Straightening-Free Algorithm for the Singularity Analysis of Stewart-Gough Platforms with Collinear/Coplanar Attachments

Júlia Borràs, Federico Thomas, and Carme Torras

Abstract An algorithm to derive the pure condition of any double-planar Stewart-Gough platform in a standard form suitable for comparison is presented. By applying the multilinear properties of brackets directly to the superbracket encoding of the pure condition, no straightening is required. It is then shown that any 3-3 platform has a corresponding 6-6 platform having its same superbracket, meaning that they have identical singularity loci. In general, the superbracket of any double-planar platform can be written as a linear combination of the superbrackets of 3-3 platforms, leading to a direct singularity assessment by inspecting the resulting decomposition.

1 Introduction

The usual method for identifying the singular configurations of a Stewart-Gough platform is to find those configurations in which the vectors of Plücker coordinates of its leg-lines become linearly dependent [1]. This condition is equivalent to assessing whether the determinant of the kinematic transformation matrix **K**, relating prismatic joint velocities with end-effector twists, vanishes. An alternative method to investigate the singularities of a Stewart-Gough platform is to consider its *pure condition*, a polynomial of 4×4 determinants, termed *brackets*, the four columns of which correspond to the coordinates of four leg attachments written in homogeneous coordinates. The pure condition for a Stewart-Gough platform becoming zero is equivalent to the determinant of **K** vanishing.

The pure condition for the general Stewart-Gough platform is an expression involving 16 different bracket monomials, each monomial consisting of the product of three brackets. This pure condition, with the labeling of Fig. 1, can be expressed as:

The authors are with the Institut de Robòtica i Informàtica Industrial (CSIC-UPC), Llorens Artigas 4-6, 08028 Barcelona, Spain, e-mail: {jborras, fthomas, ctorras}@iri.upc.edu

$$
\begin{aligned}
&[abgi][cdhk][efjl]-[ghac][ijbe][kldf]-[abgi][cdhl][efjk]+[ghac][ijbf][klde]\\
&-[abgj][cdhk][efil]+[ghad][ijbe][klcf]+[abgj][cdhl][efik]-[ghad][ijbf][klce]\\
&-[abhi][cdgk][efjl]+[ghbc][ijae][kldf]+[abhi][cdgl][efjk]-[ghbc][ijaf][klde]\\
&+[abhj][cdgk][efil]-[ghbd][ijae][klcf]-[abhj][cdgl][efik]+[ghbd][ijaf][klce]. \qquad (1)
\end{aligned}
$$

It can be proved that the pure condition equals $\frac{1}{l_1 l_2 l_3 l_4 l_5 l_6}\det(\mathbf{K})$, where $l_1,\ldots,l_6$ are the six leg lengths.

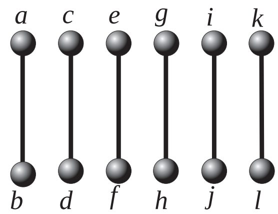
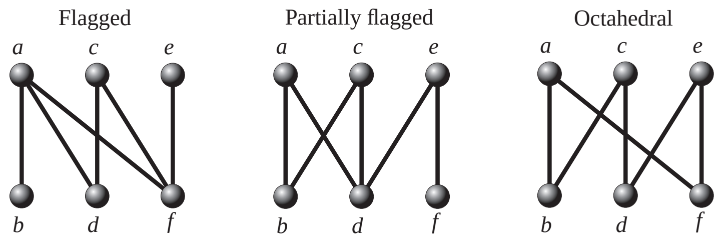


Fig. 1 Labeling for the attachments of a general 6-6 Stewart-Gough platform.

The advantage of the bracket representation of the linear dependence of leg-lines is seen when investigating simplified forms of the general Stewart-Gough platform in which some legs share attachments and/or some attachments are collinear or coplanar. Placing constraints on the geometrical structure of the platform reduces the number of bracket terms to a manageable level, thus offering the opportunity for simple geometrical interpretations of the singularities. For example, the pure conditions of the three possible 3-3 Stewart-Gough platforms in Fig. 2 [2, 3, 4] reduce to the bracket expansions appearing in the same figure.

