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On the N-bar Mechanism, or How to Find Global Solutions to Redundant Single Loop Spatial Kinematic Chains *

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Abstract

This paper investigates the global sets of solutions for single loop inverse kinematic problems containing only independent rotational and translational degrees of freedom, providing a rational and compact method to obtain these sets.

The presented methodology relies on a deep understanding of spherical redundant mechanisms, shedding light on the basic algebraic structure underlying inverse kinematic problems associated with single loop kinematic chains. This methodology consists of finding the inverse kinematic solution to the so-called n-bar mechanism. It is shown that one can model any kinematic loop equation as the loop equation derived from this mechanism after taking as many bars as needed and constraining some of the resulting degrees of freedom.

The solution of any kinematic equation can be factored into a solution of both its rotational and translational components. While the solution of the former can be obtained quite easily, the latter leads, in general, to inextricable formulae. This is why this factorization strategy has been used with very limited range of practical application in the past. Herein some important relationships between both components are obtained, which make it simpler to obtain the solution to the translational component once the solution to the rotational one has been found.

Obtaining the global sets of solutions has applications in, among other problems, the mobility analysis of spatial mechanisms, the inverse kinematics of redundant kinematic chains and, in general, in automatic geometric reasoning.

1 Introduction

A forward kinematic function, $\mathbf{F}$, is defined as a non-linear vector function which relates a set of n joint coordinates, Ω, of a closed kinematic chain so that

$$\mathbf{F}(\Omega) = \mathbf{I}, \tag{1}$$

where $\mathbf{I}$ is the identity displacement.

One of the primary problems of practical interest in kinematics is determining the inverse kinematic function, $\mathbf{F}^{-1}$, which computes the sets of joint coordinates that keep the mechanism closed. The problem is especially hard to solve when dealing with redundant kinematic chains, i.e. kinematic chains with extra degrees of freedom (d..o.f.) which have been used, mainly in the robotics domain, to satisfy some additional criteria such as singularity avoidance, obstacle avoidance, joint and torque limit avoidance, etc. This paper essentially deals with the problem of obtaining these solution sets, keeping in mind that, for redundant mechanisms, there is an infinite number of joint coordinates that satisfy (1).

Kinematics of interconnected rigid bodies may lead to very complex computations and it is important to perform these computations in the most compact form and to search for their most rational organization. This goal motivates a great deal of research on the fundamental operations and the algebraic structures underlying kinematic methods. Nevertheless, no general satisfactory solution, convenient for practical use, has been found for the general positional inverse kinematic problem. This problem is highly complicated because of its non-linearity, the non-uniqueness of the solution, and the existence of singularities. This is why the redundant manipulator literature has focused on the linearized first order instantaneous kinematic relation between joint velocities, that is

$$\mathbf{J}(\Omega)\dot{\Omega} = 0 \tag{2}$$

where $\mathbf{J}(\Omega) = d\mathbf{F}(\Omega)/d\Omega$ is the Jacobian of the kinematic chain. In this literature, the inverse solution of (2) is often referred to as the inverse kinematic solution, rather than that of (1). Thus, given the position and velocity states, the set of joint coordinates can be obtained either by directly solving positional equations (1), or by solving the first-order differential equations (2). Since (2) at a particular position is linear, numerical solutions to the inverse velocity problem are relatively easier than that of the inverse position problem. Nevertheless, practical applications, including most industrial robot coordination algorithms, avoid numerical inversion of the Jacobian by using analytical inverses developed on an ad hoc basis.

Two main approaches may be distinguished within the ad hoc methods: the purely geometrical methods and the matrix methods operating on coordinates. The purely geometric methods contain the essential feature of kinematics but require a great deal of algebraic skill. On the other hand, matrix algebra, though of simple use, is somewhat blind and often leads to inextricable formulae.

The first foundation for a unified theory of analysis of spatial mechanisms was provided in [7]. Nevertheless, the algebra used

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prevents any intuitive interpretation of the results, making any further insight difficult. The present work presents an alternative foundation for a unified theory of analysis of spatial mechanisms based on simple results.

Finally, it is important to realize that, depending on the kind of problem to be solved, some simplified descriptions of the solution space are often enough. Actually, there are two basic descriptions of the solution space of a kinematic equation, namely: (a) explicit algebraic descriptions; and (b) projections of the solution space onto the coordinate axes defined by the d.o.f.

The former is the most common kind of description sought. Nevertheless, it is in general difficult to obtain, if even possible, for kinematic chains of arbitrary complexity.

