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Towards an Efficient Interval Method for Solving Inverse Kinematic Problems

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Abstract

In this paper we present a new algebraic analysis of the closure equation obtained for arbitrary single loop spatial kinematic chains, which allows us to design an efficient interval method for solving their inverse kinematics.

The solution of a kinematic equation can be factored into a solution of both its rotational and its translational components. We have obtained general and simple expressions for these equations and their derivatives that are used to perform Newton cuts. A branch and prune strategy is used to get a set of boxes as small as desired containing the solutions. If the kinematic chain is redundant, this approach can also provide a discretized version of the solution set.

The mathematics of the proposed approach are quite simple and much more intuitive than continuation or elimination methods. Yet it seems to open a promising field for further developments.

1 Introduction

Solving loops of kinematic constraints [12] is a basic requirement when dealing with inverse kinematics, task-level robot programming, assembly planning, or constraint-based modelling problems.

The problem of solving a loop of kinematic constraints can be easily proved to be equivalent to solve the following matrix equation:

$$\prod_{i=1}^{n} \begin{pmatrix} 0 & -1 & 0 & d_i \\ \cos\phi_i & 0 & -\sin\phi_i & 0 \\ \sin\phi_i & 0 & \cos\phi_i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{I}, \quad (1)$$

where $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)^t$ and $\mathbf{d} = (d_1, \ldots, d_n)^t$ are called the vector of rotations and translations, respectively [13]. In general, some of the elements of these vectors are fixed and then the problem consists in finding which values for the remaining elements – within certain given ranges – satisfy the equation. This problem is difficult due to its inherent computational complexity (i.e., it is NP-complete) and due to the numerical issues involved to guarantee correctness and to ensure termination.

Two basic approaches have been used for solving this problem: continuation and elimination methods [11, 8]. The former is based upon homotopy techniques to solve a system of polynomial equations [16]. They compute the solutions of the algebraic system by following paths in the complex space. They are robust but slow. The latter approach is based on an algebraic formulation, eliminating variables from a system of equations, and is used along with algorithms for finding roots of univariate polynomials [10]. They can be slow because of symbolic expansion and usually do not work for kinematic chains with special geometries.

Recently, interval methods for solving systems of non-linear equations have attracted much attention and have been explored by a variety of authors [5]. They have already been used to solve some instances of the above problem, proving to be robust but sometimes slow compared to continuation methods [6]. In our case, an interval method would receive a box, i.e. two interval tuples $<I_{d_1}, \ldots, I_{d_n}>$ and $<I_{\phi_1}, \ldots, I_{\phi_n}>$ specifying the initial range of the elements in $\mathbf{d}$ and $\boldsymbol{\phi}$, respectively, and it would return a set of boxes of specified accuracy containing all solutions. When the kinematic chain is redundant, this method would also be able to provide a discretized version, up to a given resolution, of the underlying self-motion manifold (i.e., the set of all solutions [1]).

For example, solving the inverse kinematic problem for a PUMA 560 would be done by applying an interval method to equation (1) with $n = 15$, $\mathbf{d}$=<[0][0][0][0][431.8][0][20.32][149.09][0][433.07][0][0][d_{13}][d_{14}][d_{15}] > and ϕ=<[180][20, 340][90][−43, 223][180][−232, 52][90][−104, 46][−90][80, 280][90][0, 360][ϕ_{13}][ϕ_{14}][ϕ_{15}]>, where the last three bars are used to close the chain from the end effector to the first bar.

Interval methods manipulate upper and lower bounds on variables and have been explored by a variety of authors. The heart of most of these methods consists in improving the known bounds using a set of inference rules called *interval cuts* [6], so that the global efficiency heavily depends on how these cuts are computed.

The purpose of this paper is to present a novel algebraic analysis of equation (1) that allows the implementation of an interval-based method able to efficiently generate a particular set of cuts called Newton cuts.

This paper is organized as follows. Section 2 briefly reviews some basic facts on equation (1) used to introduce our novel results. Section 3 explains how these results are relevant for implementing an interval method. Section 4 describes the adopted branch and prune strategy and, finally, Section 5 contains the conclusions.