It is worth nothing that a pure condition can be written in many different equivalent forms. Obtaining the shortest form for each case may need applying *syzygies* (see [4] for details).



$[abdf][acdf][acef]$ $\quad$ $[abdc][acde][bdef]$ $\quad$ $[abfc][acde][bdef] + [abfd][acbe][cdef]$

Fig. 2 The three possible topologies for 3-3 Stewart-Gough platforms –flagged, partially flagged and octahedral– and their corresponding pure conditions.

There is a direct connection between the exterior calculus (Grassmann-Cayley algebra [5]) and bracket algebra. For instance, the exterior product of four points in space in Grassmann-Cayley algebra translates directly to a bracket. It is generally accepted that the conversion of Grassmann-Cayley algebra to bracket algebra is straightforward. Moreover, bracket algebra can be converted to coordinate algebra by replacing points in brackets with their homogeneous coordinates and expanding the determinants into polynomials. Unfortunately, the reverse procedures are not trivial. The conversion from polynomials of coordinates to bracket polynomials is, in general, extremely difficult. The operation that converts a bracket expression into a Grassmann-Cayley expression is called Cayley factorization, for which no general algorithm exists yet.

The Grassmann-Cayley expression for the pure condition is known as the *superbracket* [6]. Given the general 6-6 Stewart-Gough platform in Fig. 1, its superbracket is represented as $[ab, cd, ef, gh, ij, kl]$, where each term rs is the Grassmann-Cayley algebra entity representing the line defined by points r and s.

In [7], the multilinear properties of brackets were exploited to obtain simplified pure conditions for platforms with aligned attachments in the base and/or the platform. The algorithm presented in [7] uses the so-called *straightening* procedure to obtain canonical bracket forms that can be compared. Unfortunately, the obtained bracket expressions are long and it is difficult to state if they are in their shortest form. Here, by applying the multilinear properties of brackets directly to the superbracket instead of the brackets in the pure condition, an important simplification is obtained. Actually, the straightening procedure is avoided.

The proposed algorithm permits to express the pure condition of any double-planar platform (that is, any parallel platform with its platform and base attachments coplanar) as a linear combination of the pure conditions of 3-3 platforms with flagged, partially-flagged and octahedral topology. As a consequence, the presented algorithm permits detecting singularity equivalences between double-planar platforms in a straightforward way.

This paper is structured as follows. Section 2 presents the algorithm. Based on it, section 3 shows how platforms with different topologies can have equivalent pure conditions and, as a consequence, the same singularity locus. As another possible application, section 4 shows how the pure condition of an arbitrary double-planar platform can always be expressed as a linear combination of pure conditions of 3-3 platforms. Finally, section 5 summarizes the main results.

2 Decomposing a superbracket into simple superbrackets

Using the same notation as in [7], the collinearity condition of point a lying on line pq can be expressed as:

$$a = k_1 p + (1 - k_1) q. \tag{2}$$

In what follows, the points that are used to define an alignment will be called *characteristic points* (p and q in the expression above), and those expressed as a linear combination of them, *composite points* (a in the expression above).

Now, using the distributive property of the join operation and the scalar multiplication, the expression of a line through points a and b can be written as $ab = a \vee b = (k_1 p + (1 - k_1) q) \vee b = k_1 pb + (1 - k_1) qb$. Then, using the distributive property again, a superbracket containing point a can be expanded into two superbrackets as follows:

$$[ab, cd, ef, gh, ij, kl] = k_1[pb, cd, ef, gh, ij, kl] + (1-k_1)[qb, cd, ef, gh, ij, kl]. \quad (3)$$

By recursively applying this operation to a superbracket involving composite points, it is possible to express it as the sum of superbrackets depending only on characteristic points, called *simple superbrackets*.

Finally, since the elements in a superbracket can be permuted using the same rules as those used when permuting columns or rows in ordinary determinants, the elements of the resulting simple superbrackets can be put in lexicographic order to make their comparison possible.