The latter actually provides a set of valid ranges for each d.o.f. (equivalent to a mobility analysis of the mechanism). When one d.o.f. is fixed to a value within the corresponding range, we are sure to find a set of solutions inside the range of the other d.o.f. Note that the ranges obtained define an enclosing box (possibly a set of boxes) within which the solution space is included. This suggests an intermediate description based on a recursive decomposition of the solution space described in terms of an n-ary tree, mainly if an effective algorithm for range propagation is available [3]. In the limit this description would be as accurate as the former representation.

When a set of feasible ranges is obtained, some important information is lost. For example, a single range for all the d.o.f. does not mean a single solution set, thus we have no guarantee of finding a continuous motion linking two arbitrary solutions inside the feasible ranges of motion.

Despite the kind of description sought, one can take advantage of the results presented herein.

This paper is structured as follows. Section 2 states the inverse positional kinematic problem in terms of 3×3 dual-number matrices. Section 3 is devoted to finding the expressions of the components of rotation and translation of a kinematic equation used throughout this paper. Section 4 investigates the sets of solutions of the rotations equation and their possible parametrizations. It is shown that this problem is equivalent to that of finding a global solution to the positional kinematic problem of the orthogonal spherical mechanisms. Section 5 is devoted to obtaining the links between the equation of rotations and the equation of translations and how the solution to the latter can be *easily* found if the solution for the former has already been obtained. Section 6 presents an example whose analysis is carried out using the developed methodology and, finally, Section 7 provides a summary of the main points in the paper, as well as the conclusions and prospects for future research.

2 Stating the problem in terms of 3×3 dual-number matrices

The general inverse kinematic problem will be formulated in terms of 3×3 dual matrices (see, for example [11] or [7]). This is an alternative method to the more well-known vector approach using the 4×4 homogeneous matrix representation. The dual matrix formulation is preferred here to the popular homogeneous coordinate form because, for analytical manipulation, the formulation obtained for the separation of the rotational and translational parts is essential herein.

A kinematic chain is defined as a set of n links in series. The proportions of link i will be specified by a constant dual angle $\hat{\phi}_i = \phi_i + \varepsilon d_i$ between the two adjacent joint axes. The dual operator is defined by $\varepsilon^m = 0$ where m is any integer greater than one. The parameters ϕ_i and d_i are referred to as the twist angle and the length of link i, respectively.

Neighboring links have a common joint axis between them. One parameter of interconnection is the distance along this common axis from one link to the next. Thus, the displacement linking the reference frame of element i with that of element $i+1$ can be expressed as:

$$\mathbf{Rx}(\alpha_i + \varepsilon x_i)\ \mathbf{Ry}(\beta_i + \varepsilon y_i), \tag{3}$$

or, in other words,

$$\mathbf{Rx}(\alpha_i + \varepsilon x_i)\ \mathbf{X}\ \mathbf{Rx}(\beta_i + \pi + \varepsilon y_i)\ \mathbf{X}\ \mathbf{Rx}(\pi). \tag{4}$$

where

$$\mathbf{X} = \mathbf{Rz}(\pi/2) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5}$$

Thus, any kinematic chain can be described kinematically by giving the values of four quantities for each link. Two describe the link itself, and the other two describe the link's connection to a neighboring link [6 p. 64]. In what follows, we will consider the four quantities as variables. This way we can model any kinematic chain by constraining the appropriate rotational or translational d.o.f.

If we define

$$\mathbf{B}(\hat{\phi}_i) = \mathbf{Rx}(\hat{\phi}_i)\ \mathbf{X} =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_i - \varepsilon d_i \sin\phi_i & -\sin\phi_i - \varepsilon d_i \cos\phi_i \\ 0 & \sin\phi_i + \varepsilon d_i \cos\phi_i & \cos\phi_i - \varepsilon d_i \sin\phi_i \end{pmatrix} \mathbf{X} =$$

$$\left(\mathbf{Rx}(\phi_i) + d_i \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin\phi_i & -\cos\phi_i \\ 0 & \cos\phi_i & -\sin\phi_i \end{pmatrix}\right) \mathbf{X}, \tag{6}$$

the kinematic equation for a closed kinematic chain is an equation of the form:

$$\mathbf{F}(\hat{\Phi}) = \prod_{i=1}^{n} \mathbf{B}(\hat{\phi}_i) = \mathbf{I} \tag{7}$$

where

$$\hat{\Phi} = (\hat{\phi}_1, \hat{\phi}_2, \ldots, \hat{\phi}_n) = (\phi_1 + \varepsilon d_1, \phi_2 + \varepsilon d_2, \ldots, \phi_n + \varepsilon d_n)$$

is called the *vector of displacements*; $\Phi = (\phi_1, \phi_2, \ldots \phi_n)$, the *vector of rotations*; and $D = (d_1, d_2, \ldots d_n)$, the *vector of translations*.