2 Background

Any closed kinematic chain can be described as a circular list of screws $X_1, X_2, \ldots, X_n$, each one being orthogonal to the next one, so that its configuration is determined by the angles ϕ_i around X_i and the offsets d_i along X_i. Then, its associated loop equation can be expressed as

$$\prod_{i=1}^{n} \mathbf{T}(d_i)\mathbf{R}(\phi_i)\mathbf{Z} = \mathbf{I}, \tag{2}$$

where $\mathbf{T}(d_i)$ stands for a translation along the x-axis, $\mathbf{R}(\phi_i)$, a rotation around the x-axis and $\mathbf{Z}$, a rotation of $\pi/2$ radians around the z-axis. This equation, equivalent to (1), corresponds to the loop equation of what in [13] is called the *n-bar mechanism* (*Fig. 1*).

If we rename $\alpha_{(i-1)/2} = \phi_i + \pi$ and $a_{(i-1)/2} = d_i$ when i is odd and $\theta_{i/2} = \phi_i + \pi$ and $t_{i/2} = d_i$ when i is even, α_i, a_i, θ_i and t_i are the Denavit-Hartenberg parameters of the mechanism, where the odd bars correspond to links and the even bars to joints [3]. Therefore, any single loop mechanism with $n/2$ links can be represented by an n-bar mechanism by restricting some of its degrees of freedom.

Equation (2) can be factored into the following two equations [13]:

$$\mathbf{F}(\phi) = \prod_{i=1}^{n} \mathbf{R}(\phi_i)\mathbf{Z} = \mathbf{I} \tag{3}$$

and

$$\nabla\mathbf{F}(\phi) \cdot \mathbf{d} = \sum_{i=1}^{n} \frac{\partial \mathbf{F}(\phi)}{\partial \phi_i} d_i = 0, \tag{4}$$

which are called the *rotational and translational matrix equations*, respectively.

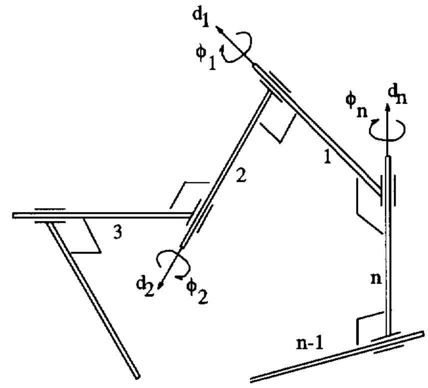

Figure 1: *The n-bar mechanism.*

Next, the solution to both equations is discussed. Some properties are given without proof, but the reader is addressed to proper references.

2.1 The rotational matrix equation

The rotational matrix equation (3) can be derived by simply removing all translations from (2). Note that it corresponds to the loop equation of an orthogonal spherical mechanism. Then, its solution can be seen as a subset of the configuration space of this mechanism, which is formed by the n-fold product of the variables of rotation, that is, a torus of dimension n (T^n).

A spherical mechanism becomes redundant for $n>3$. Then, its inverse kinematic solution can be described as an (n-3)-dimensional algebraic set or *self-motion set* (SS^{n-3}) embedded in T^n [15]. This self-motion set, however, is not a (n-3)-manifold, but rather a pseudo-manifold or a *punched manifold* because of the presence of singular points.

Singular points correspond to those situations in which the mechanism becomes planar, that is, when all axes of rotation lie on the same plane. Then, from a topological point of view, removing all these points yields a (n-3)-manifold, the *self-motion manifold* (SM^{n-3}). Thus, SS^{n-3} can be obtained from SM^{n-3} by pinching it at certain points.

Since an orthogonal spherical mechanism can degenerate to planar only when n is even, we conclude that there are no singularities when n is odd. It is also easy to see that, when n is even, the number of singularities is 2^{n-2} and it can be proved that, when $n>4$, the self-motion set remains connected after removing its singular points [15].

SM^{n-3} is an (n-3)-dimensional manifold of class C^{∞}. Among all possible parameterizations, we can take n-3 coordinates of the surrounding space, T^n, as local coordinates in the neighborhood of each point $\boldsymbol{\phi}_0 \in SM^{n-3}$. Actually, this is the implicit function theorem formulated in convenient terms, whose proof can be found in any textbook on differential geometry.

