Chapter 4

Graphs of kinematic constraints

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When a set of kinematic constraints are imposed between several rigid bodies, finding out the set of configurations that satisfy all these constraints is a matter of special interest. The problem is not new and has been discussed, not only in Kinematics, but also in the design of object level robot programming languages for assembly tasks.

This chapter deals with the problem of finding out how constrained movements, or kinematic constraints, are propagated and how some workpieces in an assembly reduce their degrees of freedom after this propagation, and how inconsistencies between constraint movements can be found. Special attention is paid to those problems which can be solved using a simple topological analysis derived from the Theory of Continuous Groups of Transformations.

Part of the material presented herein has already appeared in [16]. Here important points have been clarified and some modifications have been introduced. Also, an important part of this chapter is devoted to the propagation of kinematic constraints using part of the material appeared in [17].

This chapter is structured as follows. Section 4.1 shows the important role of kinematic constraints in the assembly domain. Section 4.2 provides all basic theory about kinematic constraints needed in this chapter. Section 4.3 essentially deals with the basic operations to be carried out on a graph of kinematic constraints, namely composition, intersection and star-polygon transform. Section 4.4 introduces a basic algorithm for constraint propagation, which avoids the application of the star-polygon transform when obtaining the equivalent constraint between any two bodies in a graph of kinematic constraints with arbitrary topology. Section 4.5 presents an example and, finally, Section 4.6 gives a brief summary of the main points in this chapter.

4.1 The role of kinematic constraints in the assembly domain

In the assembly domain, it does not suffice to make the workpiece models produced by a CAD system available in the programming environment, but a description of the way the different pieces should be fitted together is also required. This description can be provided in full detail by either the designer or the programmer, or rather be automatically inferred, at least in part, from constraints derived from both the shapes of the workpieces involved in the assembly, after trying to find matings of complementary subparts between them, and the mechanics of the assembly operations themselves.

Matings of complementary subparts of different workpieces have a direct translation into constrained movements, or kinematic constraints. In general, this translation assumes that the legal motion for compatible pairs of predefined subparts, or features, is provided by the user and thus already known. Alternatively, it would be possible to infer legal motions directly from geometric models of predefined features. This problem has been reduced to find local symmetries [9] or, when working with polyhedral workpieces, to find cycles of edges [18]. The recognition and extraction of expected patterns of geometry and topology, corresponding to particular engineering functionality, as described in [20], will play an important role in this area in a near future.

In the assembly domain, kinematic constraints are not only relevant when mating complementary subparts, but also when specifying relative locations between workpieces, specially when using an interactive graphics system. Let us look at a simple example. In order to specify the location of the block with reference to the box in fig. 4.1, we impose that faces P_1 and P_2 of the block be against P_4 and P_3 of the box, respectively. Then, we might ask: Is there any configuration satisfying both constraints? In other words, are they consistent? If the answer is yes, how many degrees of freedom remain between the block and the box? Which are the values of the constrained degrees of freedom? It will be shown that a directed graph of kinematic constraints, that is, a graph whose nodes correspond to workpieces and whose arcs are labeled with a set of

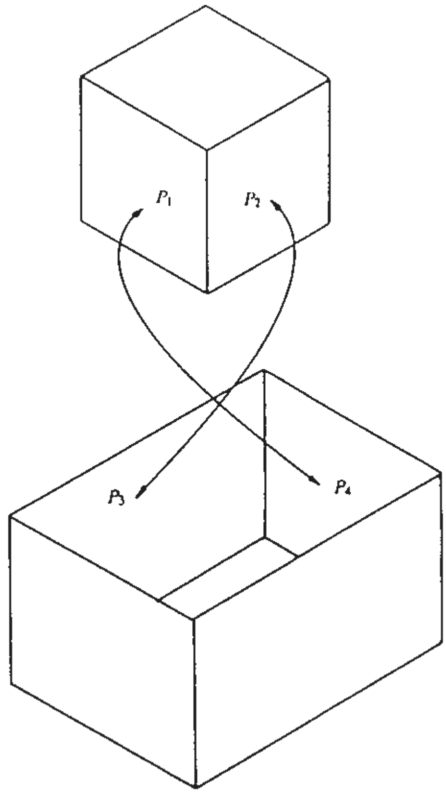


Figure 4.1: Specifying the location of a block with reference to a box using a set of kinematic constraints.

legal transformations linking the coordinate reference frame of the corresponding workpieces, is a proper data structure to represent these problems.

If we want to deal with graphs of kinematic constraints with arbitrary topology and constraints, then we must be able to find a solution to any inverse kinematic problem. Nevertheless, no general satisfactory solution, convenient for practical use, has been found for the general inverse kinematic problem. This problem is highly complicated because of its non-linearity, non uniqueness of the solution and existence of singularities. Fortunately, most kinematic graphs arising in the assembly domain are *quite* simple, since most planes and axis of symmetry of the involved geometric features are parallel and orthogonal in the final assembly.

The automatic manipulation of kinematic constraints has attracted a lot of attention not only in Kinematics, but also in the design of object level robot programming languages, such as RAPT [14] or LM-Geo [12]. Several algebraic symbolic approaches have emerged, among which we will mention a system of rewriting rules [14] and a table look-up procedure [8].

Algebraic symbolic (as opposed to a numerical) methods for dealing with kinematic constraints can shed light on basic aspects of the problem. For example, as it is shown in [17], the way they propagate provide useful information on the sequence of assembly.

The algebraic symbolic method used by the RAPT interpreter can be factored into a solution for the rotation which will determine angles, followed by the formation of real equations involving variables representing linear displacements and sines and cosines of the angle variable, which, in general, are difficult to deal with.

The approach presented herein distinguishes between *topological* and *geometrical* analysis of a set of kinematic constraints. The described topological analysis, well suited for the assembly domain, is derived from the Theory of Continuous Groups of Transformations, and it was essentially devised by Hervé in [7] for obtaining the number of degrees of freedom in mechanisms (see [1] for a revision). This analysis takes advantage of the fact that the legal relative motions resulting from mating two complementary subparts, such as pegs and holes or grooves and tongues, constitute cosets of subgroups of the Euclidean group, leading to a procedure based on a set of look-up tables.

4.2 The Euclidean group and kinematic constraints

It is well known that a rigid body in 3-dimensional space has 6 degrees of freedom, and, given a reference frame, any displacement can be obtained by a pure rotation about the origin followed by a pure translation.

The set of all displacements of a rigid body, with the composition operation, is isomorphic to the Special Euclidean group $SE(3)$. The decomposition $SE(3) = \Re^3 \times SO(3)$ shows the aforementioned fact that for any $\mathbf{D} \in SE(3)$,

$$\mathbf{D} = \mathbf{Trans}(\mathbf{v})\mathbf{Rot}(\mathbf{u}, \theta),$$

where $\mathbf{Trans}(\mathbf{v})$ is a translation along the vector $\mathbf{v} \in \Re^3$ and $\mathbf{Rot}(\mathbf{u}, \theta) \in SO(3)$ is a rotation of angle θ about the axis $\mathbf{u}$. Rotations about the axes $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are denoted by **Twix, Twiy** and **Twiz**, respectively. An arbitrary rotation can be written, using Euler's decomposition, as:

$$\mathbf{Rot}(\mathbf{u}, \theta) = \mathbf{Twix}(\varphi)\mathbf{Twiz}(\phi)\mathbf{Twix}(\psi).$$

A rotation can also be expressed using only the **Twix** operator and constant rotations as follows:

$$\mathbf{Rot}(\mathbf{u}, \theta) = \mathbf{Twix}(\alpha)\ \mathbf{XTOY}\ \mathbf{Twix}(\beta)\ \mathbf{XTOY}\ \mathbf{Twix}(\gamma)$$

where the constant rotation **XTOY** is defined as $\mathbf{Twiz}(\pi/2)$.

While the results presented below do not depend on a particular representation of $SE(3)$, we will use the well known 4×4-matrix representation of homogeneous transforms [13], which has become fashionable because its simplicity. Let us see a brief overview to this representation (see [2] for alternative representations such as screw coordinates, quaternions, dual numbers, etc.)

4.2.1 Homogeneous transformations. An overview

The representation of objects in an n-dimensional space using homogeneous coordinates needs a space of dimension $n+1$ from which the original space is recovered by projection. For example, the vector $\mathbf{v} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors along the Cartesian coordinate axes, is represented using homogeneous coordinates as a column vector:

$$\mathbf{v} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

so that

$$\begin{aligned} x_1 &= x/t \\ y_1 &= y/t \\ z_1 &= z/t \end{aligned}$$

Henceforth we will normalize $t = 1$.

A transformation $\mathbf{H}$ is a 4×4-matrix so that, the image of a given point $\mathbf{v}$ under this transformation is represented by the matrix product $\mathbf{u} = \mathbf{H}\mathbf{v}$.

Translations

A transformation $\mathbf{H}$ representing a translation by a vector $\mathbf{d} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ will be:

$$\mathbf{H} = \mathbf{Trans}\ (\mathbf{d}) = \mathbf{Trans}\ (a, b, c) = \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus, given a vector $\mathbf{v} = (x, y, z, 1)^t$, its image $\mathbf{u}$ under $\mathbf{H}$ will be

$$\mathbf{u} = \mathbf{H}\ \mathbf{v} = \begin{pmatrix} x+a \\ y+b \\ z+c \\ 1 \end{pmatrix}$$

It is easy to prove that the set of all translations constitutes a group under the matrix product operation, which will be denoted by T.

Rotations

The transformations representing rotations about the x, y, and z axes by angles ψ, θ or ϕ, respectively, are:

$$\mathbf{Rotx}(\psi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi & 0 \\ 0 & \sin\psi & \cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{Roty}(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{Rotz}(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Each element ij of the 3×3 upper left submatrix is equal to the cosine of the angle between the i-axis of the original coordinate frame and the j-axis of the rotated one.

These matrices, as well as their products, are orthogonal matrices with determinant equal to $+1$. They also constitute a group under matrix multiplication which will be denoted by S_o.