Equation (7) can be interpreted as the loop equation of the n-bar mechanism appearing in *fig. 1*. It should be clear that by constraining some of the d.o.f. of this mechanism and taking as many bars as needed, one can model any kinematic chain with independent translational and rotational d.o.f.

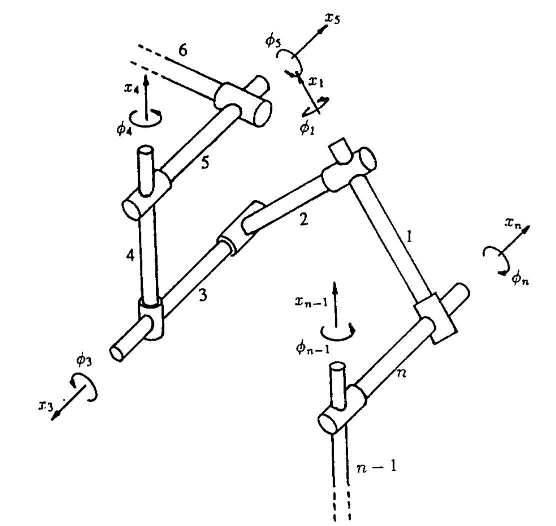

Fig. 1 *The n-bar mechanism.*

3 Two important equations

Some proposed symbolic methods for solving simple inverse kinematic problems can be factored into a solution for the rotational component and a solution for the translational component [9]. Notice that the rotation component can be extracted directly from the original equation by simply removing all the translations, but not the translational component. The approach taken here also follows this sequencing strategy. As is shown below, under the dual number formalism, both components can be obtained by equating the real and dual parts of (7), respectively.

Equating the real parts in (7), we have:

$$\mathbf{F}(\Phi) = \prod_{i=1}^{n} \mathbf{B}(\phi_i) = \mathbf{I} \tag{8}$$

which will be called the *equation of rotations*.

Let us now obtain the partial derivatives of $\mathbf{F}(\Phi)$ with respect to ϕ_i. Since

$$\frac{\partial \mathbf{F}(\Phi)}{\partial \phi_i} = \mathbf{B}(\phi_1) \cdots \mathbf{B}(\phi_{i-1}) \left(\frac{\partial \mathbf{Rx}(\phi_i)}{\partial \phi_i} \mathbf{X} \right) \mathbf{B}(\phi_{i+1}) \cdots \mathbf{B}(\phi_n) \tag{9}$$

and

$$\frac{\partial \mathbf{Rx}(\phi_i)}{\partial \phi_i} \mathbf{X} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin\phi_i & -\cos\phi_i \\ 0 & \cos\phi_i & -\sin\phi_i \end{pmatrix} \mathbf{X} = \mathbf{Q}\,\mathbf{B}(\phi_i), \tag{10}$$

where

$$\mathbf{Q} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{11}$$

then, after postmultiplying (9) by the transpose of (8), we have

$$\frac{\partial \mathbf{F}(\Phi)}{\partial \phi_i} = \mathbf{A}_i\,\mathbf{Q}\,\mathbf{A}_i^t \tag{12}$$

where

$$\mathbf{A}_i = \begin{cases} \mathbf{I}, & i = 1 \\ \prod_{j=1}^{i-1} \mathbf{B}(\phi_j), & i > 1 \end{cases} \tag{13}$$

Thus, we define $\nabla\mathbf{F}(\Phi)$ as

$$\nabla\mathbf{F}(\Phi) = (\mathbf{A}_1\mathbf{Q}\mathbf{A}_1^t,\ \mathbf{A}_2\mathbf{Q}\mathbf{A}_2^t, \ldots, \mathbf{A}_n\mathbf{Q}\mathbf{A}_n^t). \tag{14}$$

Since $\mathbf{Q}$ is a skew symmetric matrix, then $\mathbf{HQH}^t$ is also skew symmetric for any $\mathbf{H}$. Thus, all the components of $\nabla F(\Phi)$ are skew symmetric matrices. This fact will be useful later.