Let us take n-3 consecutive variables as parameters. Without loss of generality, let $\boldsymbol{\psi} = \{\psi_1, \psi_2, \ldots, \psi_{n-3}\}$, $\psi_i = \phi_i$ $(i = 1, \ldots, n-3)$, be the set of parameters. Hence, the rotational matrix equation can be expressed as

$$\mathbf{R}(\phi_{n-2})\,\mathbf{Z}\,\mathbf{R}(\phi_{n-1})\,\mathbf{Z}\,\mathbf{R}(\phi_n) = \mathbf{A}(\boldsymbol{\psi}) \ ,$$

which has a solution for any proper orthogonal matrix $\mathbf{A}(\boldsymbol{\psi})$ encompassing all the parameters. This equation has the following two discrete solutions

$$\begin{aligned} \phi_{n-2} &= \operatorname{atan2}(\pm a_{21}, \mp a_{31}) \\ \phi_{n-1} &= \mp \operatorname{acos}(-a_{11}) \\ \phi_n &= \operatorname{atan2}(\mp a_{12}, \mp a_{13}) \ , \end{aligned} \qquad (5)$$

where a_{ij} denotes the element (i,j) of $\mathbf{A}(\boldsymbol{\psi})$. These three equations will be called the *rotational equations.*

When $a_{11}=\pm 1$, there appear infinite solutions. The points of SM^{n-3} where this happens are called *singularities of the parameterization* and it can be shown that they correspond to those situations in which the last three bars are coplanar.

It is worth mentioning that SS^{n-3} is highly symmetrical. There are symmetrical points with respect to any singularity of the parameterization, which correspond to the two solutions of (5). For instance, given a point $\boldsymbol{\phi}_0 = (\phi_1, \ldots, \phi_n)$ on SS^{n-3}, points $\boldsymbol{\phi}_i = (\phi_1, \ldots, -\phi_{i-1}, \phi_i + \pi, -\phi_{i+1}, \ldots, \phi_n)$ are also on it. The proof is straightforward by analyzing (5). It can be easily seen that the effect over the translations is a change of sign of d_i. The iterative computation of all these symmetries lead to 2^n symmetric points for any point on SS^{n-3} [15].

2.2 The translational matrix equation

Equation (4) can be derived from (2) using the fact that $\partial\mathbf{F}(\boldsymbol{\phi})/\partial\phi_i$, when $\boldsymbol{\phi}$ is restricted to SM^{n-3}, can be expressed as:

$$\left.\frac{\partial \mathbf{F}(\boldsymbol{\phi})}{\partial \phi_i}\right|_{\boldsymbol{\phi}\in SS^{n-3}} = \begin{pmatrix} 0 & -n_{iz} & n_{iy} \\ n_{iz} & 0 & -n_{ix} \\ -n_{iy} & n_{ix} & 0 \end{pmatrix} = \mathbf{N}_i, \qquad (6)$$

where $\mathbf{n}_i = (n_{ix}, n_{iy}, n_{iz})$ is a unit vector pointing in the positive direction of the i^{th} bar with respect to the first one of the n-bar mechanism [13].

The most important consequence of this formulation is that the solution of the translational equation is the tangent bundle of the solution of the rotational equation, as graphically shown in *Fig. 2.*

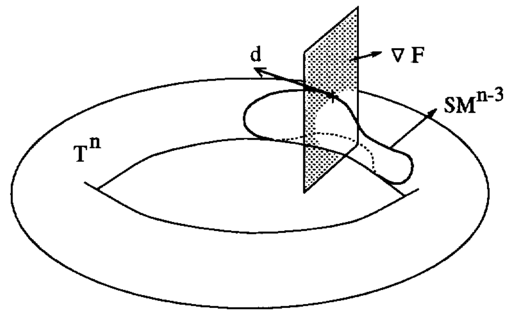

Figure 2: **d** *must be in the tangent space of* SM^{n-3} *of the corresponding spherical mechanism.*

Since the tangent space is a linear space of dimension $r = n-3$, we can find r vectors as a basis of this space. Using the parameterization described in the previous subsection, this basis can be obtained by computing $\frac{\partial \boldsymbol{\phi}}{\partial \psi_i}$, where $i = 1, \ldots, r$. If the first r rotations are taken as parameters, most of these derivatives are straightforwardly computed as

$$\frac{\partial \phi_i}{\partial \psi_i} = 1 \quad \text{and} \quad \frac{\partial \phi_i}{\partial \psi_j} = 0, \ \ i \neq j$$

for $i, j = 1, \ldots, r$.