Displacements

The transformations representing rotations and translations can be multiplied, and the resulting matrices are said to describe *displacements*.

The following properties must be emphasized:

- *Decomposition of a displacement.* Every displacement $\mathbf{H}$ can be decomposed into the product of a translation and a rotation, so that

$$\mathbf{H} = \mathbf{Trans}(\mathbf{d})\,\hat{\mathbf{H}} = \mathbf{Trans}(a,b,c)\,\hat{\mathbf{H}} \quad , \quad \forall \mathbf{H} \in SE(3)$$

  where $\hat{\mathbf{H}}$ is the rotation component of the displacement $\mathbf{H}$ or, in other words, is the matrix resulting from setting the first three elements – a, b and c – of the last column of $\mathbf{H}$ to zero.

- *Composition of n displacements.*

$$\mathbf{H}_1 \cdots \mathbf{H}_i \cdots \mathbf{H}_n = \text{Trans}\ (\mathbf{d}_1)\ \hat{\mathbf{H}}_1 \cdots\ \text{Trans}\ (\mathbf{d}_n)\ \hat{\mathbf{H}}_n =$$

$$\text{Trans}\ (\mathbf{d}_1 + \hat{\mathbf{H}}_1\mathbf{d}_2 + \ldots + \hat{\mathbf{H}}_1\hat{\mathbf{H}}_2 \cdots \hat{\mathbf{H}}_{n-1}\mathbf{d}_n)\ \hat{\mathbf{H}}_1\ \hat{\mathbf{H}}_2 \cdots\ \hat{\mathbf{H}}_n\ ,$$

$$\forall \mathbf{H}_1 \cdots \mathbf{H}_n \in SE(3)$$

If a transformation is postmultiplied by another transformation, the latter is applied with respect to the transformed frame described by the former. Conversely, if a transformation is premultiplied by another one, the latter is applied with respect to the reference frame [13]. Other authors [14], in using the transposes of the above defined transformations, adhere to the inverse rule.

- *Inverse displacement.* Because of the properties of orthogonal matrices, the inverse displacement of $\mathbf{H}$ is:

$$\mathbf{H}^{-1} = \hat{\mathbf{H}}^t\ \text{Trans}\ (-a, -b, -c)\ \ ,\ \ \forall \mathbf{H} \in SE(3)$$

where $\hat{\mathbf{H}}^t$ denotes the transpose matrix of $\hat{\mathbf{H}}$.

A given displacement has been denoted using a upper case bold letter. Hereafter, sets of displacements, possibly subgroups, will be denote using just an upper case letter.

4.2.2 Subgroups of the group of displacements

It is well known that a *group* is a set of elements closed under an associative operation with an identity and inverse elements, as is the group $SE(3)$ of displacements. A *subgroup* $S \subset SE(3)$ is a subset of $SE(3)$ which is itself a group under the same operation. The composition of elements of $SE(3)$ can be extended to the composition of elements and subgroups. If $S \subseteq SE(3)$ and $\mathbf{D} \in SE(3)$, then the *right coset* $S \cdot \mathbf{D}$ is the set $\{\mathbf{H} \cdot \mathbf{D} \mid \mathbf{H} \in S\}$. The *left coset* $\mathbf{D} \cdot S$ and the *two-sided coset* $\mathbf{D}_1 \cdot S \cdot \mathbf{D}_2$ can be similarly defined. More generally, the composition of two subgroups $S_1 \cdot S_2$ is defined as $\{\mathbf{D}_1 \cdot \mathbf{D}_2 \mid \mathbf{D}_1 \in S_1, \mathbf{D}_2 \in S_2\}$.

Definition 1 (Conjugation classes of subgroups of SE(3)) *Every such class is an equivalence class with respect to the relation:*

$$S_1 \sim S_2 \Leftrightarrow \exists \mathbf{D} \in SE(3)\ \mid\ S_2 = \mathbf{D}S_1\mathbf{D}^{-1}$$

S_1 and S_2 being subgroups of $SE(3)$.

Table 4.1: Classification of the subgroups of *SE(3)* into conjugation classes

Dimension (d.o.f.)	Notation	Conjugation class and associated lower pair	Geometric elements of definition	Canonical subgroup
0	I	Identity displacement		I
1	$T_{\mathbf{v}}$	Rectilinear translation (P) Prismatic	A direction of translation given by a vector **v**	{**Trans** (x,0,0) | x $\in \Re$}
	$R_{\mathbf{u}}$	Rotation around an axis (R) Revolution	An axis of revolution **u**	{**Twix** (ψ) | $\psi \in (-\pi, +\pi]$}
	$H_{\mathbf{u},p}$	Helicoidal movement (H) Screw	An axis of revolution **u** and a thread pitch p	{**Trans** (x,0,0)**Twix**(px) | x $\in \Re$, p = constant }
2	T_P	Planar translation	A plane P	{**Trans** (0,y,z) | x, y $\in \Re$}
	$C_{\mathbf{u}}$	Lock movement (C) Cylindrical	An axis **u**	{**Trans** (x,0,0) **Twix** (ψ) | x $\in \Re, \psi \in (-\pi, +\pi]$}
3	T	Spatial translation		{**Trans** (x,y,z) | x, y, z $\in \Re$}
	G_P	Planar sliding (E) Plane	A plane P	{**Trans** (0,y,z) **Twix** (ψ) | y, z $\in \Re, \psi \in (-\pi, +\pi]$}
	S_o	Spheric rotation (S) Spherical	A point o in the space	{**Twix** (ψ) **XTOY Twix**(ξ) **XTOY Twix** (η) | $\psi, \xi, \eta \in (-\pi, +\pi]$}
	$Y_{\mathbf{v},p}$	Translating screw	A direction of revolution **v** and a thread pitch p	{**Trans** (x,y,z) **Twix**(px) | x, y, z $\in \Re$, p = constant }
4	$X_{\mathbf{v}}$	Translating gimbal	A direction of revolution **v**	{**Trans** (x,y,z) **Twix**(ψ) x, y, z $\in \Re, \psi \in (-\pi, +\pi]$}

There exists infinite subgroups of $SE(3)$, but they can be classified into a finite number of conjugation classes. This suggest that we can represent each conjugation class by a *canonical subgroup*, so that all subgroups of the same class can be expressed as a conjugate of it.

An exhaustive classification of the continuous subgroups of $SE(3)$ into conjugation classes can be carried out using classic methods of analysis of finite dimension continuous groups [3]. A list of the classes thus obtained and a canonical subgroup for each of them is shown in table 4.1 (adapted from [7]). Note that all lower pairs are included in this classification. Let us recall that a lower pair exists when one element is coupled to the other via a wrapping action and contact takes place along a surface.

The notation used for these conjugation classes appears in the second column of table 4.1. Each class can be characterized by a set of geometric elements of definition which appear as subindices in the notation of the class. A geometric element of definition of a given subgroup is an affine space of $\Re^3$ of dimension 0, 1 or 2 (a point, line or a plane) which characterize the subgroup. A scalar is also required to characterize the $H_{\mathbf{u},p}$ and $Y_{\mathbf{v},p}$ subgroups. An instance of this elements leads to a subgroup belonging to the class. Instances will be denoted using numerical subindices. For example, T_P denotes the conjugation class of planar translations and T_{P_1} denotes a given subgroup belonging to this class.

The canonical subgroups are chosen in such a way that their geometric elements of definition satisfy the following conditions:

- if it is a point, it coincides with the origin of the reference frame;
- if it a line, it passes through the the origin of the reference frame and the x axis is aligned with it; and
- if it is plane, it passes through the origin of the reference frame and the x axis is orthogonal to it.

The election of canonical subgroups is thus arbitrary. If S_i is a canonical subgroup, it will be denoted $\hat{S}_i$. Given a subgroup S_1, $(S_1)^G$ denotes the canonical subgroup in the same class.

The *degree of freedom* of a kinematic chain is defined as the necessary and sufficient number of variables that define uniquely the position and orientation of all the workpieces involved. The *dimension* of one of the foregoing subgroups is defined as the degree of freedom of the constrained motion it allows. A set of variables is thus associated with every subgroup. The dimension of a subgroup is indicated as $dim(\cdot)$, where $(\cdot)$ denotes one of those subgroups. Obviously, $dim(SE(3)) = 6$.

When the geometric elements of definition of two different subgroups satisfy some kind of spatial relationship – such as parallelism, collinearity or perpen-

Table 4.2: Conditions of inclusion of one subgroup of $SE(3)$ into another

$\mathbf{u}_0 \perp P_0 \Rightarrow \mathbf{u}_0$ perpendicular to P_0
$\mathbf{u}_1 \bowtie \mathbf{u}_0 \Rightarrow \mathbf{u}_1$ and $\mathbf{u}_0$ collinear
$\mathbf{u}_0 \parallel \mathbf{v}_0 \Rightarrow \mathbf{u}_0$ and $\mathbf{v}_0$ parallel

	T_{P_0}	$C_{\mathbf{u}_1}$	T	G_{P_1}	S_{o_0}	$Y_{\mathbf{v}_0,p_1}$	$X_{\mathbf{v}_1}$
$T_{\mathbf{u}_0}$	$\mathbf{u}_0 \parallel P_0$	$\mathbf{u}_0 \parallel \mathbf{u}_1$	$\forall \mathbf{u}_0$	$\mathbf{u}_0 \parallel P_1$		$\mathbf{u}_0 \perp \mathbf{v}_0$	$\forall \mathbf{u}_0$
$R_{\mathbf{u}_0}$		$\mathbf{u}_0 \bowtie \mathbf{u}_1$		$\mathbf{u}_0 \perp P_1$	$o_0 \in \mathbf{u}_0$		$\mathbf{u}_0 \parallel \mathbf{v}_1$
$H_{\mathbf{u}_0,p_0}$		$\mathbf{u}_0 \bowtie \mathbf{u}_1$				$\mathbf{u}_0 = \mathbf{v}_0, p_0 = p_1$	$\mathbf{u}_0 \parallel \mathbf{v}_1$
T_{P_0}			$\forall P_0$	$P_0 \parallel P_1$		$P_0 \perp \mathbf{v}_0$	$\forall P_0, \forall \mathbf{v}_1$
$C_{\mathbf{u}_0}$							$\mathbf{u}_0 \parallel \mathbf{v}_1$
T							$\forall \mathbf{v}_1$
G_{P_0}							$P_0 \perp \mathbf{v}_1$
$Y_{\mathbf{v}_0,p_0}$							$\mathbf{u}_0 \parallel \mathbf{v}_1$
$X_{\mathbf{v}_0}$							

dicularity –, one may become subgroup of the other. The conditions of inclusion of one subgroup into another appear in table 4.2 (adapted from [7]).