On the other hand, the linearization of equation (8) around a solution $\Phi_0 = (\phi_1^0, \ldots, \phi_n^0)$ can be expressed as:

$$\sum_{i=1}^{n} \Delta\phi_i\ \mathbf{A}_i\mathbf{Q}\mathbf{A}_i^t = 0 \tag{15}$$

where

$$\Delta\phi_i = (\phi_i - \phi_i^0) \tag{16}$$

In other words,

$$\nabla\mathbf{F}(\Phi_0)(\Phi - \Phi_0)^t = \nabla\mathbf{F}(\Phi_0)\Delta\Phi^t = 0 \tag{17}$$

This is called the *equation of approximation*, which agrees with the result obtained in [10]. Actually, equation (15) is the first order approximation of $\mathbf{F}(\Phi) = \mathbf{I}$ around Φ_0, which defines a *hyperplane of approximation*. Thus, the first order Taylor's approximation of $\mathbf{F}(\Phi_0 + \Delta\Phi)$ can be expressed as:

$$\mathbf{F}(\Phi_0 + \Delta\Phi) = \mathbf{I} + \nabla\mathbf{F}(\Phi_0)\Delta\Phi^t, \tag{18}$$

which will also be useful later.

On the other hand, noting from (6) that

$$\mathbf{Rx}(\hat{\phi}_i) = \mathbf{Rx}(\phi_i) + d_i\varepsilon\ \mathbf{Q}\ \mathbf{Rx}(\phi_i) = (\mathbf{I} + d_i\varepsilon\mathbf{Q})\ \mathbf{Rx}(\phi_i), \tag{19}$$

equation (7) becomes:

$$\mathbf{F}(\hat{\Phi}) = \prod_{i=1}^{n} (\mathbf{I} + d_i\varepsilon\mathbf{Q})\ \mathbf{Rx}(\phi_i)\ \mathbf{X} = \mathbf{I}. \tag{20}$$

Equating the dual parts in (20) and taking into account that $\varepsilon^2 = 0$, it can be easily shown that

$$\begin{array}{lllll} d_1 & \mathbf{Q}\ \mathbf{Rx}(\phi_1)\ \mathbf{X} & \mathbf{Rx}(\phi_2)\ \mathbf{X} & \cdots & \mathbf{Rx}(\phi_n)\ \mathbf{X} + \\ d_2 & \mathbf{Rx}(\phi_1)\ \mathbf{X} & \mathbf{Q}\ \mathbf{Rx}(\phi_2)\ \mathbf{X} & \cdots & \mathbf{Rx}(\phi_n)\ \mathbf{X} + \\ \vdots & & & & \\ d_n & \mathbf{Rx}(\phi_1)\ \mathbf{X} & \mathbf{Rx}(\phi_2)\ \mathbf{X} & \cdots & \mathbf{Q}\ \mathbf{Rx}(\phi_n)\ \mathbf{X} = 0. \end{array} \tag{21}$$

In other words,

$$\nabla\mathbf{F}(\Phi)D^t = 0 \tag{22}$$

which will be called the *equation of translations*.

Considering that there are three translational degrees of freedom in space, it should be expected that there be three equations representing this information. Actually, if

$$\mathbf{A}_i = (\mathbf{n}_i\ \mathbf{o}_i\ \mathbf{a}_i) = \begin{pmatrix} \mathbf{n}_{ix} & \mathbf{o}_{ix} & \mathbf{a}_{ix} \\ \mathbf{n}_{iy} & \mathbf{o}_{iy} & \mathbf{a}_{iy} \\ \mathbf{n}_{iz} & \mathbf{o}_{iz} & \mathbf{a}_{iz} \end{pmatrix}, \qquad (23)$$

then it can be easily proved that

$$\mathbf{A}_i\mathbf{Q}\mathbf{A}_i^t = \begin{pmatrix} 0 & -(\mathbf{o}_i \times \mathbf{a}_i)_z & (\mathbf{o}_i \times \mathbf{a}_i)_y \\ (\mathbf{o}_i \times \mathbf{a}_i)_z & 0 & -(\mathbf{o}_i \times \mathbf{a}_i)_x \\ -(\mathbf{o}_i \times \mathbf{a}_i)_y & (\mathbf{o}_i \times \mathbf{a}_i)_x & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & -\mathbf{n}_{iz} & \mathbf{n}_{iy} \\ \mathbf{n}_{iz} & 0 & -\mathbf{n}_{ix} \\ -\mathbf{n}_{iy} & \mathbf{n}_{ix} & 0 \end{pmatrix}. \qquad (24)$$