In order to obtain the remaining derivatives let us define

$$\varphi_{n-2}(\boldsymbol{\psi}) = \phi_{n-2},\ \varphi_{n-1}(\boldsymbol{\psi}) = \phi_{n-1},\ \varphi_n(\boldsymbol{\psi}) = \phi_n$$

when $\mathbf{F}(\boldsymbol{\phi}) = \mathbf{I}$.

By the implicit function theorem, we can differentiate $\mathbf{F}(\boldsymbol{\psi})$ with respect to any ψ_i as follows:

$$\frac{\partial \mathbf{F}(\boldsymbol{\psi})}{\partial \psi_i} = \frac{\partial \mathbf{F}(\boldsymbol{\phi})}{\partial \phi_i} + \frac{\partial \mathbf{F}(\boldsymbol{\phi})}{\partial \phi_{n-2}}\frac{\partial \varphi_{n-2}(\boldsymbol{\psi})}{\partial \psi_i} + \frac{\partial \mathbf{F}(\boldsymbol{\phi})}{\partial \phi_{n-1}}\frac{\partial \varphi_{n-1}(\boldsymbol{\psi})}{\partial \psi_i} +$$

$$+ \frac{\partial \mathbf{F}(\boldsymbol{\phi})}{\partial \phi_n}\frac{\partial \varphi_n(\boldsymbol{\psi})}{\partial \psi_i} = 0.$$

In other words, using (6),

$$\mathbf{N}_{n-2}\frac{\partial\varphi_{n-2}(\psi)}{\partial\psi_i}+\mathbf{N}_{n-1}\frac{\partial\varphi_{n-1}(\psi)}{\partial\psi_i}+\mathbf{N}_n\frac{\partial\varphi_n(\psi)}{\partial\psi_i}=-\mathbf{N}_i.$$

Solving this linear system using Cramer's rule, we get

$$\begin{array}{rcl}\frac{\partial\phi_{n-2}}{\partial\psi_i} & = & -\frac{|\mathbf{n}_i\ \mathbf{n}_{n-1}\ \mathbf{n}_n|}{|\mathbf{n}_{n-2}\ \mathbf{n}_{n-1}\ \mathbf{n}_n|}\\ \frac{\partial\phi_{n-1}}{\partial\psi_i} & = & -\frac{|\mathbf{n}_{n-2}\ \mathbf{n}_i\ \mathbf{n}_n|}{|\mathbf{n}_{n-2}\ \mathbf{n}_{n-1}\ \mathbf{n}_n|}\\ \frac{\partial\phi_n}{\partial\psi_i} & = & -\frac{|\mathbf{n}_{n-2}\ \mathbf{n}_{n-1}\ \mathbf{n}_i|}{|\mathbf{n}_{n-2}\ \mathbf{n}_{n-1}\ \mathbf{n}_n|}.\end{array} \tag{7}$$

As a consequence, the solution of the translational equation can be expressed as [2]:

$$\mathbf{d}=\mathbf{K}\boldsymbol{\lambda},\ \ \forall\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_r)^t\in\Re^r, \tag{8}$$

where

$$\mathbf{K}=\left[\begin{array}{ccc}1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 1\\ -\frac{|\mathbf{n}_1\ \mathbf{n}_{n-1}\ \mathbf{n}_n|}{|\mathbf{n}_{n-2}\ \mathbf{n}_{n-1}\ \mathbf{n}_n|} & \cdots & -\frac{|\mathbf{n}_r\ \mathbf{n}_{n-1}\ \mathbf{n}_n|}{|\mathbf{n}_{n-2}\ \mathbf{n}_{n-1}\ \mathbf{n}_n|}\\ -\frac{|\mathbf{n}_{n-2}\ \mathbf{n}_1\ \mathbf{n}_n|}{|\mathbf{n}_{n-2}\ \mathbf{n}_{n-1}\ \mathbf{n}_n|} & \cdots & -\frac{|\mathbf{n}_{n-2}\ \mathbf{n}_r\ \mathbf{n}_n|}{|\mathbf{n}_{n-2}\ \mathbf{n}_{n-1}\ \mathbf{n}_n|}\\ -\frac{|\mathbf{n}_{n-2}\ \mathbf{n}_{n-1}\ \mathbf{n}_1|}{|\mathbf{n}_{n-2}\ \mathbf{n}_{n-1}\ \mathbf{n}_n|} & \cdots & -\frac{|\mathbf{n}_{n-2}\ \mathbf{n}_{n-1}\ \mathbf{n}_r|}{|\mathbf{n}_{n-2}\ \mathbf{n}_{n-1}\ \mathbf{n}_n|}\end{array}\right].$$