Now, we can introduce a formal definition of kinematic constraint.

Definition 2 (Constraints and linking displacements) *A constraint R is a set of displacements which can be expressed as a composition of cosets of canonical subgroups. That is,*

$$R = \mathbf{L}_0 \hat{S}_1 \mathbf{L}_1 \hat{S}_2 \mathbf{L}_2 \cdots \mathbf{L}_{n-1} \hat{S}_n \mathbf{L}_n \tag{4.1}$$

where $\mathbf{L}_1, \ldots, \mathbf{L}_{n-1}$ are defined as linking displacements. A constraint is said to be trivial *when it can be reduced to a single coset.*

The interest of most mechanisms is to provide a constrained motion which cannot be expressed as a constraint in the way it has been defined here. Nevertheless, we are not interested in analyzing mechanisms, but reasoning about constrained motions in the assembly domain.

Hereafter, we will assume that our constraints are trivial. In this particular case, if $R_i = \mathbf{L}_0 \hat{S}_i \mathbf{L}_1$, then R_i^G will denote the canonical subgroup $\hat{S}_i$, thus extending the notation introduced for subgroups.

Constraints will be denoted by R_i, where i is a subindex that identifies it. If R_i is the set of legal transformations from the reference frame of $\mathcal{B}_1$ to the reference frame of $\mathcal{B}_2$, R_i^{-1} denotes the set of legal transformations in the way around, i.e. from $\mathcal{B}_2$ to $\mathcal{B}_1$. Note that $R_i^G = (R_i^{-1})^G$ for all R_i.

A constraint R_i has the variables and geometric elements of definition inherited from $\hat{S}_i$. Given a reference frame, the subgroup with the same geometric elements of definition as a given constraint R_i will be called its associated subgroup, which will be denoted by R_i^A. Obviously, $R_i^A \sim R_i^G$.

4.3 Operations on a graph of kinematic constraints

A directed graph of kinematic constraints – or GR graph, for short – is defined as a graph whose nodes correspond to workpieces and whose directed arcs are labeled with constraints. The two basic operations on a graph of kinematic constraints are composition and intersection of constraints. The former (fig. 4.2a) involves finding the constraint between bodies $\mathcal{B}_1$ and $\mathcal{B}_3$ that results from composing the constraint between $\mathcal{B}_1$ and $\mathcal{B}_2$ – say R_i – with that between $\mathcal{B}_2$ and $\mathcal{B}_3$ – say R_j –, which will be denoted by $R_i \cdot R_j$. The latter operation (fig. 4.2b) permits combining two given constraints, R_i and R_j, between the same two workpieces into a single resulting constraint, which will be denoted by $R_i \cap R_j$. Let us analyze both operations in terms of composition and intersection of subgroups.

4.3.1 Composition

Let us assume a universe of three bodies – $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ – linked by two trivial constraints

$$R_{12} = \mathbf{A}_1 \hat{S}_1 \mathbf{A}_2$$

$$R_{23} = \mathbf{B}_1 \hat{S}_2 \mathbf{B}_2.$$

Then, the equivalent constraint between bodies $\mathcal{B}_1$ and $\mathcal{B}_3$, that results from composing R_{12} and R_{23}, is:

$$R_{13} = R_{12} R_{23} = \mathbf{A}_1 \hat{S}_1 \mathbf{A}_2 \mathbf{B}_1 \hat{S}_2 \mathbf{B}_2 = \mathbf{A}_1 \mathbf{A}_2 S_1 S_2 \mathbf{B}_1 \mathbf{B}_2 \tag{4.2}$$

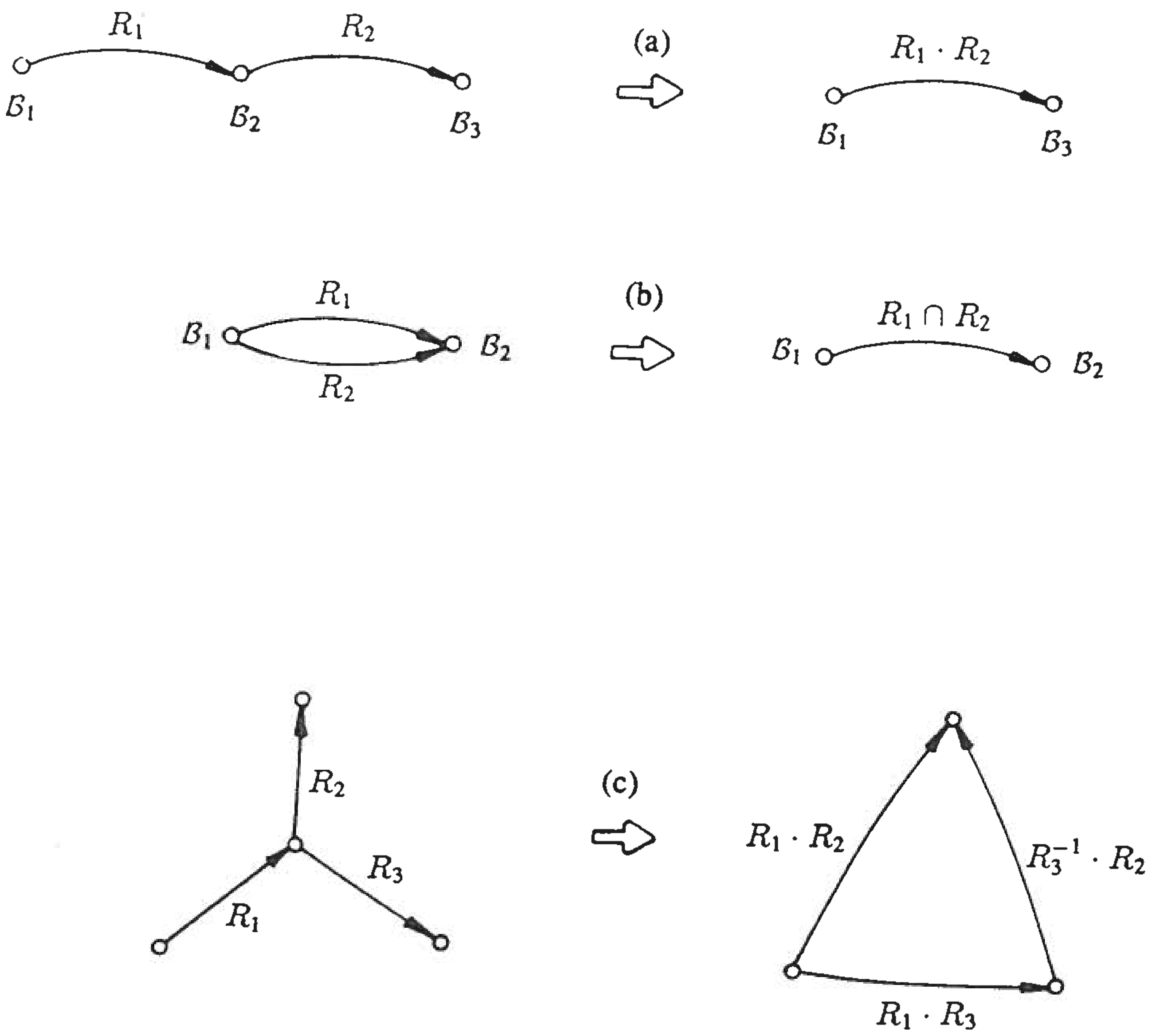


Figure 4.2: Operations on a graph of kinematic constraints: (a) composition; (b) intersection; and (c) star-polygon transform.

where

$$S_1 \sim \hat{S}_1 \text{ and } S_2 \sim \hat{S}_2$$

We will denote $(R_{12}R_{23})^C = S_1S_2$ according to 4.2.

Thus, the problem of composing two trivial constraints can be reduced to the problem of composing two subgroups, and the outcome of the composition of two continuous subgroups of $SE(3)$ can be tabulated as shown in table 4.3 (adapted from [7]).

Clearly, the composition of two trivial constraints needs not be a trivial constraint itself, and the only information we need to find it out is the spatial relationships between their geometric elements of definition of both constraints.

When we compose two constraints expressed in terms of canonical subgroups, the linking displacement ($\mathbf{A}_2\mathbf{B}_1$ in (4.2)) captures the information about the spatial relationship between their geometric elements of definition. Taking advantage of this fact, we can check the linking displacement to find whether the composition of two trivial constraints can be reduced to a trivial constraint.

4.3.2 Intersection

If body $\mathcal{B}_3$, still in the same example above, is rigidly linked to $\mathcal{B}_1$ forming a closed kinematic chain, the intermediate body $\mathcal{B}_2$ will only have the possibilities of motion given by $R_{12} \cap R_{23}^{-1}$.

We can write,

$$R_{12} \cap R_{23}^{-1} = \mathbf{A}_1\hat{S}_1\mathbf{A}_2 \cap \mathbf{B}_2^{-1}\hat{S}_2\mathbf{B}_1^{-1} = (S_1' \cap S_2'\mathbf{B}_2^{-1}\mathbf{B}_1^{-1}\mathbf{A}_2^{-1}\mathbf{A}_1^{-1})\mathbf{A}_1\mathbf{A}_2 =$$

$$= (S_1' \cap S_2'\mathbf{C})\mathbf{A}_1\mathbf{A}_2$$

If S_1' and S_2' are subgroups of $SE(3)$, then $(S_1' \cap S_2'\mathbf{C})$ is either null or is a coset of $S_1' \cap S_2'$ (proposition 2 of [15]). Then, we have

$$R_{12} \cap R_{23}^{-1} = (S_1' \cap S_2')\mathbf{D}\mathbf{A}_1\mathbf{A}_2 \ \text{ iff } \ R_{12} \cap R_{23}^{-1} \neq \emptyset \tag{4.3}$$

where

$$\mathbf{D} = \mathbf{E}\mathbf{C}, \ \mathbf{E} \in S_2', \ \mathbf{D} \in S_1', \tag{4.4}$$

S_1' being a conjugate subgroup of $\hat{S}_1$, and S_2' of $\hat{S}_2$.

