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SOME NEW RESULTS ON THE KINEMATICS OF 3R SERIAL ROBOTS USING NESTED DETERMINANTS

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ABSTRACT

It has been recently shown that the singularity locus of a 3R robot, and in particular its nodes and cusps, can be algebraically characterized in terms of nested determinants. This neat and structured formulation contrasts with the huge and often meaningless formulas generated using computer algebra systems. In this paper we explore further this kind of formulation. We present two new results which we think are of interest by themselves. First, it is shown how Chrystal's method, used to obtain the resultant of two quadratic polynomials, can be formulated as nested determinants. Second, it is also shown how the coefficients of the harmonic conic of two given conics, can also be expressed in the same form. These results lead to new formulations for the inverse kinematics of 3R robots, their singularity loci, their nodes, and some of their high-order singularities.

INTRODUCTION

The regional part of most wrist-partitioned robots is based on a 3R robot meeting some geometric conditions to make its inverse kinematics reduce to the solution of quadratic equations. One important consequence of this simplification is that these robots have to pass through a singularity to change their working mode (change from one inverse kinematic solution to another). Nevertheless, it has been shown how generic 3R robots can change their working modes without meeting any singularity, if at least one point in its workspace has exactly three inverse kinematic solutions (corresponding to a cusp point in its singularity locus). Actually, the number of cusps provides a lot of information about the topology of the singularity locus and, as a result, about the global properties of the manipulator such as the existence of voids and of 4-solution regions [1–3]. This observation reveals that a precise understanding of the nature of the singularity loci of generic 3R robots can assist in the design of industrial robots [4].

Despite what it might seem at first glance, the analysis of the singularities of generic 3R robots is a huge task. This is why this analysis has been limited to 3R robots whose consecutive joint axes are mutually orthogonal (usually known as orthogonal 3R robots). The first attempt to classify 3R manipulators with orthogonal joints was presented in [5]. Five surfaces were found to divide the manipulator parameter space into cells with constant number of cusp points. The equations of these surfaces were derived as polynomials in the DH-parameters using Groebner bases. A physical interpretation of this theoretical work was conducted in [6] where the existence of extraneous surface equations was detected, and where additional features in the classification such as genericity [7] and the number of aspects were took into account. The complete classification of orthogonal 3R manipulators was established for the first time on the basis of the number of cusps and nodes in the singularity locus in [2, 8].

The use of algebraic geometry methods applied on algebraizations of the problem based on DH parameters thus has important limitations. This motivated our quest for finding alternative formulations that could be used to study general 3R

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robots. As a result, we proposed a distance-based formulation in [9]. One important feature of this formulation is that that the nodes and cusps in the robots' singularity loci can be algebraically characterized in terms of nested determinants. In this paper we explore further this kind of formulation. At a pure theoretical level, it is shown how the coefficients of the resultant of two quadratic polynomials, and the coefficients of the harmonic conic of two given conics, can be both formulated as nested determinants. These results lead to new formulations for the inverse kinematics of 3R robots, their singularity loci, their nodes, and some of their high-order singularities.

The rest of this paper is structured as follows. In the next section, we briefly review the results presented in [9] that are expressible as nested determinants. They refer to the robot's singularity locus, and its nodes and cusps. Then, we show how the coefficients of the closure polynomial of generic 3R robots can also be expressed as nested determinants using Chrystal's method. This result leads to a new formulation for the singularity locus as the discriminant of this closure polynomial, and a simple way to detect swallowtail higher-order singularities. All obtained necessary and sufficient conditions are also expressed as nested determinants. Then, we show how the coefficients of the harmonic conic of two given conics can be expressed as nested determinants. This result permits to have a new way to characterize the nodes of the singularity locus of 3R robots. Finally, we present two examples to clarify some of the presented results. Finally, we conclude with some prospect for future research.

SUMMARY OF KNOWN RESULTS EXPRESSED AS NESTED DETERMINANTS

Consider the 3R robot depicted in Fig. 1. We have placed in each revolute axis two points. Their exact location along the revolute axes is irrelevant as long as they are far apart to avoid numerical instabilities. Let us denote these points defining the joints locations $P_1,\ldots,P_7$. Now, observe that the distances between the points between two consecutive axes do not depend on the robots' configurations.