The above procedure in algorithmic form appears in Algorithm 1. The function $ZeroSB(sbr)$ checks whether a simple superbracket is identically zero. The easiest way to do this, without writing coordinates, is to expand the superbracket in terms of brackets using (1) and setting to zero all brackets containing the same point at least twice. The function $SortSB(sbr)$ sorts the lines of a superbracket in a lexicographic order and computes the sign of the corresponding permutation.

Algorithm 1 *ExpandSB*(): Expansion of a superbracket containing composite points

Input: $sbr = [a, b, c, d, e, f, g, h, i, j, k, l]$: list of 12 points representing a superbracket.
Output: a linear combination of simple superbrackets.

```
if sbr is a simple superbracket then
  if ZeroSB(sbr) then
    return 0;
  else
    return SortSB(sbr);
  end if
else
  for i=1 to 12 do
    if sbr[i] is a composite point then
      coe, points ← Split(sbr[i]);{Ex: in (2) it would be, coe = [k_1, (1 − k_1)], points = [p, q]}
      comp_1 ← coe[1] · ExpandSB(sbr_{sbr[i]=points[1]});{Substitution of the i-th member}
      comp_2 ← coe[2] · ExpandSB(sbr_{sbr[i]=points[2]});
      return comp_1 + comp_2;
    end if
  end for
end if
```

Once a linear combination of simple superbrackets is obtained, one can substitute each of them by a bracket expression using the general formula in (1) to obtain the pure condition for the analyzed platform. Note that the obtained output is useful, even before expressing it in terms of brackets, to detect singularity equivalences between platforms. This is shown in Section 3.

Finally, note that a coplanarity among attachments can be expressed as a linear combination of three points: if a belongs to the plane pqv, then $a = k_{11}p + k_{12}q + (1 - k_{11} - k_{12})v$ for some scalars k_{11} and k_{12} and, therefore, the proposed algorithm can be easily extended to deal with coplanarities. In double-planar platforms, all attachments, either in the base or the platform, can be expressed in terms of only three characteristic points. As a consequence, the output of the proposed algorithm would always be a linear combination of superbrackets corresponding to platforms with flagged, partially flagged and octahedral topology. Thus, it can be said that the three possible 3-3 platforms behave as a basis for all double-planar Stewart-Gough platforms. This is exemplified in Section 4.

3 Platforms with different topologies and the same singularities

Figure 3 displays three inputs supplied to the proposed algorithm, together with the obtained outputs. Each input consists of the twelve points representing the superbracket of a 6-6 platform, and all outputs are in the form of a simple superbracket multiplied by a constant K_i. These three outputs are very special cases, corresponding to the three topologies for 3-3 platforms appearing in Fig. 2. Thus, there exist 6-6 platforms that, with the adequate alignments for their leg attachments, give a single-term decomposition. Moreover, they have singularities equivalent to those of 3-3 platforms with flagged, partially-flagged or octahedral topologies. And, to the best of our knowledge, there are no other 6-6 platforms than those shown in Fig. 3 with singularities directly equivalent to those of a 3-3 platform.

The constant factors K_1, K_2, and K_3 multiplying the obtained decompositions (which, in these cases, consist of a single superbracket), when equated to zero, give the condition for each 6-6 platform to become architecturally singular. Indeed, if one of these constants is zero, the corresponding superbracket is identically zero and, as a consequence, the platform is always singular independently of the lengths of its legs. $K_1 = 0$ and $K_2 = 0$ correspond to the cross-ratio condition of the line-line component [8], as already noted in [9, 10, 7, 11], and $K_3 = 0$ corresponds to the architecturally singular condition of the Griffis-Duffy platform, in concordance with results in [12, 13].