Or simply,

$$\begin{array}{rcl}d_{n-2} & = & -\sum_{i=1}^r d_i\frac{|\mathbf{n}_i\ \mathbf{n}_{n-1}\ \mathbf{n}_n|}{|\mathbf{n}_{n-2}\ \mathbf{n}_{n-1}\ \mathbf{n}_n|}\\ d_{n-1} & = & -\sum_{i=1}^r d_i\frac{|\mathbf{n}_{n-2}\ \mathbf{n}_i\ \mathbf{n}_n|}{|\mathbf{n}_{n-2}\ \mathbf{n}_{n-1}\ \mathbf{n}_n|}\\ d_n & = & -\sum_{i=1}^r d_i\frac{|\mathbf{n}_{n-2}\ \mathbf{n}_{n-1}\ \mathbf{n}_i|}{|\mathbf{n}_{n-2}\ \mathbf{n}_{n-1}\ \mathbf{n}_n|},\end{array} \tag{9}$$

which will be called the *translational equations.* This is an important result because of its simplicity (compare it to the development in [4] using spherical trigonometry).

Equation (9) is valid for points away from the singularities of the chosen parameterization (i.e., when $|\ \mathbf{n}_{n-2}\ \mathbf{n}_{n-1}\ \mathbf{n}_n\ |\neq 0$, or equivalently, $\sin\phi_{n-1}\neq 0$). This is not a problem because it is always possible to find three consecutive bars not contained in a plane, provided that all n bars are non-coplanar.

We can conclude that the system of equations composed of the three rotational equations (5) and the three translational equations (9) are equivalent to the original matrix equation (2) for a proper parameterization. For reasons that will become clear in Section 4, a key point for obtaining an efficient interval method relies on the existence of explicit expressions for the partial derivatives of these equations with respect to the parameters. This is accomplished in the following section.

3 Partial derivatives in terms of rotations

Let us take three consecutive bars of the n-bar mechanism. It is easy to show (*Fig. 3*) that

$$\begin{array}{l}|\ \mathbf{n}_{a-1}\ \mathbf{n}_a\ \mathbf{n}_{a+1}\ |\ =\ \sin\phi_a\\ \mathbf{n}_{a-1}\cdot\mathbf{n}_{a+1}\ =\ -\cos\phi_a\\ \mathbf{n}_a=-\cos\phi_{a-1}\mathbf{n}_{a-2}+\sin\phi_{a-1}(\mathbf{n}_{a-2}\times\mathbf{n}_{a-1})\end{array} \tag{10}$$

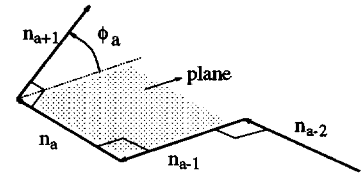

Figure 3: ϕ_a *in terms of the directions of the bars* $\mathbf{n}_{a-1}$, $\mathbf{n}_a$ *and* $\mathbf{n}_{a+1}$.

Let us also consider the following two relations:

$$\begin{array}{l}(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})\\ |\mathbf{a\ b\ c}||\mathbf{a\ d\ e}|-|\mathbf{a\ b\ d}||\mathbf{a\ c\ e}|+|\mathbf{a\ b\ e}||\mathbf{a\ c\ d}|=0\end{array} \tag{11}$$

While the former is a classic vectorial relation, the latter is a reduced form of the Grassmann-Plücker relations [14].