According to the notation used in Fig. 1, the distances between the set of points $\{P_1,P_2,P_3,P_4,P_7\}$ or $\{P_3,P_4,P_5,P_6,P_7\}$ are not independent because they are embedded in $\mathbb{R}^3$. This dependency, using the theory of Cayley-Menger determinants, translates into the following algebraic conditions:

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & s_{1,2} & s_{1,3} & s_{1,4} & s_{1,7} \\ 1 & s_{2,1} & 0 & s_{2,3} & s_{2,4} & s_{2,7} \\ 1 & s_{3,1} & s_{3,2} & 0 & s_{3,4} & s_{3,7} \\ 1 & s_{4,1} & s_{4,2} & s_{4,3} & 0 & s_{4,7} \\ 1 & s_{7,1} & s_{7,2} & s_{7,3} & s_{7,4} & 0 \end{vmatrix} = 0 \qquad (1)$$

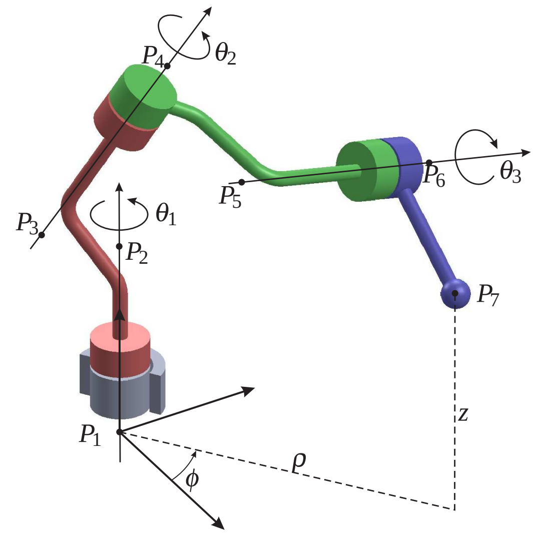

FIGURE 1. A GENERAL 3R ROBOT AND ASSOCIATED NOTATION.

and

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & s_{3,4} & s_{3,5} & s_{3,6} & s_{3,7} \\ 1 & s_{4,3} & 0 & s_{4,5} & s_{4,6} & s_{4,7} \\ 1 & s_{5,3} & s_{5,4} & 0 & s_{5,6} & s_{5,7} \\ 1 & s_{6,3} & s_{6,4} & s_{6,5} & 0 & s_{6,7} \\ 1 & s_{7,3} & s_{7,4} & s_{7,5} & s_{7,6} & 0 \end{vmatrix} = 0, \qquad (2)$$

where $s_{i,j}$ stands for the squared distance between P_i and P_j. The above two equations are quadratic forms in the unknown distances $s_{3,7}$ and $s_{4,7}$. They actually represent two real ellipses, $\mathscr{A}: \mathbf{x}\mathbf{A}\mathbf{x}^T = 0$ and $\mathscr{B}: \mathbf{x}\mathbf{B}\mathbf{x}^T = 0$, where $\mathbf{x} = (s_{3,7}, s_{4,7}, 1)$ and

$$\mathbf{A} = \begin{pmatrix} a_1 & c_1 & d_1 \\ c_1 & b_1 & e_1 \\ d_1 & e_1 & f_1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} a_2 & c_2 & d_2 \\ c_2 & b_2 & e_2 \\ d_2 & e_2 & f_2 \end{pmatrix}. \qquad (3)$$

The entries of **A** and **B** can, in turn, be expressed in terms of determinants (see Table I).

A 3R robot is in a singularity if, and only if, the following discriminant vanishes

$$\Delta_1 = \begin{vmatrix} 2\delta_1 & \delta_2 \\ \delta_2 & 2\delta_3 \end{vmatrix} \qquad (4)$$

where

$$\delta_1 = \begin{vmatrix} l_3 & l_2 \\ l_2 & l_1 \end{vmatrix},\ \delta_2 = \begin{vmatrix} l_3 & l_1 \\ l_2 & l_0 \end{vmatrix},\ \text{and}\ \delta_3 = \begin{vmatrix} l_2 & l_1 \\ l_1 & l_0 \end{vmatrix}. \qquad (5)$$