Thus, it is clear that all components of $\nabla\mathbf{F}(\Phi)$ are skew symmetric matrices, and, if we define the vectors

$$\widehat{\mathbf{A}_i\mathbf{Q}\mathbf{A}_i^t} = \begin{pmatrix} \mathbf{n}_{ix} \\ \mathbf{n}_{iy} \\ \mathbf{n}_{iz} \end{pmatrix}, \qquad (25)$$

and we express the equation of approximation and the equation of translations in terms of them, we get the three equations we were expecting. Actually, a deeper insight reveals that the vector (25) in $\Re^3$ is a unit vector pointing in the positive direction of translation d_i with respect to the first bar of the n-bar mechanism. Thus, the equation of translations and the equation of approximation can also be expressed as:

$$\begin{pmatrix} \mathbf{n}_{1x} & \mathbf{n}_{2x} & \dots & \mathbf{n}_{nx} \\ \mathbf{n}_{1y} & \mathbf{n}_{2y} & \dots & \mathbf{n}_{ny} \\ \mathbf{n}_{1z} & \mathbf{n}_{2z} & \dots & \mathbf{n}_{nz} \end{pmatrix} D^t = \mathbf{J}(\Phi)D^t = 0 \qquad (26)$$

and

$$\begin{pmatrix} \mathbf{n}_{1x} & \mathbf{n}_{2x} & \dots & \mathbf{n}_{nx} \\ \mathbf{n}_{1y} & \mathbf{n}_{2y} & \dots & \mathbf{n}_{ny} \\ \mathbf{n}_{1z} & \mathbf{n}_{2z} & \dots & \mathbf{n}_{nz} \end{pmatrix} \Delta\Phi^t = \mathbf{J}(\Phi)\Delta\Phi^t = 0, \qquad (27)$$

respectively. It can be easily checked that $\mathbf{J}(\Phi)$ is the $3\times n$ Jacobian matrix of the spatial transformation $\mathbf{F}(\phi)$ [8], that is:

$$\mathbf{J}(\Phi) = \frac{\mathrm{d}\mathbf{F}(\Phi)}{\mathrm{d}\Phi}. \qquad (28)$$

This formulation, which is clearly much simpler than the one used in [10], leads to the following important remark.

Remark I. The solution of the translations equation of the n-bar mechanism (17) is the Jacobian null space of the corresponding rotations equation (8).

If the following condition is satisfied

$$\max_{\Phi} \left(\text{rank } \mathbf{J}(\Phi)\right) = 3, \qquad (29)$$

it is said that the degree of redundancy of the closed spherical kinematic chain is $r = n - 3$. If

$$\text{rank } \mathbf{J}(\Phi) = 2, \qquad (30)$$

for some Φ, then we say that the closed kinematic chain is in a singular state (all the n bars are on a plane). In others words, the vectors $\widehat{\mathbf{A}_i\mathbf{Q}\mathbf{A}_i^t}$, i.e. the columns of the Jacobian matrix, define a subspace of $\Re^3$ which coincides with $\Re^3$ iff the mechanism is not in a singular state.

Since all the axes of rotations of an n-bar mechanism can never be arranged so that they all keep aligned, rank $\left(\mathbf{J}(\Phi)\right)$ cannot be lower than two.

Of course, a unique inverse of equation (26) is feasible only if $n = 3$. If $n > 3$ and we are just looking for a solution according to some optimality criteria, pseudo-inverse techniques can be applied [5].

Let $\mathcal{N}(\mathbf{J})$ be the null space of the linear mapping $\mathbf{J}$. Any element of this subspace is mapped into the zero vector. If the Jacobian is of full rank, the dimension of the null space, $\dim(\mathcal{N}(\mathbf{J}))$, is the same as r, the degrees of redundancy. When the Jacobian is degenerate, its rank decreases, and the dimension of the null space increases by the same amount. The sum of the two is always equal to n, that is:

$$\text{rank}(\mathbf{J}) + \dim\left(\mathcal{N}(\mathbf{J})\right) = n \qquad (31)$$

The solution to equation (26) involves the same number of arbitrary parameters as the dimension of the null space.

4 The self-motion manifolds of the equation of rotations

Equation (8) corresponds to the loop equation of a orthogonal spherical mechanism. The configuration space, C, for a spherical mechanism is a product space formed by the n-fold product of the individual variables of rotation, that is:

$$C = S^1 \times S^1 \times \dots \times S^1 = T^n \qquad (32)$$

where T^n is an n-torus, which is a compact n-dimensional manifold. The space of pointing directions in three-space is two-dimensional and can be represented by the set of unit vectors in three-space or, equivalently, by the surface of a sphere. In contrast, the configuration space is an n-torus. The mismatch between the topological properties of a sphere and a torus prevents the construction of a singularity-free inverse kinematic mapping [1].