Using equations (10) and (11), we can express any determinant of the type $|\mathbf{n}_i\ \mathbf{n}_{n-1}\ \mathbf{n}_n|$, $|\mathbf{n}_{n-2}\ \mathbf{n}_i\ \mathbf{n}_n|$ or $|\mathbf{n}_{n-2}\ \mathbf{n}_{n-1}\ \mathbf{n}_i|$ in terms of the parameters. To this end, if we define

$$\begin{array}{rcl}v_a^i & = & |\mathbf{n}_i\ \mathbf{n}_{a-1}\ \mathbf{n}_a|\\ w_a^i & = & \frac{|\mathbf{n}_{a-2}\ \mathbf{n}_i\ \mathbf{n}_a|}{|\mathbf{n}_{a-2}\ \mathbf{n}_{a-1}\ \mathbf{n}_a|}\ =\ \frac{|\mathbf{n}_{a-2}\ \mathbf{n}_i\ \mathbf{n}_a|}{\sin\phi_{a-1}},\end{array}$$

the following two recursive expressions can be obtained:

$$\begin{array}{rcl}v_a^i & = & v_{a-1}^i\cos\phi_{a-1}\ +\ w_{a-1}^i\sin\phi_{a-1}\\ w_a^i & = & v_{a-2}^i\sin\phi_{a-2}\ -\ w_{a-2}^i\cos\phi_{a-2}\end{array} \tag{12}$$

Now, let $\mathbf{z}_a^i=(v_a^i\ w_a^i\ v_{a-1}^i\ w_{a-1}^i)^t$. Then, we can write (12) in matrix form:

$$\mathbf{z}_a^i=\mathbf{M}_{a-1}\mathbf{z}_{a-1}^i,$$

where

$$\mathbf{M}_a=\left(\begin{array}{cccc}\cos\phi_a & \sin\phi_a & 0 & 0\\ 0 & 0 & \sin\phi_{a-1} & -\cos\phi_{a-1}\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{array}\right).$$

Since $\mathbf{z}_{i+1}^i=(0\ 1\ 0\ 0)^t$, it is clear that

$$\mathbf{z}_a^i=\left(\prod_{k=1}^{a-i-1}\mathbf{M}_{a-k}\right)\left(\begin{array}{c}0\\1\\0\\0\end{array}\right). \tag{13}$$

Now, the partial derivatives of (5) can be written as follows

$$\begin{pmatrix} \frac{\partial \phi_{n-2}}{\partial \psi_i} \\ \frac{\partial \phi_{n-1}}{\partial \psi_i} \\ \frac{\partial \phi_n}{\partial \psi_i} \end{pmatrix} = \mathbf{L}\, \mathbf{z}_n^i \qquad (14)$$

where

$$\mathbf{L} = \begin{pmatrix} -\frac{1}{\sin \phi_{n-1}} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sin \phi_{n-1}} & 0 \end{pmatrix}.$$

Note that $\sin \phi_{n-1} \neq 0$, because it is assumed that we are using a proper parameterization.

Then, the translational equations (9) can be rewritten as

$$\begin{pmatrix} d_{n-2} \\ d_{n-1} \\ d_n \end{pmatrix} = \mathbf{L} \sum_{i=1}^{n-3} \left(\prod_{k=1}^{n-i-1} \mathbf{M}_{n-k} \right) \begin{pmatrix} 0 \\ d_i \\ 0 \\ 0 \end{pmatrix}. \qquad (15)$$

The derivatives of these equations with respect to ψ_i $(i = 1, \ldots, n-3)$ are now straightforward:

$$\begin{pmatrix} \frac{\partial d_{n-2}}{\partial \phi_j} \\ \frac{\partial d_{n-1}}{\partial \phi_j} \\ \frac{\partial d_n}{\partial \phi_j} \end{pmatrix} = \mathbf{L} \sum_{i=1}^{j-1} \mathbf{R}_{ij} \begin{pmatrix} 0 \\ d_i \\ 0 \\ 0 \end{pmatrix}$$

where

$$\mathbf{R}_{ij} = \prod_{k=1}^{j} \mathbf{M}_{n-k} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \prod_{k=j+1}^{n-i-1} \mathbf{M}_{n-k}.$$

4 Applying a branch and prune strategy

We have seen how the system of equations (2) is equivalent to three rotational equations (5) and three translational equations (15) using a valid parameterization. Next, we describe an interval method to solve this set of equations for a box of the variables of rotation and translation. In general, this box is highly degenerate, since usually most of the variables are fixed by the mechanism's geometry. Only the variables corresponding to the degrees of freedom of the mechanism will vary within a range delimited by design constraints (just remember the example for the PUMA 560 in the introduction).