We will denote $(R_{12} \cap R_{23}^{-1})^I = S_1' \cap S_2'$ according to 4.3 and 4.4.

Table 4.3: Intersection and regular representation for the composition of all pairs of subgroups of $SE(3)$ whose intersection is different from the identity displacement or one is not subgroup of the other

Groups to be composed	Conditions on the geometric elements	Conditions on the linking displacement	Intersection	Regular representation
$T_{P_0} \cdot T_{P_1}$			$T_{\mathbf{v}_0}$ $\mathbf{v}_0 = P_0 \cap P_1$	T
$T_{P_0} \cdot G_{P_1}$			$T_{\mathbf{v}_0}$ $\mathbf{v}_0 = P_0 \cap P_1$	$X_{\mathbf{v}_0}$ $\mathbf{v}_0 \perp P_1$
$G_{P_0} \cdot G_{P_1}$			$T_{\mathbf{v}_0}$ $\mathbf{v}_0 = P_0 \cap P_1$	$R_{\mathbf{u}_0} \cdot T \cdot R_{\mathbf{u}_1}$ $\mathbf{u}_0 \perp P_0, \mathbf{u}_1 \perp P_1$
$Y_{\mathbf{v}_0 p_0} \cdot T_{P_0}$	$\mathbf{v}_0 \not\perp P_0$	$l_{11} \neq \pm 1$	$T_{\mathbf{v}_1}$ $\mathbf{v}_1 \parallel P_0,\ \mathbf{v}_1 \perp \mathbf{v}_0$	$X_{\mathbf{v}_0}$
$Y_{\mathbf{v}_0 p_0} \cdot G_{P_0}$	$\mathbf{v}_0 \not\perp P_0$	$l_{11} \neq \pm 1$	$T_{\mathbf{v}_1}$ $\mathbf{v}_1 \parallel P_0,\ \mathbf{v}_1 \perp \mathbf{v}_0$	$X_{\mathbf{v}_0} \cdot R_{\mathbf{u}_0}$ $\mathbf{u}_0 \perp P_0$
$Y_{\mathbf{v}_0 p_0} \cdot Y_{\mathbf{v}_1 p_1}$	$\mathbf{v}_0 \nparallel \mathbf{v}_1$	$l_{11} \neq \pm 1$	$T_{\mathbf{v}_2}$ $\mathbf{v}_2 \perp \mathbf{v}_0,\ \mathbf{v}_2 \perp \mathbf{v}_1$	$R_{\mathbf{u}_0} \cdot T \cdot R_{\mathbf{u}_1}$ $\mathbf{u}_0 \parallel \mathbf{v}_0,\ \mathbf{u}_1 \parallel \mathbf{v}_1$
$Y_{\mathbf{v}_0 p_0} \cdot C_{\mathbf{u}_0}$	$\mathbf{u}_0 \perp \mathbf{v}_0$	$l_{11} = 0$	$T_{\mathbf{v}_0}$	$Y_{\mathbf{v}_0 p_0} \cdot R_{\mathbf{u}_0}$
$C_{\mathbf{u}_0} \cdot C_{\mathbf{u}_1}$	$\mathbf{u}_0 \parallel \mathbf{u}_1$	$l_{11} = \pm 1$ $l_{24} \neq 0$ or $l_{34} \neq 0$	$T_{\mathbf{u}_0}$	$C_{\mathbf{u}_0} \cdot R_{\mathbf{u}_1}$
$T_{P_0} \cdot C_{\mathbf{u}_0}$	$\mathbf{u}_0 \parallel P_0$	$l_{11} = 0$	$T_{\mathbf{u}_0}$	$T_{P_0} \cdot R_{\mathbf{u}_0}$
$T \cdot C_{\mathbf{u}_0}$			$T_{\mathbf{u}_0}$	$X_{\mathbf{v}_0}$ $\mathbf{v}_0 \parallel \mathbf{u}_1$
$G_{P_0} \cdot C_{\mathbf{u}_0}$	$\mathbf{u}_0 \parallel P$	$l_{11} = 0$	$T_{\mathbf{u}_0}$	$G_{P_0} \cdot R_{\mathbf{u}_0}$
$X_{\mathbf{v}_0} \cdot C_{\mathbf{u}_0}$	$\mathbf{u}_0 \nparallel \mathbf{v}_0$	$l_{11} \neq \pm 1$	$T_{\mathbf{u}_0}$	$X_{\mathbf{v}_0} \cdot R_{\mathbf{u}_0}$
$Y_{\mathbf{v}_0 p_0} \cdot C_{\mathbf{u}_0}$	$\mathbf{u}_0 \parallel \mathbf{v}_0$	$l_{11} = \pm 1$	$H_{\mathbf{v}_0 p_0}$	$X_{\mathbf{v}_0}$
$G_{P_0} \cdot C_{\mathbf{u}_0}$	$\mathbf{u}_0 \perp P_0$	$l_{11} = \pm 1$	$R_{\mathbf{u}_0}$	$X_{\mathbf{v}_0}$ $\mathbf{v}_0 \perp P_0$
$S_{o_0} \cdot C_{\mathbf{u}_0}$	$o_0 \in$ axis $\mathbf{u}_0$	$l_{11} = \pm 1$ $l_{24} = 0$ $l_{34} = 0$	$R_{\mathbf{u}_0}$	$S_{o_0} \cdot T_{\mathbf{u}_0}$
$S_{o_0} \cdot G_{P_0}$			$R_{\mathbf{u}_0}$ $o_0 \in$ axis $\mathbf{u}_0$ $\mathbf{u}_0 \perp P_0$	$S_{o_0} \cdot T_{P_0}$
$S_{o_0} \cdot X_{\mathbf{v}_0}$			$R_{\mathbf{u}_0}$ $o_0 \in$ axis $\mathbf{u}_0$ $\mathbf{u}_0 \parallel \mathbf{v}_0$	$SE(3)$
$S_{o_0} \cdot S_{o_1}$			$R_{\mathbf{u}_0}$ $\mathbf{u}_0 = \frac{o_0 o_1}{\lvert o_0 o_1 \rvert}$	$S_{o_0} \cdot R_{\mathbf{u}_0} \cdot R_{\mathbf{u}_1}$ $o_1 \in$ axis $\mathbf{u}_0$ $o_1 \in$ axis $\mathbf{u}_1$
$Y_{\mathbf{v}_0 p_0} \cdot Y_{\mathbf{v}_1 p_1}$ $(p_0 \neq p_1)$	$\mathbf{v}_0 \parallel \mathbf{v}_1$	$l_{11} = \pm 1$	T_{P_0} $P_0 \perp \mathbf{v}_0$	$X_{\mathbf{v}_0}$
$Y_{\mathbf{v}_0 p_0} \cdot X_{\mathbf{v}_1}$	$\mathbf{v}_0 \nparallel \mathbf{v}_1$	$l_{11} \neq \pm 1$	T_{P_0} $P_0 \perp \mathbf{v}_0$	$X_{\mathbf{v}_0} \cdot R_{\mathbf{u}_1}$ $\mathbf{u}_1 \parallel \mathbf{v}_1$
$G_{P_0} \cdot Y_{\mathbf{u}_1 p_0}$	$\mathbf{v}_0 \perp P_0$	$l_{11} = \pm 1$	T_{P_0}	$X_{\mathbf{v}_0}$
$G_{P_0} \cdot X_{\mathbf{v}_0}$	$\mathbf{v}_0 \not\perp P_0$	$l_{11} \neq \pm 1$	T_{P_0}	$R_{\mathbf{u}_0} \cdot T \cdot R_{\mathbf{u}_1}$ $\mathbf{u}_0 \perp P_0,\ \mathbf{u}_1 \parallel \mathbf{v}_0$
$G_{P_0} \cdot T$			T_{P_0}	$X_{\mathbf{v}_0}$ $\mathbf{v}_0 \perp P_0$
$Y_{\mathbf{v}_0 p_0} \cdot T$			T_{P_0}	$X_{\mathbf{v}_0}$
$X_{\mathbf{v}_0} \cdot X_{\mathbf{v}_1}$	$\mathbf{v}_0 \nparallel \mathbf{v}_1$	$l_{11} \neq \pm 1$	T	$R_{\mathbf{u}_0} \cdot T \cdot R_{\mathbf{u}_1}$ $\mathbf{u}_0 \parallel \mathbf{v}_0,\ \mathbf{u}_1 \parallel \mathbf{v}_1$

Note that, although the intersection of two subgroups is at least the identity displacement, the intersection of two constraints may be the empty set.

When the intersection of two constraints is null, i.e. it is not possible to find a set of displacements satisfying (4.4), it implies that both kinematic constraints can not be simultaneously satisfied. This situation can not be detected through the intersection of subgroups. Roughly speaking, if we state our problems of kinematic constraints purely in terms of compositions and intersections of subgroups, we will be unable to detect inconsistencies. As it has been pointed out in [1], Group Theory provides the means for a *topological analysis* of the behavior of a set of bodies linked by a set of kinematic constraints, but a *geometric analysis* is required if we care about dimensions.

Thus, the problem of intersecting two constraints, say $\mathbf{A}_1\hat{S}_1\mathbf{A}_2$ and $\mathbf{B}_2^{-1}\hat{S}_2\mathbf{B}_1^{-1}$, can be expressed, if their intersection is different from null, in terms of the intersection of two subgroups, and the information about the spatial relationships between their geometric elements of definition can be obtained either from $\mathbf{A}_2 \cdot \mathbf{B}_1$ or $\mathbf{B}_2 \cdot \mathbf{A}_1$. Obviously, both information must be consistent.

