the values of the derivatives at particular points are required and these can be recursively evaluated using the formulas above.

These derivatives conform the matrix **K** in equation (3), leading to the solution for the translational component of the n-bar mechanism.

In the next section we will show how the solution of the n-bar mechanism can be used to find the inverse kinematics of arbitrary spatial mechanisms.

IV. Inverse kinematics of Arbitrary Single Closed-Loop Mechanisms

By constraining some of the variables in (1), one can model any closed kinematic loop containing independent translational and rotational pairs. Thus, we define $\mathcal{T}$

and $\mathcal{R}$ as the set of indices of the constrained translational and rotational d.o.f. and $|\mathcal{T}|$ and $|\mathcal{R}|$ their cardinalities, respectively. Note that, if the Denavit-Hantenberg parameters are properly taken, $|\mathcal{T}|+|\mathcal{R}|\leq n$.

Let $\tilde{\Phi} = (\ldots, \phi_i, \ldots)$, $i \in \mathcal{R}$, and $\tilde{D} = (\ldots, d_j, \ldots)$, $j \in \mathcal{T}$, where the elements of $\tilde{D}$ are scaled so that $\|\tilde{D}\| = 1$.

Once we choose a stating point, $\Phi^0 = (\phi_1^0, \ldots, \phi_n^0)$, on the self-motion manifold of the spherical indicatrix of the corresponding n-bar mechanism and a set of values for the constrained d.o.f., we introduce two errors, called translational and rotational, that will allow us to direct the search from Φ^0 towards a solution of the mechanism under analysis.

In general, we have to find Λ so that

$$\tilde{D} = \tilde{\mathbf{K}}\Lambda, \tag{16}$$

where

$$\tilde{\mathbf{K}} = \begin{bmatrix} \vdots & & \vdots \\ \frac{\partial \phi_i}{\partial \psi_1} & \cdots & \frac{\partial \phi_i}{\partial \psi_r} \\ \vdots & & \vdots \end{bmatrix}, \quad i \in \mathcal{T}. \tag{17}$$

Nevertheless, this is not always possible and the value of Λ that provides the closest value of $\tilde{D}$ to the desired one, in the least squares sense, which will be called Λ^0 is that which minimizes the residual $r = \|\tilde{\mathbf{K}}\Lambda - \tilde{D}\|$. Then, the translational error $\mathcal{E}_t$ is defined as:

$$\mathcal{E}_t(\Phi) = \|\tilde{\mathbf{K}}\Lambda^0 - \tilde{D}\|. \tag{18}$$

There are several numerical approaches for obtaining Λ^0 (see [9]).

On the other hand, the rotational error, $\mathcal{E}_r$, is simply defined as:

$$\mathcal{E}_r(\Phi) = \frac{\sum_{i\in\mathcal{R}}(\phi_i - \phi_i^0)(\text{mod } 2\pi)}{2\pi\,|\mathcal{R}|}. \tag{19}$$

It is clear that $0 \leq \mathcal{E}_t(\Phi) \leq 1$ and $0 \leq \mathcal{E}_r(\Phi) \leq 1$, and a solution of the analyzed mechanism is found iff $\mathcal{E}_t = \mathcal{E}_r = 0$

The implementation is now at the level that permits finding inverse kinematic solutions for the n-bar mechanism when all translations are constrained and all rotations ramain free. Then, only the translational error is considered. The starting point Φ_0 is randomly generated. But, in order to improve speed, the translational error function is evaluated in all its symmetric points. Then, the symmetric point with minimum translational error is effectively used as starting point.

Since, for the moment, partial derivatives of errors are not available, we are bound to using minimization methods requiring only function evaluations, not derivatives. Obviously, this is not very efficient. The *downhill simplex method* [9] has been chosen to get something working quickly.

V. Conclusions

This paper presents a generalized solution to the inverse kinematics of single closed chains with arbitrary number of degrees of freedom using the concept of ***spatial to spherical transference***.

The procedure requires points on the self-motion manifold of the spherical indicatrix of the n-bar mechanism and their tangent planes, which can be either analytically or numerically computed. An analytic solution is provided.

A foreseen extension of this work is finding a trajectory between two predetermined configurations of the mechanism. This can become more difficult if joint limits are to be taken into account.

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