and

$$l_3 = \begin{vmatrix} a_1 & c_1 & d_1 \\ c_1 & b_1 & e_1 \\ d_1 & e_1 & f_1 \end{vmatrix} = \det(\mathbf{A}), \quad (6)$$

$$3l_2 = \begin{vmatrix} a_2 & c_1 & d_1 \\ c_2 & b_1 & e_1 \\ d_2 & e_1 & f_1 \end{vmatrix} + \begin{vmatrix} a_1 & c_2 & d_1 \\ c_1 & b_2 & e_1 \\ d_1 & e_2 & f_1 \end{vmatrix} + \begin{vmatrix} a_1 & c_1 & d_2 \\ c_1 & b_1 & e_2 \\ d_1 & e_1 & f_2 \end{vmatrix} \quad (7)$$

$$3l_1 = \begin{vmatrix} a_1 & c_2 & d_2 \\ c_1 & b_2 & e_2 \\ d_1 & e_2 & f_2 \end{vmatrix} + \begin{vmatrix} a_2 & c_1 & d_2 \\ c_2 & b_1 & e_2 \\ d_2 & e_1 & f_2 \end{vmatrix} + \begin{vmatrix} a_2 & c_2 & d_1 \\ c_2 & b_2 & e_1 \\ d_2 & e_2 & f_1 \end{vmatrix} \quad (8)$$

$$l_0 = \begin{vmatrix} a_2 & c_2 & d_2 \\ c_2 & b_2 & e_2 \\ d_2 & e_2 & f_2 \end{vmatrix} = \det(\mathbf{B}). \quad (9)$$

The robot is in a cusp singularity if, and only if, $\delta_2 = 0$ and $\delta_3 = 0$.

Finally, using Silvester's criterion, the robot is in a node singularity if, and only if, the following overconstrained system of equations has a root

$$\begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix} \begin{pmatrix} \lambda^2 \\ \lambda \\ 1 \end{pmatrix} = 0 \quad (10)$$

where

$$p_1 = \begin{vmatrix} a_1 & c_1 \\ c_1 & b_1 \end{vmatrix}, q_1 = \begin{vmatrix} a_1 & c_2 \\ c_1 & b_2 \end{vmatrix} + \begin{vmatrix} a_2 & c_1 \\ c_2 & b_1 \end{vmatrix}, r_1 = \begin{vmatrix} a_2 & c_2 \\ c_2 & b_2 \end{vmatrix}, \quad (11)$$

$$p_2 = \begin{vmatrix} c_1 & b_1 \\ d_1 & e_1 \end{vmatrix}, q_2 = \begin{vmatrix} c_1 & b_2 \\ d_1 & e_2 \end{vmatrix} + \begin{vmatrix} c_2 & b_1 \\ d_2 & e_1 \end{vmatrix}, r_2 = \begin{vmatrix} a_2 & b_2 \\ d_2 & e_2 \end{vmatrix}, \quad (12)$$

$$p_3 = \begin{vmatrix} b_1 & e_1 \\ e_1 & f_1 \end{vmatrix}, q_3 = \begin{vmatrix} b_2 & e_1 \\ e_2 & f_1 \end{vmatrix} + \begin{vmatrix} b_1 & e_2 \\ e_1 & f_2 \end{vmatrix}, r_3 = \begin{vmatrix} b_2 & e_2 \\ e_2 & f_2 \end{vmatrix}. \quad (13)$$

OBTAINING A CLOSURE POLYNOMIAL

If we eliminate, for example, $s_{3,7}$ from the system formed by Eqns. (1) and (2), a quartic closure polynomial in $s_{4,7}$ is obtained. The result, in its expanded version, cannot be included here for space limitation reasons, but it can be easily reproduce using a computer algebra system. Next, we show how this quartic closure polynomial can be compactly expressed using nested determinants by following the little known Chrystal's procedure [10, pp. 416-417].