The outcomes of the composition and intersection of two continuous subgroups of $SE(3)$, for all those cases in which the intersection is different from the identity displacement or one subgroup is a subgroup of the other, have also been tabulated in table 4.3. l_{ij} denotes the element (i, j) of the 4×4 homogeneous transformation representation for the linking displacement.

See [9] for deeper prospects on the intersection of, possibly not continuous, subgroups of $SE(3)$.

As a summary, we can say that: (a) the composition of two trivial constraints is sometimes a trivial constraint; (b) the intersection of two trivial constraints is a trivial constraint or null; and (c) the intersection of two non-trivial constraints is not necessarily a constraint, as defined here.

Definition 3 (Independence and inconsistency) *Two trivial constraints, R_1 and R_2, are said to be independent iff $(R_1 \cap R_2)^I$ is the identity displacement, and they are said to be inconsistent iff $R_1 \cap R_2$ is the empty set.*

Let us suppose that we want to find out the dimension of $R_{13} = R_{12} \cdot R_{23}$ or, in other words, the number of d.o.f. of the body $\mathcal{B}_3$ with respect to $\mathcal{B}_1$. It can be stated that:

$$dim(R_{13}) = dim(R_{12}) + dim(R_{23}) - dim(R_{12} \cap R_{23})$$

This formula can be extended to the composition of n constraints, leading to a variation of the *Chebyshev-Grübler-Kutzbach* formula

$$dim(R_{1,n+1}) = \sum_{1}^{n} dim(R_{i,i+1}) - \sum_{l=2}^{n} dim(R_{1,l} \cap R_{1,l+1}) \tag{4.5}$$

where

$$R_{ij} = \prod_{l=i}^{j-1} R_{l,l+1}$$

There are many examples of kinematic chains whose degree of freedom cannot be determined from its sole topology, i.e., they are elusive to the application of 4.5 [1, page 86].

Definition 4 (Regular representation) *The composition of two trivial constraints, $R_3 = R_1 R_2$ provide a regular representation for R_3 iff $(R_1 \cap R_2)^I = \mathbf{I}$. Then, $dim(R_3) = dim(R_1) + dim(R_2)$*

Notice that regular representations are not unique.

4.3.3 Examples

Firstly, let us analyze the composition of two constraints whose associated subgroups are G_{P_0} and $X_{\mathbf{u}_0}$. This composition can be expressed as:

$$\begin{aligned} R_c &= \mathbf{A}\ \hat{S}_1\ \mathbf{L}\ \hat{S}_2\ \mathbf{B} \\ &= \mathbf{A}\ \mathbf{Trans}(0, y, z)\ \mathbf{Twix}(\theta)\ \mathbf{L}\ \mathbf{Trans}(x', y', z')\ \mathbf{Twix}(\psi)\ \mathbf{B} \end{aligned}$$

where $\mathbf{L}$ is the linking displacement between both constraints. On the other hand, G_{P_0} and $X_{\mathbf{u}_0}$ can be decomposed into composition of subgroups as follows:

$$\left.\begin{array}{l} G_{P_0} = T_{P_0} \cdot R_{\mathbf{u}_1} = R_{\mathbf{u}_1} \cdot T_{P_0} \\ X_{\mathbf{u}_0} = T \cdot R_{\mathbf{u}_2} = R_{\mathbf{u}_2} \cdot T \end{array}\right\}$$

with $\mathbf{u}_1 \perp P_0$ and $\mathbf{u}_2 \| \mathbf{u}_0$.

If $\mathbf{u}_0 \not\perp P_0$, then $l_{11} \neq \pm 1$ (see table 4.3) and the only possible simplification for R_c is:

$$R_c = \mathbf{A}\ \mathbf{Trans}(x'', y'', z'')\ \mathbf{Twix}(\theta)\ \mathbf{L}\ \mathbf{Twix}(\psi)\ \mathbf{B}$$

The simplified term, $\mathbf{Trans}(0, y, z)$, corresponds to the intersection of G_{P_0} and $X_{\mathbf{u}_0}$. In terms of subgroups (table 4.3), we have

$$G_{P_0} \cdot X_{\mathbf{u}_0} = X_{\mathbf{u}_1} \cdot R_{\mathbf{u}_2} = R_{\mathbf{u}_1} \cdot T \cdot R_{\mathbf{u}_2} = R_{\mathbf{u}_1} \cdot X_{\mathbf{u}_2}$$

$$G_{P_0} \cap X_{\mathbf{u}_0} = T_{P_0}$$

with $\mathbf{u}_1 \perp P_0$ and $\mathbf{u}_2 \parallel \mathbf{u}_0$.

If $\mathbf{u}_0 \perp P_0$, then $\mathbf{u}_1 \parallel \mathbf{u}_0$, $l_{11} = \pm 1$, and G_{P_0} becomes a subgroup of $X_{\mathbf{u}_0}$ (table 4.2). Consequently, R_c can be expressed as

$$\begin{aligned} R_c &= \mathbf{A}\ \mathbf{Trans}(x''', y''', z''')\ \mathbf{Twix}(\theta)\mathbf{L}'\ \mathbf{Twix}(\psi)\ \mathbf{B} \\ &= \mathbf{A}\ \mathbf{Trans}(x''', y''', z''')\ \mathbf{Twix}(\theta + l'_{11}\psi)\ \mathbf{L}'\ \mathbf{B} \end{aligned}$$

where $\mathbf{L} = \mathbf{L}'\ \mathbf{Trans}(0, l_{24}, l_{34})$.

Notice that the necessary and sufficient condition for the equality

$$\mathbf{Twix}(\theta_1)\ \mathbf{L}\ \mathbf{Twix}(\theta_2) = \mathbf{Twix}(\psi)\ \mathbf{L}$$

to hold is that $l_{11} = \pm 1$, $l_{24} = 0$ and $l_{34} = 0$. In this case $\psi = \theta_1 + l_{11}\theta_2$.

Let us see another example. Imposing that the axes of the cylinders be aligned with the axes of their corresponding holes for the workpieces in fig. 4.3, the following expressions for both constraints will be obtained:

$$R_{12} = \mathbf{A}_{11}\mathbf{Trans}(x_1, 0, 0)\mathbf{Twix}(\theta_1)\mathbf{A}_{12}, \quad R_{12}^A = C_{\mathbf{u}_0}$$

$$R_{23} = \mathbf{A}_{21}\mathbf{Trans}(x_2, 0, 0)\mathbf{Twix}(\theta_2)\mathbf{A}_{22}, \quad R_{23}^A = C_{\mathbf{u}_1}$$

The composition of both constraints yields:

$$R_{12}R_{23} = \mathbf{A}_{11}\ \mathbf{Trans}(x_1, 0, 0)\ \mathbf{Twix}(\theta_1)\ \mathbf{L}\ \mathbf{Trans}(x_2, 0, 0)\ \mathbf{Twix}(\theta_2)\ \mathbf{A}_{22}$$

where the linking displacement is:

$$\mathbf{L} = \mathbf{A}_{12}\mathbf{A}_{21}.$$

Since $\mathbf{u}_0$ and $\mathbf{u}_1$ are parallel, and according to table 4.3, $l_{11} = \pm 1$; therefore, the composition of both constraints can be simplified leading to:

$$R_{12}R_{23} = \mathbf{A}_{11}\ \mathbf{Trans}(x_1 + l_{11}x_2, 0, 0)\ \mathbf{Twix}(\theta_1)\ \mathbf{L}\ \mathbf{Twix}(\theta_2)\ \mathbf{A}_{22} \qquad (4.6)$$

or, in other words,

$$R_{12}R_{23} = \mathbf{A}_{11}\ \hat{S}_1\ \mathbf{L}\ \hat{S}_2\ \mathbf{A}_{22}$$

where $\hat{S}_1 \in C_{\mathbf{u}}$ and $\hat{S}_2 \in R_{\mathbf{u}}$. Expression (4.6) is a regular representation for the composition of both constraints.

If, in addition to $l_{11} = \pm 1$, $l_{24} = 0$ and $l_{34} = 0$ ($\mathbf{u}_0 \bowtie \mathbf{u}_1$), a further simplification could be carried out and $R_{12}R_{23}$ becomes a trivial constraint. In this case $(R_{12}R_{23})^G \in C_{\mathbf{u}}$.