For non-redundant kinematic chains, there is a finite set of configurations that solve the inverse kinematic problem. For redundant ones, there is an infinite number of configurations, which can be grouped into regions of the configuration space each having a simple manifold structure [2]. Thus, the inverse kinematic solution of a redundant spherical mechanism must be some set of r-dimensional submanifolds of C. Formally, let a redundant inverse kinematic solution of a rotation equation be denoted as the union of disjoint r-dimensional manifolds

$$\mathbf{F}(\Phi) = \mathbf{I} \quad \Rightarrow \quad \Phi = \bigcup_i M_i, \qquad (33)$$

where M_i is the i^{th} r-dimensional manifold and $M_i \cap M_j = \emptyset$ when $i \neq j$. Following Burdick's terminology [2], each of these manifolds corresponds to a "self-motion" manifold which represents a continuous motion of the elements of the closed kinematic chain without requiring the chain to open.

Bounds on the number of self-motions manifolds are discussed in [2], where it is shown that a redundant kinematic chain can have no more self-motions than the maximum number of inverse kinematic solutions of a non-redundant kinematic chain of the same class.

A discrete closed-form solution exists for spherical mechanisms with up to three d.o.f. For $n = 3$, there are two discrete solutions. For $n > 3$, a spherical kinematic chain becomes redundant. Thus, regardless of the number of d.o.f., a spherical redundant mechanism can have at most two distinct self-motion manifolds. These manifolds can be parameterized by a set of r independent parameters, $\Psi = \{\psi_1, \ldots, \psi_r\}$, so that distinct self-motions can be generated by continuously sweeping ψ_i through their range.

As already noted in [7], equation (8) has a straightforward geometric interpretation as an n-sided spherical polygon. Consider a unit radius sphere centered at the coordinate origin. As a result of applying successive rotations, the $z-$axis will describe on the surface of the sphere a spherical polygon with sides (*arcs of great circles*) of length ϕ_i, and exterior angles $\pi/2$. Alternatively, the y-axis will describe a spherical polygon with sides of length $\pi/2$ and exterior angles ϕ_i. These two polygons are considered duals. All theorems from spherical trigonometry are thus applicable; in particular, the sine, cosine, and sine-cosine laws, which are the the three basic laws for spherical triangles.

By triangulating the thus obtained n-sided spherical polygon, we will obtain the global solutions to equation (8) in terms of r independent parameters. This is not the only possible parametrization. Actually, we can take any r variables as parameters.

Finally, a set of expressions of the ϕ_i, $i = 1 \ldots n$, parameterized in terms of $n-3$ independent parameters, is obtained, so that they yield one solution for every choice of the parameters.

5 Self-motion manifolds that also satisfy the equation of translations

The tangent space to M_i at some $\Phi_0 = \Phi(\Psi_0)$ in terms of the independent parameters is:

$$\frac{dM_i(\Phi_0)}{d\Psi} = \frac{d\Phi(\Psi_0)}{d\Psi} \in \Re^{n\times n-3} \tag{34}$$

where $d\Psi \in \Re^{n-3}$. Then, by applying the chain rule,

$$\frac{d\mathbf{F}(\Phi_0)}{d\Psi} = \frac{d\mathbf{F}(\Phi(\Psi_0))}{d\Phi}\frac{d\Phi(\Psi_0)}{d\Psi} = \mathbf{J}(\Phi_0)\frac{d\Phi(\Psi_0)}{d\Psi} \in \Re^{3\times n-3}. \tag{35}$$

However, when $\mathbf{F}$ is restricted to a self-motion manifold, $\mathbf{F}$ must equal the identity displacement, $d\mathbf{F}(\Phi_0)/d\Psi$ must be zero, and therefore $d\Phi(\Psi_o)/d\Psi$, which is the tangent space of $M_i(\Psi)$ at $\Psi = \Psi_0$, must be in the null space of the Jacobian evaluated at this point (see also [2]). Thus, since the null space of the Jacobian is the solution of the equation of rotations, the tangent space of $M(\Psi)$ is part of the solution of the translations equation.