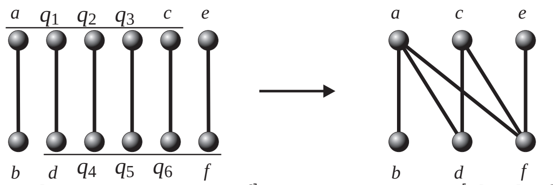
4 Double-planar platforms and the three 3-3 topologies

Consider the platform in Fig. 4. It has two alignments in the base and one alignment involving five points in the platform. When decomposing its superbracket using the proposed algorithm, five simple superbrackets are obtained (Output 1 in Fig. 4). Two of them correspond to superbrackets associated with 3-3 platforms with partially-flagged topology, and the other three with 3-3 platforms with flagged topology. Now, we can translate each of these simple superbrackets into bracket polynomials, using the expressions in Fig. 2, to obtain the pure condition of the considered platform. When performing this operation, and applying syzygies, a common factor to all terms arises: the bracket $[abce]$. This common factor tells us that the considered platform in is a singularity when point b lies on the plane ace. This is a direct consequence of the presence of a line-plane component defined by the 5-point alignment [14].

Case 1

$$q_1 = k_1 a + (1-k_1)c, \qquad q_2 = k_2 a + (1-k_2)c, \qquad q_3 = k_3 a + (1-k_3)c$$

$$q_4 = k_4 d + (1-k_4)f, \qquad q_5 = k_5 d + (1-k_5)f, \qquad q_6 = k_6 d + (1-k_6)f$$



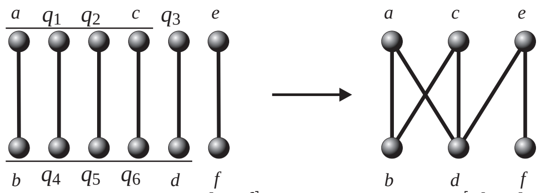
input: $[a,b,q_1,d,q_2,q_4,q_3,q_5,c,q_6,e,f]$ **output:** $K_1 \cdot [ab,ad,af,cd,cf,ef]$

$$K_1 = k_1(k_3(1-k_5)(k_6-k_4) - (1-k_4)(k_6-k_5)k_2) - k_2k_3(k_5-k_4)(1-k_6)$$

Case 2

$$q_1 = k_1 a + (1-k_1)c, \qquad q_2 = k_2 a + (1-k_2)c, \qquad q_3 = k_{31} a + k_{32} c + (1-k_{31}-k_{32})e$$

$$q_4 = k_4 b + (1-k_4)d, \qquad q_5 = k_5 b + (1-k_5)d, \qquad q_6 = k_6 b + (1-k_6)d$$



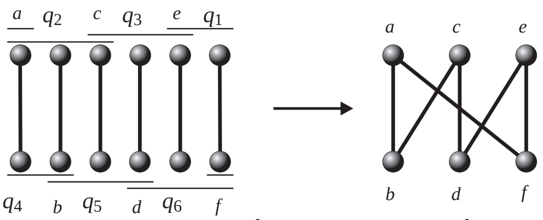
input: $[a,b,q_1,q_4,q_2,q_5,c,q_6,q_3,d,e,f]$ **output:** $K_2 \cdot [ab,ad,cb,cd,ed,ef]$

$$K_2 = (1-k_{31}-k_{32})(k_2((1-k_6)(k_4-k_5)k_1 - (1-k_5)(k_4-k_6)) - k_1(k_6-k_5)(1-k_4))$$

Case 3

$$q_1 = k_1 a + (1-k_1)e, \qquad q_2 = k_2 a + (1-k_2)c, \qquad q_3 = k_3 c + (1-k_3)e$$

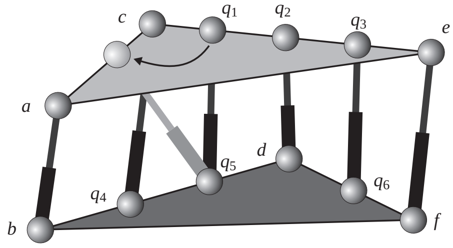
$$q_4 = k_4 b + (1-k_4)f, \qquad q_5 = k_5 b + (1-k_5)d, \qquad q_6 = k_6 d + (1-k_6)f$$



input: $[a,q_4,q_2,b,q_1,f,q_3,d,c,q_5,e,q_6]$ **output:** $K_3 \cdot [ab,af,cb,cd,ed,ef]$