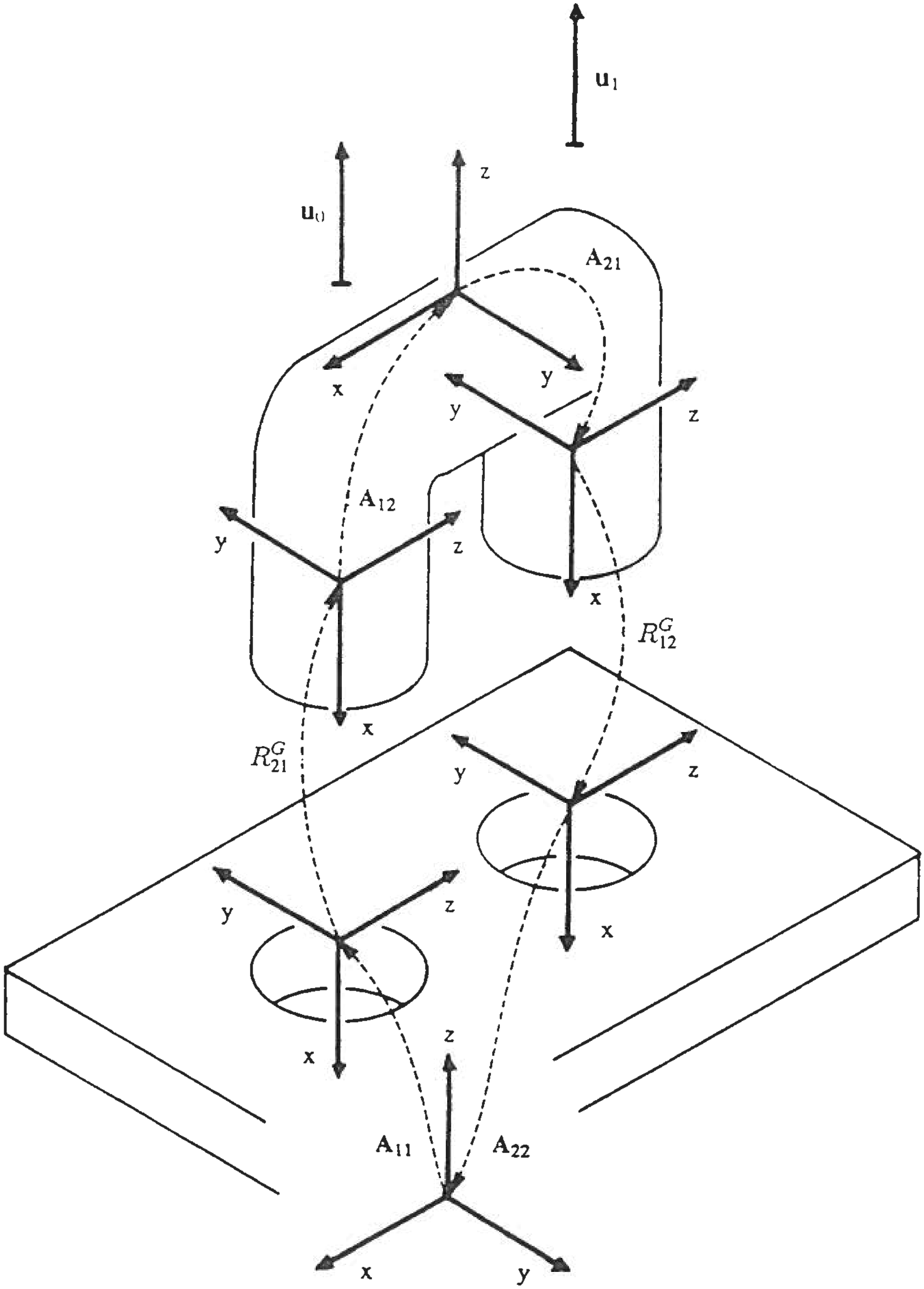


Figure 4.3: Insertion of a clamp. Geometric elements of definition, kinematic constraints and canonical subgroups involved.

Let us suppose that now we want to obtain $R_{12} \cap R_{23}^{-1}$, the equivalent constraint between $\mathcal{B}_1$ and $\mathcal{B}_2$. Then, $(R_{12} \cap R_{23}^{-1})^I$ can be easily obtained using table 4.3. Observe that the simplified term in 4.6, $\mathbf{Trans}(x_2, 0, 0)$, is $(R_{12} \cap R_{21}^{-1})^{IG}$. This term encompasses the remaining d.o.f. of body $\mathcal{B}_2$ with respect to $\mathcal{B}_1$, when $\mathcal{B}_3$ is kept rigidly linked, as in this case, to $\mathcal{B}_1$.

We have proved that the above constraints are not independent, but we have not checked their consistency. Depending on the relative dimensions of the involved workpieces, they may be inconsistent. The only thing we can say using this kind of symbolic manipulation is that, if $(R_{12} \cap R_{21}^{-1}) \neq \emptyset$, then $\mathcal{B}_2$ only have one translational degree of freedom with reference to $\mathcal{B}_1$. Checking consistency requires a geometrical analysis which requires, in turn, solving a kinematic equation. In our example, we would have to decide whether $R_{12}R_{23} = \mathbf{I}$ has a solution. Thus, although the previous ideas provide a theoretical framework within which it is easy to justify, for instance, when the composition of two constraints can be simplified, they must be complemented with an algorithm to obtain numerical values for the constrained d.o.f. if we care about dimensions. See [4] for new developments in this area.

4.3.4 Star-polygon transform

The above two basic operations are not enough for obtaining the equivalent constraint between any two bodies in an arbitrary graph of kinematic constraints. This fact can be easily proved by drawing a fully connected GR graph with four nodes and trying to obtain the equivalent constraint between any two of them through the iterative application of compositions and intersections of constraints.

The star-polygon transform is included here to provide a complete set of operations which make possible to obtain the equivalent constraint between any two bodies in an arbitrary GR graph.

The star-polygon transform consists in removing one node of the GR graph by fully connecting all the nodes connected to it with the equivalent constraint between them (fig. 4.2c). This operation can be seen as a generalized composition. Actually, when this transform is applied to a node of degree two, the result is the composition of two constraints.

The problem with this operations is that, once it has been applied, the involved constraints *share* variables. Thus, when a variable is assigned somewhere in the graph, it is necessary to take into account that it may be shared by another constraint. In the next section, an algorithm, which represents a way around this difficulty, is introduced. This algorithm is able to find the equivalent constraint between any two bodies without resorting to this operation.

4.4 Propagation of constraints

If, as the result of intersecting two constraints between the same two workpieces, the empty set is obtained, we say that they are inconsistent. The goal is now to verify the consistency of entire GR graphs. This can be stated as a problem of consistency in networks of relations. As it is pointed out in [10], any representation of the constraints that allow composition and intersection is sufficient for this purpose.

Informally, a GR graph is consistent if there exist configurations between workpieces whose defining coordinate transformations belong to the corresponding constraints. Obviously, a GR graph without cycles is always consistent; thus, it is easy to realize the important role of cycles in GR graphs.

Next, before introducing a general algorithm for propagating kinematic constraints, some few concepts on cycles in graphs are reminded.

4.4.1 Preliminaries on cycles

Two basic operations with cycles are the *union* and the *ring sum*. The union of two cycles $C_1 = (V_1, E_1)$ and $C_2 = (V_2, E_2)$ is a graph $G = C_1 + C_2$ with node set $V_3 = V_1 \cup V_2$ and arc set $E_3 = E_1 \cup E_2$. The ring sum of two cycles C_1 and C_2 (written $C_1 \bigoplus C_2$) is another cycle or a set of disjoint cycles consisting of the node set $V_1 \cup V_2$ and of arcs that are either in C_1 or C_2, but not in both.

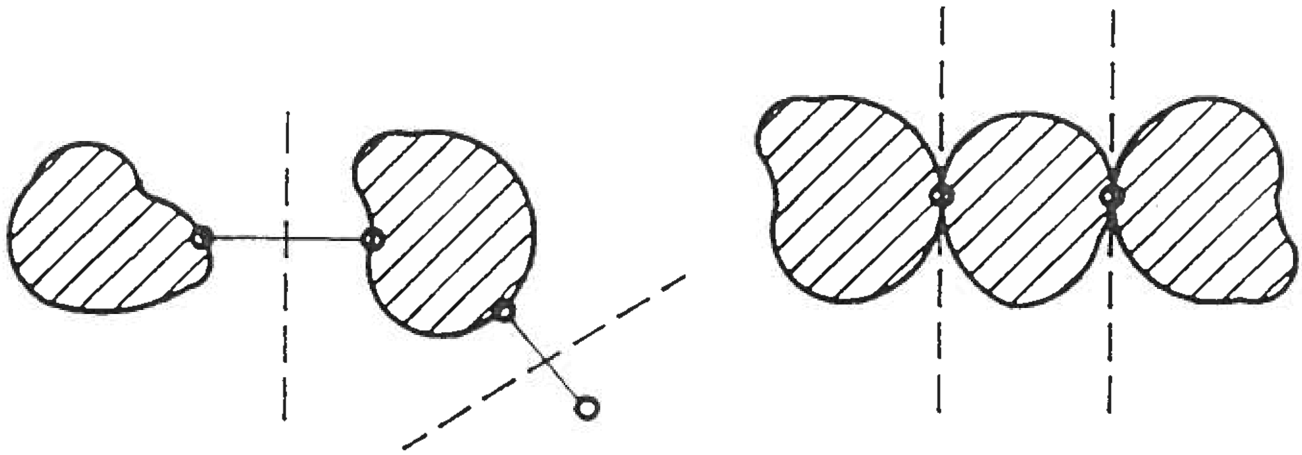
A set of cycles $\mathcal{H}$ in a graph $G = (V, E)$ is said to be a complete set of basic cycles if (i) every cycle in the graph can be expressed as a ring sum of some or all cycles in $\mathcal{H}$, and (ii) no cycle in $\mathcal{H}$ can be expressed as a ring sum of others in $\mathcal{H}$. The cardinality of a complete set of basic cycles is $\mu = \mid E \mid - \mid V \mid + 1$, which is called the cyclomatic number. Hence the maximum number of cycles is $2^\mu - 1$.

4.4.2 Isolation of blocks

When a kinematic constraint is posted, it can affect other workpieces different from those it is incident to, but, in general, a constraint is limited in its *scope*. In order to isolate subgraphs within which the effect of a constraint is limited, the following operations are applied:

1 Elimination of cutlines or bridges. This includes the elimination of pendant constraints (fig. 4.4a).

2 Split cutpoints or articulation nodes into two nodes to produce two disjoint subgraphs (fig. 4.4b).



(a) (b)

Figure 4.4: Operations applied for the isolation of blocks: (a) elimination of cutlines; and (b) splitting cutpoints.

As a result of these operations a set of subgraphs, or simply *blocks*, are obtained. A GR graph is consistent if each of its blocks is consistent.

Now, we can introduce a definition for an important subclass of graphs of kinematic constraints.

Definition 5 (Trivial GR graph) *A GR graph is said to be trivial iff the equivalent constraint between any two nodes in any of its blocks can be expressed a trivial constraint.*

It is obvious that a GR graph without cycles is always trivial.

Let us assume that the obtained blocks are planar graphs. This assumption, while not very restrictive, simplifies the treatment given below. Anyway, the provided results can be extended to non-planar graphs.