Let us define the matrix

$$\mathbf{W} = \begin{pmatrix} a_1 & b_1 & 2c_1 & 2d_1 & 2e_1 & f_1 \\ a_2 & b_2 & 2c_2 & 2d_2 & 2e_2 & f_2 \end{pmatrix} \quad (14)$$

and w_{ij}, $i \neq j$, the minor of $\mathbf{W}$ containing its columns i and j. For example, $w_{31} = 2\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}$. Then, tediously following by hand Chrystal's procedure, it is possible to prove that the resultant of Eqns.(1) and (2) can be expressed as:

$$A s_{4,7}^4 + 4B s_{4,7}^3 + 6C s_{4,7}^2 + 4D s_{4,7} + E = 0, \quad (15)$$

where

$$A = \begin{vmatrix} w_{12} & w_{13} \\ w_{32} & w_{12} \end{vmatrix},$$

$$4B = \begin{vmatrix} 2\,w_{15} & w_{35} \\ w_{13} & w_{12} \end{vmatrix} + \begin{vmatrix} w_{13} & w_{32} \\ w_{14} & w_{24} \end{vmatrix},$$

$$6C = \begin{vmatrix} w_{15} & w_{12} \\ 2\,w_{61} & w_{15} \end{vmatrix} + \begin{vmatrix} w_{14} & w_{13} \\ w_{36} & w_{24} \end{vmatrix} + \begin{vmatrix} w_{13} & w_{14} \\ w_{35} & w_{54} \end{vmatrix},$$

$$4D = \begin{vmatrix} 2\,w_{15} & w_{14} \\ w_{45} & w_{16} \end{vmatrix} + \begin{vmatrix} w_{13} & w_{14} \\ w_{36} & w_{64} \end{vmatrix},$$

$$E = \begin{vmatrix} w_{16} & w_{14} \\ w_{46} & w_{16} \end{vmatrix}.$$

The roots of Eqn. (15) determine the inverse kinematics solutions of the analyzed 3R robot. In general, this equation will have four different roots but, when some roots coincide, the following possibilities arise:

1. If two roots coincide, the robot is in a standard singularity.
2. If there are two pairs of coincident roots, the robot is in a node of its singularity locus.
3. If three roots coincide, the robot is in a cusp of its singularity locus.
4. If four roots coincide, the robot is in a so-called swallowtail singularity, a kind of high-order singularity.

In the following two sections, we derive a new characterization of standard singularities, as well as swallowtails, using the resultant polynomial in Eqn. (15).

AN ALTERNATIVE FORMULATION FOR SINGULARITIES

Using the results presented in [11, pp. 264-267], the discriminant of Eqn. (15) can be expressed as:

$$\Delta_2 = \begin{vmatrix} 3\omega_1 & 3\omega_2 & \omega_0 \\ 3\omega_2 & 9\omega_3 + \omega_0 & 3\omega_4 \\ \omega_0 & 3\omega_4 & 3\omega_5 \end{vmatrix} \quad (16)$$

TABLE 1. COEFFICIENTS OF THE ELLIPSES $\mathscr{A}$ AND $\mathscr{B}$ EXPRESSED AS DETERMINANTS OF SQUARED DISTANCES BETWEEN $P_1,\ldots,P_7$. OBSERVE THAT THE ORIGINAL PRESENTATION IN [9] CONSTAINS A TYPO IN THE DEFINITIONS OF f_1 AND d_2.

$$a_1=-\begin{vmatrix}0&1&1&1\\1&0&s_{1,2}&s_{1,4}\\1&s_{1,2}&0&s_{2,4}\\1&s_{1,4}&s_{2,4}&0\end{vmatrix},\quad b_1=-\begin{vmatrix}0&1&1&1\\1&0&s_{1,2}&s_{1,3}\\1&s_{1,2}&0&s_{2,3}\\1&s_{1,3}&s_{2,3}&0\end{vmatrix},\quad c_1=\begin{vmatrix}0&1&1&1\\1&0&s_{1,2}&s_{1,3}\\1&s_{1,2}&0&s_{2,3}\\1&s_{1,4}&s_{2,4}&s_{3,4}\end{vmatrix},$$