The adopted algorithm is a propagation process which iteratively improves the bounds on the variables using interval cuts. An interval cut is a procedure which operates on a set of constraints and a current box, reducing this box by deriving a new bound on one of the variables. After reducing the box, three possibilities arise. First, the pruning operation may have resulted in an empty box, in which case we return failure. Second, it may be the case that the interval associated with each variable has reached a width below a specified accuracy. In this case we terminate and return the box. If the pruning operation results in a box which is not of sufficient accuracy, then we split the box and two branches are generated. Then, solutions on each branch are recursively searched. This is what in [6] is called a branch and prune strategy.

Besides the wide variety of heuristics for finding useful cuts and for determining when to branch, the key point is to efficiently generate the cuts to prune the box. We will give a simplified idea of how one of these cuts, the Newton cut [9], reduces the interval for one variable.

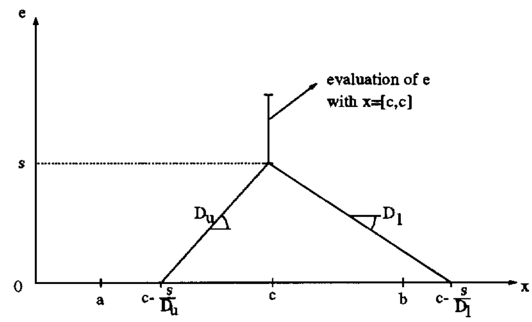

Figure 4: *A Newton cut. Here the interval* $[a, b]$ *will be reduced to* $[a, c - \frac{s}{D_u}]$.

A Newton cut is schematically represented in *fig. 4*. Let $e = 0$ be a constraint (for us one of the equations of rotations or translations), $\mathcal{B}$ a given box for the variables, x a variable appearing in e (the one we want to reduce) and $[a, b]$ be the interval of x in $\mathcal{B}$. We choose c in the interval $[a, b]$.We evaluate e for a box identical to $\mathcal{B}$ except that the variable x is set to c. Let us assume that the lower bound, s, is positive. Now we evaluate the derivative of e with respect to x in $\mathcal{B}$ and denote the resulting interval $[D_l, D_u]$. Note that the values of slopes D_u and D_l represent the fastest possible rate of descent of the value of expression e. We can ensure that there is no solution with $x \in [c, b]$ when either $D_l \geq 0$ or $D_l < 0$ and $c - \frac{s}{D_l} > b$. If $D_l < 0$ and $c - \frac{s}{D_l} < b$, we can reduce interval $[c, d]$ to $[c - \frac{s}{D_l}, b]$. Likewise, there is no solution in $[a, c]$ if $D_u \leq 0$ or if $D_u > 0$ and $c - \frac{s}{D_u} < a$ and if $D_u > 0$ and $c - \frac{s}{D_u} > a$, we can reduce interval $[a, c]$ to $[a, c - \frac{s}{D_u}]$. It can be shown that we can choose c such that we always achieve some reduction of the bounds. Although many heuristics can be used to choose c, we will usually take the middle point of the interval $[a, b]$.

Applying cuts separately to the rotational equations and the translational equations gives us an intuitive idea of what the Newton cut is doing. When reducing the box using one of the constraints imposed by the rotational equations, we are actually adjusting the box to the SM^{n-3}. This is equivalent to eliminating some values of a variable that will never close the spherical mechanism for the intervals of the others variables. If we cut a box by applying one of the constraints of the translational equations, we are eliminating values of some variable that will never close the spatial mechanism.

In the end we get a set of boxes containing the solutions: points if the mechanism is not redundant or boxes containing the set of solutions if the mechanism is redundant.

5 Conclusions

It is a well-known result that the solution of any kinematic equation can be factored into a solution of both its rotational and translational components. Nevertheless, this factorization has been used with a very limited range of practical application in the past, since it leads to inextricable formulae. Herein, some important relationships between both components that greatly simplify this factorization have been presented. This has allowed us to obtain explicit expressions for the partial derivatives of the involved equations in terms of the chosen parameters. This result is of great relevance when an interval method is applied: cuts can be efficiently computed.

The mathematics behind the described approach are quite straightforward and far less sophisticated than those underlying continuation and elimination me-thods. As a consequence, its implementation is simpler. Actually, it has already been partially implemented using PROFIL/BIAS interval libraries [7] and all expressions given herein have been checked for correctness.

Acknowlegments

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