A plane representation of a graph divides the plane into *regions*. A region is characterized by the set of arcs forming its boundary. In a plane representation of a planar connected graph the set of cycles forming the interior regions, or *region cycles*, constitutes a complete set of basic cycles. The set of region cycles is not unique. Actually, there are $\binom{\mu+1}{\mu}$ different sets of region cycles. This can be easily seen by noting that a planar graph can be embedded on the surface of a sphere. The number of region cycles in the surface of a sphere would be $\mu + 1$, which are also the shortest cycles for a planar graph.

Let X be the set of region cycles in a planar block G. The *cycle graph* of X is the graph with vertex set X and arcs joining two distinct nodes if and only if the corresponding cycles have an arc in common. This graph is denoted by $\mathcal{D}(G)$ and it can be easily proved that $\mathcal{D}(G)$ is a subgraph of the dual graph of G (see [6, page 106]). Nodes in a $\mathcal{D}(G)$ graph stand for cycles and arcs in $\mathcal{D}(G)$, for *shared arcs* in G. For extension, constraints labeling a shared arc are called *shared constraints*. Note that an arc can only be shared by two region cycles.

Let C_i be a region cycle whose arc set is labeled with the constraints

$$\{R_1, R_2, \ldots, R_j, \ldots, R_n\}$$

according to fig. 4.5. Then, the constraint R_j can be substituted by

$$R_j^i = R_j \cap (R_{j-1}^{-1} \cdots R_1^{-1} \cdot R_n^{-1} \cdots R_{j+1}^{-1})$$

without modifying the consistency of the corresponding GR graph (fig. 4.5b). In order to simplify the notation, we will write

$$R_j^i = \cap^{C_i} R_j.$$

This is the basic mechanism for constraint propagation as it is shown below.

4.4.3 A filtering algorithm for propagating kinematic constraints

A general procedure to *propagate* the effect of constraints in GR graphs has been devised, either to characterize the set of configurations that satisfy all the constraints or to find out that there exist no such configurations.

The propagation process consists in *filtering* all constraints, that is eliminating from the constraints those displacements which cannot appear in any solution. Eventually, if all constraints are reduced to only one element, a single solution is obtained.

Global consistency in a block G is checked by eliminating local inconsistencies; that is, by eliminating inconsistencies in region cycles – which is equivalent to ensure *node inconsistency* in $\mathcal{D}(G)$ –, and by eliminating inconsistencies between adjacent region cycles – which is equivalent to ensure *arc consistency* in $\mathcal{D}(G)$ (see [10] or [11]). The following procedure implements this idea.

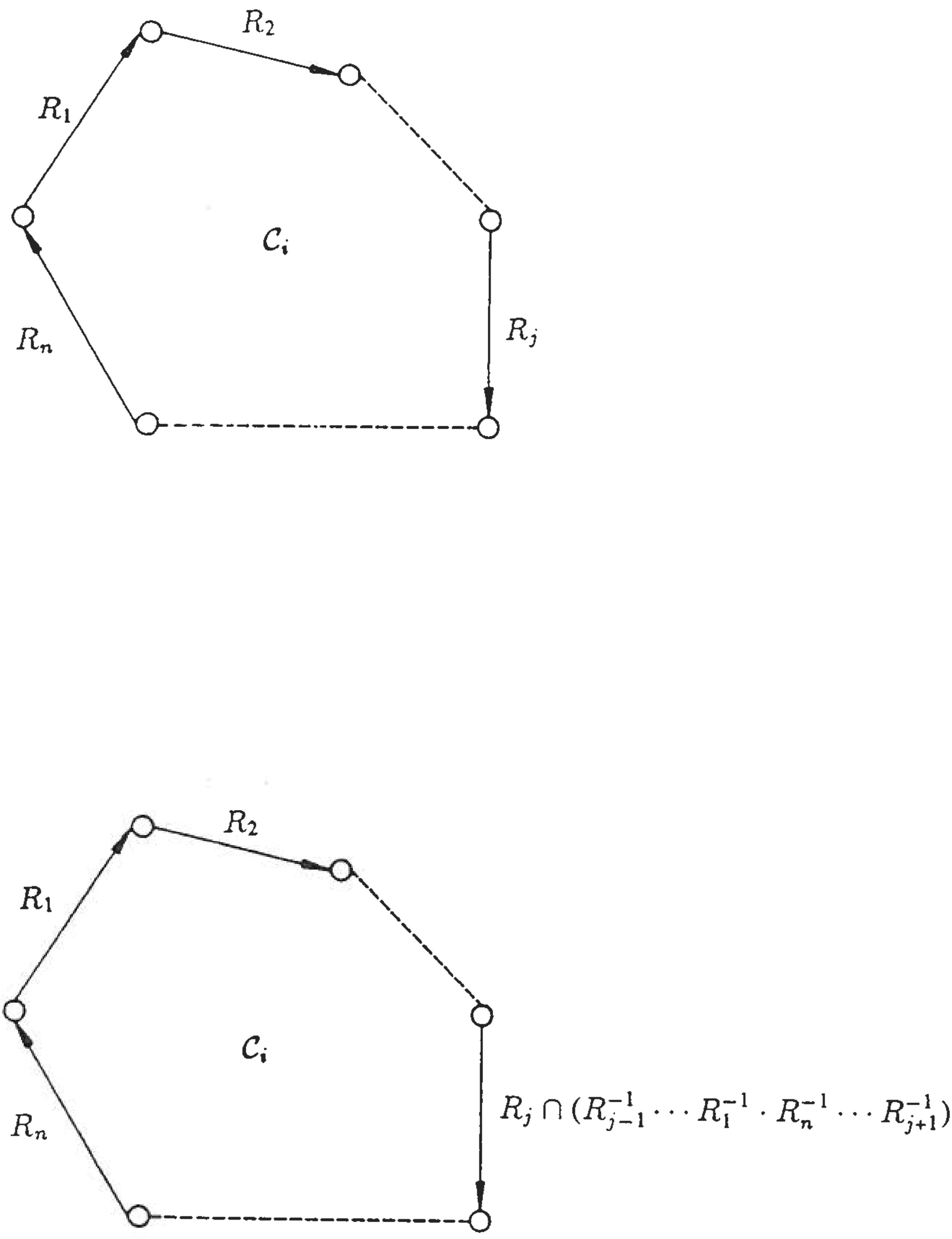


Figure 4.5: The basic mechanism for constraint propagation. A constraint R_j, labeling an arc in a cycle C_i, can be substituted by $\cap^{\mathcal{C}_i} R_j$.

```
procedure filter_constraints;
input: G; /* a block of a GR graph*/
output: G;
repeat

        stop:= true;

/* check node consistency */

        forall region cycles $C_j$ do
                forall constraints $R_i$ in $C_j$ do
/*1*/                     $R_i^j := \cap^{C_j} R_i$;
/*2*/                     if $R_i^j == \emptyset$ then exit();
                enddo;
        enddo;

/* check arc consistency */

        forall shared constraints $R_i$ do
/*3*/           $R_T := \cap_k R_i^k$;
/*4*/           if $R_T == \emptyset$ then exit();
                if $R_T \neq R_i$ then
                   stop:= false;
                   $R_i := R_T$;
                endif;
        enddo;
until stop;
end.
```

Geometric inconsistencies can be found either when $\cap^{C_j} R_i$ or when $\cap_k R_i^k$ become the empty set. In the first case, the cycle C_j becomes inconsistent; in the second one, all the cycles sharing the constraint R_i do. The problem of obtaining the minimum set of constraints that made a given GR graph become inconsistent is addressed in [17].

The above algorithm can be easily modified for its application for a topological analysis, stating the problem in terms of composition and intersections of associated subgroups instead of constraints. Then, sentences /*2*/ and /*4*/ can be removed and, if the outcome /*1*/ is not a subgroup for a particular C_j, it is not taken into account when computing /*3*/. In this case, the algorithm will halt when no progress is made, either because the graph is not trivial, or because all possible filterings have already been carried out. An example illustrating this idea is shown in the next section.

If the corresponding GR graph is planar, it is presumable that the complexity of the above algorithm is polynomial in the number of constraints [11].

The significance of the described algorithm is that it only needs to repeatedly handle a set of short cycles. Because of domain specific attributes, node and arc consistency provide a sufficient guarantee that there is a complete solution, in the same way the celebrated Waltz's filtering algorithm provides a complete solution for polyhedral scene labeling looking only for arc consistency [19].

4.5 Example

Let the workpieces in fig. 4.6a be elements of an assembly. The matings between complementary features of workpieces $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ lead to the GR graph in fig. 4.6b, which only contains one block with one cycle. Thus, the equivalent constraint between any two workpieces can be obtained by simply reduction of the graph to a single edge linking them. For example, the equivalent constraint between $\mathcal{B}_2$ and $\mathcal{B}_3$, if different from null, will be a coset of

$$(T_{\mathbf{u}_1} \cdot T_{\mathbf{u}_2}) \cap T_{\mathbf{u}_3} = T_{\mathbf{u}_1} \cap T_{\mathbf{u}_3} = I,$$

i.e. $\mathcal{B}_2$ will remain fixed with reference to $\mathcal{B}_3$. This suggest that $\mathcal{B}_2$ and $\mathcal{B}_3$ must be put together before $\mathcal{B}_1$ is assembled, providing valuable information about the assembly sequence.

Now, let us consider all the workpieces in fig. 4.6a. Given the matings between their complementary features, the problem consists in deciding whether these matings are enough for fixing the relative location of these four workpieces. The corresponding GR graph appears in fig. 4.6c. Neither composition nor intersection of constraints can be applied to reduce it.