$$d_1=-\begin{vmatrix}0&1&1&1&1\\1&0&s_{1,2}&s_{1,4}&s_{1,7}\\1&s_{1,2}&0&s_{2,4}&s_{2,7}\\1&s_{1,4}&s_{2,4}&0&0\\1&s_{1,3}&s_{2,3}&s_{3,4}&0\end{vmatrix},\quad e_1=-\begin{vmatrix}0&1&1&1&1\\1&0&s_{1,2}&s_{1,3}&s_{1,7}\\1&s_{1,2}&0&s_{2,3}&s_{2,7}\\1&s_{1,3}&s_{2,3}&0&0\\1&s_{1,4}&s_{2,4}&s_{3,4}&0\end{vmatrix},\quad f_1=\begin{vmatrix}0&1&1&1&1&1\\1&0&s_{1,2}&s_{1,3}&s_{1,4}&s_{1,7}\\1&s_{1,2}&0&s_{2,3}&s_{2,4}&s_{2,7}\\1&s_{1,3}&s_{2,3}&0&s_{3,4}&0\\1&s_{1,4}&s_{2,4}&s_{3,4}&0&0\\1&s_{1,7}&s_{2,7}&0&0&0\end{vmatrix},$$

$$a_2=-\begin{vmatrix}0&1&1&1\\1&0&s_{5,6}&s_{5,4}\\1&s_{5,6}&0&s_{6,4}\\1&s_{5,4}&s_{6,4}&0\end{vmatrix},\quad b_2=-\begin{vmatrix}0&1&1&1\\1&0&s_{5,6}&s_{5,3}\\1&s_{5,6}&0&s_{6,3}\\1&s_{5,3}&s_{6,3}&0\end{vmatrix},\quad c_2=\begin{vmatrix}0&1&1&1\\1&0&s_{5,6}&s_{5,3}\\1&s_{5,6}&0&s_{6,3}\\1&s_{5,4}&s_{6,4}&s_{3,4}\end{vmatrix},$$

$$d_2=-\begin{vmatrix}0&1&1&1&1\\1&0&s_{5,6}&s_{5,4}&s_{5,7}\\1&s_{5,6}&0&s_{6,4}&s_{6,7}\\1&s_{5,4}&s_{6,4}&0&0\\1&s_{5,3}&s_{6,3}&s_{3,4}&0\end{vmatrix},\quad e_2=-\begin{vmatrix}0&1&1&1&1\\1&0&s_{5,6}&s_{5,3}&s_{5,7}\\1&s_{5,6}&0&s_{6,3}&s_{6,7}\\1&s_{5,3}&s_{6,3}&0&0\\1&s_{5,4}&s_{6,4}&s_{3,4}&0\end{vmatrix},\quad f_2=\begin{vmatrix}0&1&1&1&1&1\\1&0&s_{5,6}&s_{5,3}&s_{5,4}&s_{5,7}\\1&s_{5,6}&0&s_{6,3}&s_{6,4}&s_{6,7}\\1&s_{5,3}&s_{6,3}&0&s_{3,4}&0\\1&s_{5,4}&s_{6,4}&s_{3,4}&0&0\\1&s_{5,7}&s_{6,7}&0&0&0\end{vmatrix}.$$

where

$$\omega_1=\begin{vmatrix}A&B\\B&C\end{vmatrix},\ \omega_2=\begin{vmatrix}A&C\\B&D\end{vmatrix},\ \omega_3=\begin{vmatrix}B&C\\C&D\end{vmatrix},$$
$$\omega_4=\begin{vmatrix}B&D\\C&E\end{vmatrix},\ \omega_5=\begin{vmatrix}C&D\\D&E\end{vmatrix},\ \omega_6=\begin{vmatrix}A&D\\B&E\end{vmatrix}. \qquad (17)$$

Then, the robot will be in a singularity if, and only if, Δ_2 vanishes. This is an alternative formulation to that given, as explained above, by $\Delta_1=0$.

Although $\Delta_1=0$ and $\Delta_2=0$ are equivalent for our purposes, the former is preferable because it is obtained from the discriminant of a polynomial of order three, instead of order four. The interest of the latter condition comes from the fact that all the entries of Δ_2 vanish in a swallowtail singularity, as explained below.