If $\Phi(\Psi_0)$ is not a singular point, then

$$\text{rank}\left(\frac{d\Phi(\Psi_0)}{d\Psi}\right) = N\left(\mathbf{J}\left(\Phi(\Psi_0)\right)\right) = r. \tag{36}$$

If $\Phi(\Psi_0)$ is a singular point, then the dimension of $N(\mathbf{J}(\Phi(\Psi_0)))$ is greater than r and $d\Phi(\Psi_o)/d\Psi$ is not strictly defined. This leads to the following remark:

Remark II. For an n-bar mechanism, the vector of translations that satisfies the equation of translations at $\Phi(\Psi_0)$, restricted to a self-motion manifold of the equation of rotations excluding its singular points, is any vector in the subspace of dimension $n-3$ defined by $d\Phi(\Psi_o)/d\Psi$.

Thus, at a singular point: (1) the associated spherical mechanism becomes planar; (2) the Jacobian becomes degenerate, that is, the dimension of its null space is greater than r; and (3) the tangent space of M_i is not strictly defined.

In general, singular points can be grouped into manifolds of dimension $r-1$.

The solution to the equation of translations can be found by obtaining D satisfying:

$$\text{rank}\left(D^t \mid \frac{d\Phi(\Psi_0)}{d\Psi}\right) = r. \tag{37}$$

In other words,

$$D = \lambda_1\frac{\partial\Phi(\Psi_0)}{\partial\psi_1} + \ldots + \lambda_r\frac{\partial\Phi(\Psi_0)}{\partial\psi_r}, \qquad \forall\lambda_1, \ldots, \lambda_r \in \Re. \tag{38}$$

Noting that the same solution for the equation of translations would be obtained for Φ if D is affected by a scale factor k, a simple geometric interpretation of this result is obtained by rewriting (22) using (18) as follows:

$$\mathbf{F}(\Phi + kD) = \mathbf{I}, \quad \text{for } \; k << . \tag{39}$$

Thus, all points on M_i that, after being shifted a differential quantity in the direction given by D remain on M_i, are rotations that satisfy the equation of translations and rotations simultaneously.

The analysis of singular points might consist in studying the n-bar mechanism when it degenerates to planar. Since this can only happen when n is even, it is not necessary to care about singular points when n is odd.

6 Example

The kinematic equation to solve for the 4-bar mechanism is:

$$\prod_{i=1}^{4}\mathbf{Rx}(\phi_i + \varepsilon d_i)\ \mathbf{X} = \mathbf{I} \tag{40}$$

It is a well-known result that a rotation can be resolved into three succesive rotations about perpendicular axes of rotation [4]. In doing so, we get $\phi_1 = \pm\phi_3$ and $\phi_2 = \pm\phi_4$. The stratification of this algebraic set leads to 4 strata of dimension 1 and 4 of dimension 0. As shown in the following diagram, the strata of dimension 1 are connected through those of dimension 0 making up a single manifold,

$$\begin{array}{ccccc}
\left.\begin{array}{c}\phi_1 - \phi_3 = 0\\ \phi_2 = \phi_4 = 0\end{array}\right\} & \longleftarrow & \Phi_0 = (0,0,0,0) & \longrightarrow & \left.\begin{array}{c}\phi_2 - \phi_4 = 0\\ \phi_1 = \phi_3 = 0\end{array}\right\} \\
\uparrow & & & & \uparrow \\
\Phi_1 = (\pi,0,\pi,0) & & & & \Phi_2 = (0,\pi,0,\pi) \\
\downarrow & & & & \downarrow \\
\left.\begin{array}{c}\phi_2 + \phi_4 = 0\\ \phi_1 = \phi_3 = \pi\end{array}\right\} & \longleftarrow & \Phi_3 = (\pi,\pi,\pi,\pi) & \longrightarrow & \left.\begin{array}{c}\phi_1 + \phi_3 = 0\\ \phi_2 = \phi_4 = \pi\end{array}\right\}
\end{array} \tag{41}$$

Since all the obtained strata are linear, their tangent spaces are constant. For example, the tangent space of the upper left stratum on diagram (41) is $(1, 0, -1, 0)$. Thus,

$$D_1 = \lambda_1(1, 0, 1, 0), \qquad \lambda_1 \in \Re, \tag{42}$$

is the only possible solution of the corresponding equation of translations for any point on this stratum. For the other strata, the solutions are:

$$\begin{array}{ll} D_2 = \lambda_2(0, 1, 0, 1) & \text{upper right stratum} \\ D_3 = \lambda_3(1, 0, -1, 0) & \text{lower left stratum} \\ D_4 = \lambda_4(0, 1, 0, -1) & \text{lower right stratum} \end{array} \tag{43}$$

Although we must analyze all the singular points because n is even, let's just see what happens at $\Phi_0 = (0, 0, 0, 0)$. The hyperplane of approximation around this point is:

$$\Delta\phi_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + \Delta\phi_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} +$$

$$\Delta\phi_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} + \Delta\phi_4 \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = 0 \tag{44}$$

The equation of translations can be readily obtained by substituting d_i for $\Delta\phi_i$ in (44). On the other hand the Jacobian is:

$$\mathrm{J}(\Phi_0) = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{45}$$

which is clearly singular, as expected. Since its rank is two, the mechanism is planar at this point, also as expected.