Using the algorithm proposed in the last section, we can write the following table:

	R_i^A	C_1	C_2	C_3	results 1^{st} iteration	C_1	C_2	C_3	results 2^{nd} iteration
R_1	$T_{\mathbf{u}_1}$	$T_{\mathbf{u}_1}$			$T_{\mathbf{u}_1}$	I			I
R_2	$T_{\mathbf{u}_2}$	$T_{\mathbf{u}_2}$		I	I	I		I	I
R_3	$T_{\mathbf{u}_3}$	I	I		I	I	I		I
R_4	$C_{\mathbf{u}_4}$		?	?	$C_{\mathbf{u}_4}$		$C_{\mathbf{u}_4}$	$C_{\mathbf{u}_4}$	$C_{\mathbf{u}_4}$
R_5	$C_{\mathbf{u}_5}$		?		$C_{\mathbf{u}_5}$		$C_{\mathbf{u}_4}$		$C_{\mathbf{u}_4}$
R_6	$C_{\mathbf{u}_6}$			?	$C_{\mathbf{u}_6}$			$C_{\mathbf{u}_4}$	$C_{\mathbf{u}_4}$

taking into account that:

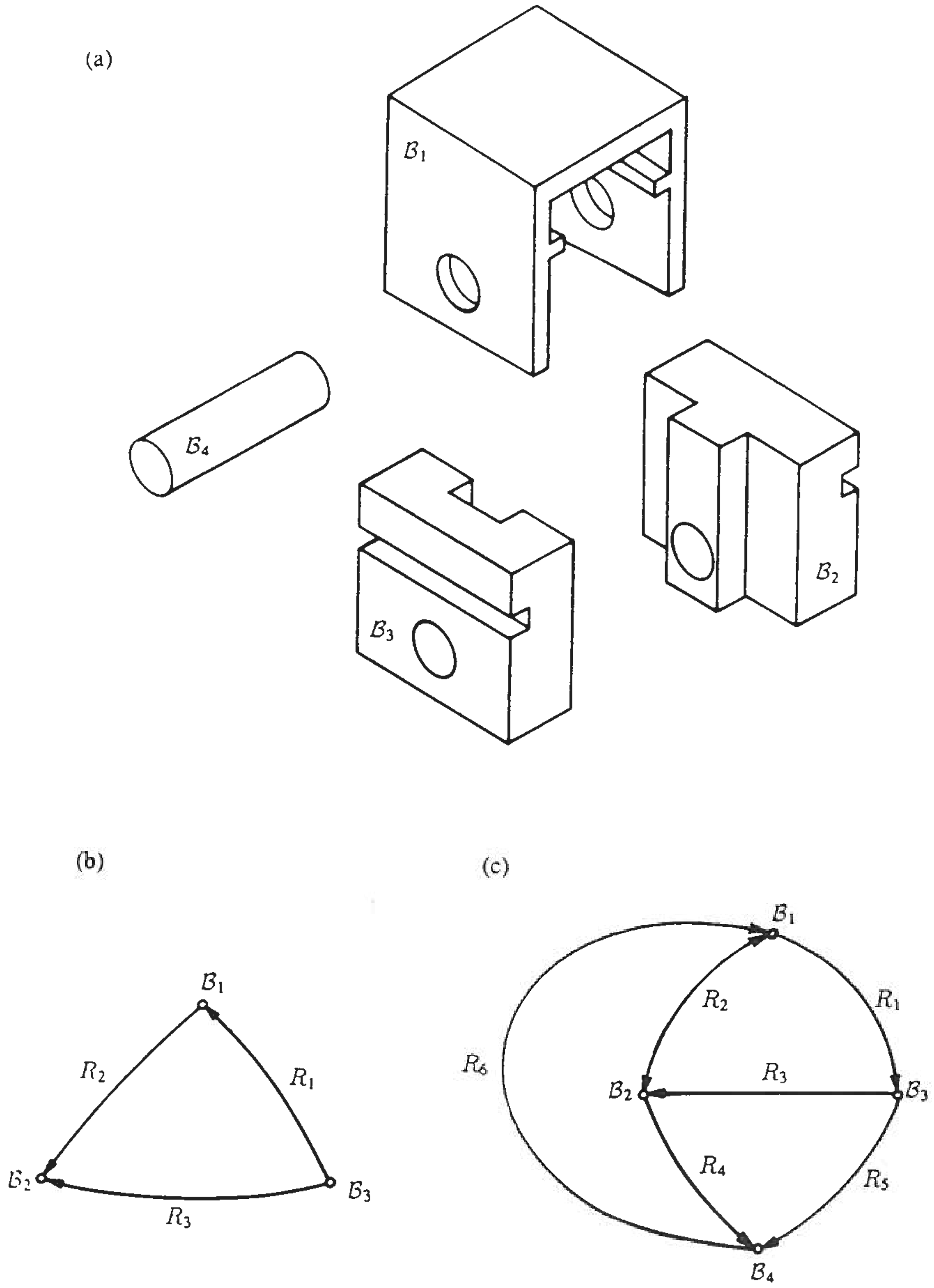


Figure 4.6: (a) A set of workpieces to be assembled; (b) the corresponding GR graph involving workpieces $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$; and (c) the GR graph involving all workpieces.

$u_1 \parallel u_2$	$u_4 \bowtie u_5$
$u_1 \perp u_3$	$u_4 \bowtie u_6$
$u_2 \perp u_1$	$u_5 \bowtie u_6$
$u_3 \perp u_4$	$u_2 \perp u_6$
$u_3 \perp u_5$	$u_1 \perp u_6$

which can be directly inferred from the linking displacements.

The outcomes of $(\cap^{C_i} R_j)^A$ appear in row R_j and column C_i. In the first iteration, some of this subgroups cannot be obtained since the corresponding constraint is not trivial. In the third iteration no new progresses can be carried out and, since it was possible to compute $(\cap^{C_i} R_j)^A$ for $i = 1..3$ and $j = 1..6$, the algorithm finishes after propagating all the constraints. Then, it can be said that, if all introduced constraints are geometrically consistent, then the bodies $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ will remain fixed and, if they are considered as a subassembly, then the body $\mathcal{B}_4$ will have two d.o.f. with reference to it.

4.6 Summary

A kinematic constraint has been defined as a set of displacements which can be expressed as a composition of cosets of Euclidean subgroups. A constraint is said to be trivial when it can be reduced to a single coset. Trivial constraints include all kinematic lower pairs.

A characterization of the spatial relationships between bodies in assemblies as trivial kinematic constraints, as well as a tabulation of the outcomes of the composition and intersection of the corresponding subgroups has been given. The theoretical foundation for this systematization has been taken from [7].

A graph of kinematic constraints has been defined as a graph whose nodes correspond to workpieces and whose directed arcs are labeled with trivial kinematic constraints.

It has been shown that it is not always possible to obtain the equivalent constraint between any two bodies in a graph of kinematic constraints by simply composing and intersecting constraints, so that the graph is reduced to a single arc linking both bodies. An algorithm that provides a way around this difficulty has been proposed. This algorithm filters all the constraints in a graph of kinematic constraints. This process consists in eliminating from the constraints those displacements which cannot appear in any solution.

It has been shown how – relying on the composition and intersection of subgroups – it is possible to carry out a topological analysis of the motion possibility for a set of bodies linked by a set of trivial kinematic constraints. It has also been shown that it is not possible to derive geometric inconsistencies from this analysis.

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References

[1] J. Angeles, *Rational Kinematics.* Springer-Verlag, New York, 1988.

[2] O. Bottema and B. Roth, *Theoretical Kinematics.* Dover Publications, New York, 1979.

[3] J.E. Campbell. *Introductory Treatise on Lie's Theory of Finite Continuous Transformations Groups.* Clarendon Press, Oxford, 1903.

[4] E. Celaya and C. Torras. Finding Object Configurations that Satisfy Spatial Relationships. *ECAI*, Stockholm, 1990.

[5] N. Christofides. *Graph Theory. An Algorithmic Approach.* Academic Press, N.Y., 1975.

[6] N. Deo. *Graph Theory with Applications to Engineering and Computer Science.* Prentice-Hall, Englewood Cliffs, N.J.,1974.

[7] J.M. Hervé. Analyse Structurelle des Mécanismes par Groupe des Déplacements. *Mechanism and Machine Theory*, Vol. 13, pp. 437-450, 1978.

[8] N. Koutsou. *Planning Motion in Contact to Achieve Parts Mating.* Ph.D. dissertation, University of Edinburgh, 1986.

[9] Y. Liu. *Symmetry Groups in Robotic Assembly Planning.* Ph.D. dissertation, University of Massachusetts, 1990.

[10] A.K. Mackworth. Consistency in Networks of Relations. *Artificial Intelligence Journal*, Vol. 8, pp. 99-118, 1977.

[11] A.K. Mackworth and E.C. Freuder, The Complexity of Some Polynomial Network Consistency Algorithms for Constraint Satisfaction Problems. *Artificial Intelligence Journal*, Vol. 25, pp. 65-74, 1985.

[12] E. Mazer. LM-GEO Geometric Programming of Assembly Robots. In *Advanced Software in Robotics*, pp. 99-125, Elsevier Science Publishers B.V. (North-Holland), 1984.

[13] R.P. Paul. *Robot Manipulators: Mathematics, Programming, and Control.* Cambridge, MA. MIT Press, 1981.

[14] R.J. Popplestone, A.P. Ambler and I.M. Bellos. An interpreter for a language for describing assemblies. *Artificial Intelligence Journal*, Vol. 14, pp. 79-107, 1980.

[15] R.J. Popplestone, Group Theory and Robotics. In *Robotics Research. The First International Symposium*, edited by M. Brady and R. Paul. MIT Press, 1984.

[16] F. Thomas and C. Torras. A Group Theoretic Approach to the Computation of Symbolic Part Relations. *IEEE Journal of Robotics and Automation*, Vol.4, No.6, December 1988.

[17] F. Thomas and C. Torras. Inferring Feasible Assemblies from Spatial Constraints. *Technical Report of the Institute of Cybernetics*, IC-DT-1989.03,June 1989.

[18] J.M. Valade. Geometric Reasoning and Automatic Synthesis of Assembly Trajectory, *International Conference on Advanced Robots*, Tokyo, Japan, Sept. 9-10, 1985.

[19] D. Waltz. Generating Semantic Descriptions from Drawings of Scenes with Shadows. AI-TR-271, MIT, reprinted in *The Psychology of Computer Vision*, Winston (ed.), McGraw Hill, 1975.

[20] J.R. Woodwark. Some Speculations on Feature Recognition. In *Geometric Reasoning*, edited by D. Kapur and L. Mundy, MIT Press, 1989.