SWALLOWTAIL SINGULARITIES

In [9], it is said that higher-order singularities correspond to those cases in which the inverse kinematics of a 3R robot has four repeated solutions. Nevertheless, this is only a necessary and sufficient condition for a kind of higher-order singularities to occur known as swallowtail singularities. Two other kinds of higher-order singularities can occur in the root loci of 3R robots, known as beaks and lips, which do not satisfy this condition (see [12] for details). Since a swallowtail (four coincident roots) can be seen as the coincidence of a cusp (three coincident roots) and a node (two pairs of coincident roots), we can readily characterize them using the criteria presented above. However, we can take advantage of the explicit expression we have just derived for the resultant polynomial in Eqn. (15) and its discriminant in Eqn. (16). Indeed, when the inverse kinematics of the 3R robot has four coincident solutions, the polynomial in Eqn. (15) becomes a perfect quartic. That is, it can be expressed as:

$$(s_{4,7}+\lambda)^4=s_{4,7}^4+4\lambda\, s_{4,7}^3+6\lambda^2 s_{4,7}^2+4\lambda^3 s_{4,7}+\lambda^4=0. \qquad (18)$$

Then, identifying the coefficients of this polynomials with those of the polynomial in Eqn. (15), we conclude that

$$\frac{A}{B}=\frac{B}{C}=\frac{C}{D}=\frac{D}{E}=\frac{1}{\lambda} \tag{19}$$

where $-\lambda$ is the root of the perfect quartic. Therefore, in a perfect quartic all ω_i's vanish, but observe that they are not independent. Actually, they should satisfy, for example, the syzygy [11, p. 266]

$$\omega_1\omega_5-\omega_2\omega_4+\omega_0\omega_3=0. \tag{20}$$

As a consequence, out of the total six, only three need to vanish. This is a much more convenient condition than imposing, at the same time, the conditions for a node and a cusp. This will be exemplified in the second example given below.

AN ALTERNATIVE FORMULATION FOR NODES

In our previous work [9], it is said that Salmon gave, in 1848, the necessary and sufficient condition for a node to occur in our singularity loci but this condition was not expressible as nested determinants [13]. This lead us to use a formulation derived from a little known result presented by Sylvester in 1850 [14]. However, a deeper analysis of Salmon's condition has revealed that, as it is explained below, it can indeed be expressed in terms of nested determinants.

Two arbitrary conics define what is known as their harmonic conic [15, p. 157-8]. This conic is the locus of a point such that the tangents from it to two given conics form an harmonic pencil. Although not explicitly referenced by this name, this conic was introduced by Salmon in his treatise on conic sections [13, Art. 334]. The important result for us is that, if the harmonic conic defined by the two conics given by the two matrices in Eqn. (3) is in the pencil defined by these two conics, then the robot is in a node of its singularity locus. This condition is next expressed as nested determinants.

Using somewhat tedious manipulations, it can be verified that the harmonic conic defined by the two conics with the matrices given in Eqn. (3) can be expressed as $\mathscr{H}:\mathbf{x}\mathbf{H}\mathbf{x}^T=0$ where

$$\mathbf{H}=\begin{pmatrix}\mathtt{a} & \mathtt{h} & \mathtt{g}\\ \mathtt{h} & \mathtt{b} & \mathtt{f}\\ \mathtt{g} & \mathtt{f} & \mathtt{c}\end{pmatrix} \tag{21}$$

and

$$\mathtt{a}=\begin{vmatrix}B_1 & F_2\\ F_1 & C_2\end{vmatrix}+\begin{vmatrix}B_2 & F_1\\ F_2 & C_1\end{vmatrix},\qquad \mathtt{b}=\begin{vmatrix}A_1 & G_2\\ G_1 & C_2\end{vmatrix}+\begin{vmatrix}A_2 & G_1\\ G_2 & C_1\end{vmatrix},$$
$$\mathtt{c}=\begin{vmatrix}A_1 & H_2\\ H_1 & B_2\end{vmatrix}+\begin{vmatrix}A_2 & H_1\\ H_2 & B_1\end{vmatrix},\qquad \mathtt{f}=\begin{vmatrix}A_1 & H_2\\ G_1 & F_2\end{vmatrix}+\begin{vmatrix}A_2 & H_1\\ G_2 & F_1\end{vmatrix},$$
$$\mathtt{g}=\begin{vmatrix}H_1 & B_2\\ G_1 & F_2\end{vmatrix}+\begin{vmatrix}H_2 & B_1\\ G_2 & F_1\end{vmatrix},\qquad \mathtt{h}=\begin{vmatrix}H_1 & F_2\\ G_1 & C_2\end{vmatrix}+\begin{vmatrix}H_2 & F_1\\ G_2 & C_1\end{vmatrix}.$$