We conjecture that the solution at a singular point, because of the continuity of the solutions, can be expressed as a linear combination of those at the non-singular points on their boundaries. While this seems easy to prove or disprove, at the time of this writing, we have not been able to find any proof or refutation. In this particular example, we would get:

$$D_5 = \lambda_1(0, 1, 0, 1) + \lambda_2(1, 0, 1, 0), \quad \forall\lambda_1, \lambda_2 \in \Re, \tag{46}$$

which is cleary the solution sought.

7 Conclusions

It has been shown the great relevance of deepening on the structure of the self-motion manifolds of the orthogonal redundant spherical mechanisms, and how a thorough understanding of them is very helpful in the study of spatial mechanisms.

Any kinematic loop equation can be modeled as the loop equation derived from an n-bar mechanism by taking as many bars as needed and constraining some of the resulting d.o.f. It has also been shown that the set of angle solutions of a n-bar mechanism can be obtained by computing the tangent space of the self-motion manifolds of the orthogonal redundant spherical mechanism. If the number of bars considered is even, the singular points must be separately analyzed. This analysis consists in applying the same technique but considering the mechanism to be planar.

The obtained results are independent of the representation used for the displacements. Nevertheless, a representation based on 3×3 dual matrices has been used because it provides a compact formulation upon which it is easy to obtain the presented results.

Some important theoretical questions, concerning the solvability of kinematic chains, remain open. The most important refers to the possibility that all kinematic equations, regardless the number of d.o.f., admit an explicit description of the solution space in terms of a set of parameters. Also, the possibility that the solution at a singular point can be expressed as a linear combination of the solutions at the non-singular points in its boundary is another theoretical point that deserves further attention.

References

[1] D.R. Baker and C.W. Wampler, On the Inverse Kinematics of Redundant Manipulators, *The International Journal of Robotics Research*, Vol. 7, No. 2, pp. 3-21, Mach/April 1988.

[2] J.W. Burdick, On the Inverse Kinematics of Redundant Manipulators: Characterization of the Self-Motion Manifolds, *1989 IEEE Int. Conf. on Robotics and Automation*, pp. 264-270.

[3] E. Celaya and C. Torras, Finding Object Configurations that Satisfy Spatial Relationships. *ECAI*, Stockholm, 1990.

[4] C.H. Chen, Application of Algebra of Rotations in Robot Kinematics, *Mechanisms and Machine Theory*, Vol. 22, No. 1, pp. 77-83, 1987.

[5] H. Cheng and K.C. Gupta, A Study of Robot Inverse Kinematics Based Upon the Solution of Differential Equations, *Journal of Robotic Systems* 8(2), pp. 159-175 (1991).

[6] J.J. Craig, *Introduction to Robotics. Mechanics and Control*, Addison-Wesley Publish. Comp., 1986.

[7] J. Duffy and J. Rooney, A Foundation for a Unified Theory of Analysis of Spatial Mechanisms, *Transactions of the ASME, Journal of Engineering for Industry*, November 1975, pp. 1159-1165.

[8] R.P. Paul, B. Shimano and G.E. Mayer, Differential Kinematic Control Equations for Simple Manipulators, *IEEE Trans. on Systems, Man and Cybernetics*, Vol. SMC-11, No. 6, pp. 456-460. June 1981.

[9] F. Thomas and C. Torras, A Group Theoretic Approach to the Computation of Symbolic Part Relations, *IEEE Journal of Robotics and Automation*, vol. 4, no. 6, December 1988.

[10] J.J. Uicker, Jr., J. Denavit and R.S. Hartenberg, An Iterative Method for the Displacement Analysis of Spatial Mechanisms, *Transactions of the ASME, Journal of Applied Mechanics*, June 1964, pp. 309-314.

[11] A.T. Yang, Displacement Analysis of Spatial Five-Link Mechanisms Using (3 × 3) Matrices With Dual-Number Elements *Transactions of the ASME, Journal of Engineering for Industry*, February. 1969, pp. 152-157.