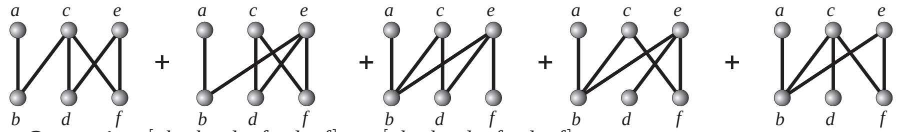
$$K_3 = k_4k_1(1-k_2)(1-k_3)(1-k_5)(1-k_6) - k_2k_3k_5k_6(1-k_4)(1-k_1)$$

Fig. 3 The 6-6 platforms on the left have the same singularities as their respective 3-3 platforms on the right, provided that the constant multiplying the obtained simple superbracket, K_i, is not identically zero, in which case the corresponding 6-6 platform is architecturally singular.

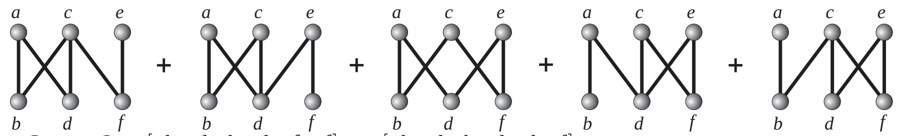


$$q_1 = k_1 c + (1-k_1)e \qquad q_2 = k_2 c + (1-k_2)e$$
$$q_3 = k_3 c + (1-k_3)e \qquad q_4 = k_4 b + (1-k_4)d$$
$$q_5 = k_5 b + (1-k_5)d \qquad q_6 = k_6 d + (1-k_6)f$$

Input: $[a,b,c,q_4,q_1,q_5,q_2,d,q_3,q_6,e,f]$



Output 1: $c_1[ab,cb,cd,cf,ed,ef]+c_2[ab,eb,cd,cf,ed,ef]$
$+c_3[ab,cb,cd,eb,ed,ef]+c_4[ab,cb,cf,eb,ed,ef]+c_5[ab,cb,cd,cf,eb,ef]$



Output 2: $c_1[ab,ad,cb,cd,cf,ef]+c_2[ab,ad,cb,cd,ed,ef]$
$+c_3[ab,ad,cb,cf,ed,ef]+c_4[ab,ad,cd,cf,ed,ef]+c_5[ab,cb,cd,cf,ed,ef]$

Fig. 4 Using the proposed algorithm, the decomposition of the superbracket associated with the platform in this figure is that in **Output 1**. If q_1 is redefined as $k_1 a + (1-k_1)c$, the resulting superbracket decomposition is that in **Output 2**. In both cases, the 3-3 platform associated with each simple superbracked is represented in the same order of the decomposition.

If we change the location of point q_1 to break the 5-point alignment and we align it with c and a, the line-plane component disappears. Applying our algorithm to the resulting superbracket, the obtained decomposition now contains a superbracket corresponding to a 3-3 platform with octahedral topology (Output 2 in Fig. 4). Coherently, the resulting expression, when translated into brackets, contains no common factors.

5 Conclusions

The proposed algorithm permits a straightforward singularity analysis of any Stewart-Gough manipulator with planar base and platform by just inspecting its resulting superbracket decomposition. In particular, if it factorizes (implying that no octahedral topology appears among its components), then each common factor corresponds to a rigid substructure (point-plane, line-line, line-plane) of the manipulator and, therefore, such factor becoming null indicates an architectural singularity of the corresponding type.

The presented algorithm has been implemented in Maple 10 within a package that also contains procedures for representing graphically the superbracket decompositions in terms of 3-3 topologies similar to those appearing in Figs. 3 and 4.

We are currently addressing the derivation of entire families of Stewart-Gough platforms sharing the same singularity structure, in particular, those for the three 3-3 topologies. Moreover, we would like to extend the potential of the algorithm by permitting the appearance of virtual points (playing an important role as regards to singularities) in the brackets, in order to push the decomposition further.

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