and

$$A_i=\begin{vmatrix}b_i & f_i\\ f_i & c_i\end{vmatrix},\qquad B_i=\begin{vmatrix}a_i & g_i\\ g_i & c_i\end{vmatrix},\qquad C_i=\begin{vmatrix}a_i & h_i\\ h_i & b_i\end{vmatrix},$$
$$F_i=\begin{vmatrix}a_i & g_i\\ h_i & f_i\end{vmatrix},\qquad G_i=\begin{vmatrix}h_i & b_i\\ g_i & f_i\end{vmatrix},\qquad H_i=\begin{vmatrix}h_i & g_i\\ f_i & c_i\end{vmatrix}.$$

Now, let us define the matrix

$$\mathbf{X}=\begin{pmatrix}a_1 & b_1 & c_1 & f_1 & g_1 & h_1\\ a_2 & b_2 & c_2 & f_2 & g_2 & h_2\\ \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{f} & \mathtt{g} & \mathtt{h}\end{pmatrix} \tag{22}$$

$\xi_{i,j,k}$, $1\le i<j<k\le 6$, the minor of $\mathbf{X}$ containing the columns $i,\ j$, and k.

Finally we conclude that $\mathscr{C}_1$, $\mathscr{C}_2$ and $\mathscr{H}$ are on a pencil of conics if, and only if, $\mathbf{X}$ is not full rank. In other words, in a node of the singularity locus $\xi_{i,j,k}=0$ for all possible values of $i,\ j$, and k.

1 EXAMPLE I: PARADOXICAL NODES IN ORTHOGONAL 3R ROBOTS

Let us consider an orthogonal 3R robot with the following DH parameters

θ_i	d_i	a_i	α_i
θ_1	0	1	$\pi/2$
θ_2	3	3	$\pi/2$
θ_3	0	9	0

The singularities of this robot plotted both in the $(s_{1,7},s_{2,7})$ plane and in the robot's workspace appear in Fig. 2(a) and Fig. 2(b), respectively. Besides the nodes located on the z-axis of the workspace, which are obtained when mapping from the distance space to the workspace [9], this singularity locus have 4 nodes and 4 cusps. If we apply the condition derived from Sylvester's criterion, which reduces to the computation of the intersection between two pairs of a conic and a line, the result appears in Fig. 2(c) and Fig. 2(d). Two nodes are real (Fig. 2(c))

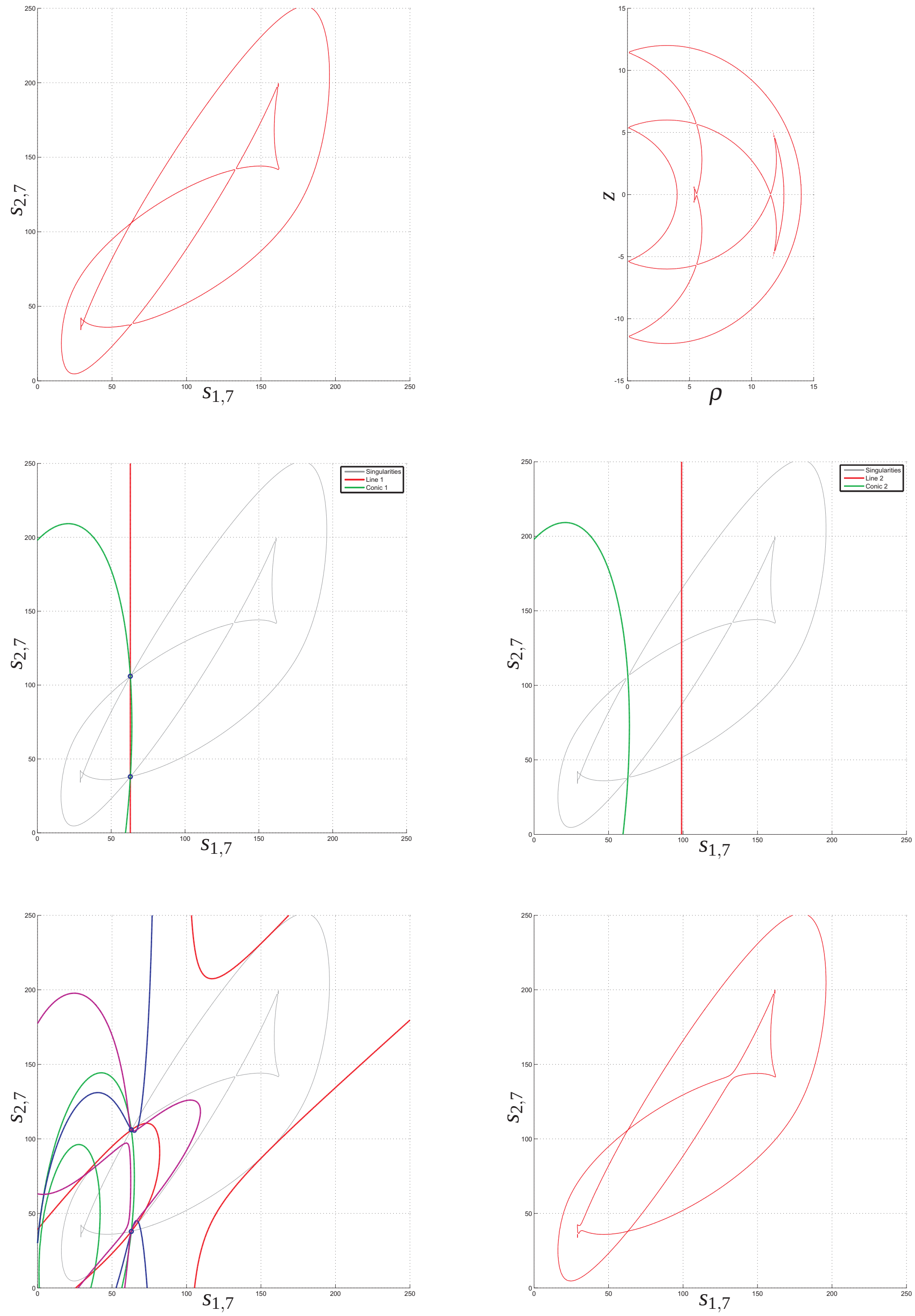

FIGURE 2. PRESENCE OF PARADOXICAL NODES IN THE SINGULARITY LOCUS OF THE 3R ROBOT ANALYZED IN EXAMPLE I. (A) SINGULARITY LOCUS IN THE DISTANCE SPACE. (B) SINGULARITY LOCUS MAPPED ONTO THE WORKSPACE. (C) AND (D) NODES DETECTED USING SYLVESTER'S CRITERION. (E) NODES DETECTED USING SALMON'S CRITERION. (F) THE UNDETECTED NODES ARE ACTUALLY HIGHER-ORDER SINGULARITIES WHICH DESAPPEAR WHEN INTRODUCING A PERTURBATION IN THE ORTHOGONALITY OF THE ROBOT.

and the other two are imaginary (Fig. 2(c)). If we apply the condition presented here, derived from Salmon's criterion, the result appears in Fig. 2(e). In this case, we have plotted the curves defined by $\xi_{1,2,3}=0$, $\xi_{2,3,4}=0$, $\xi_{3,4,5}=0$, and $\xi_{4,5,6}=0$. These four curves intersect at the same two points. That is, both methods detect the same two nodes while the other two remain undetected. The problem with these undetected nodes is that they behave as nodes as long as the robot is orthogonal. If we perturb the orthogonality of the robot —for example, if we set α_2 to $\pi/2.05$ instead of $\pi/2$— the new singularity locus appears in Fig. 2(f). Observe how the previously undetected nodes have disappeared, but we know that standard nodes are stable with respect to "small perturbations". The explanation to this apparent paradox is that the undetected nodes are not standard nodes but higher-order singularities. Moreover, we can also say that they are not swallowtail singularities because a swallowtail appears as the coincidence of a node and a cusp and we have just seen that they are not detected as nodes.

What seem to be stable nodes and cusps in a given subspace are not necessarily so in the ambient space of all possible 3R robots. This seems to be a flaw in some previous analyses of orthogonal 3R robots. As a consequence of this, not all apparent nodes in the singularity locus of an orthogonal 3R robot behave in the same way. While some nodes are stable and they are detected by either using Sylverter's or Salmon's criterion, some others are unstable as they disappear under small perturbations in the robot's orthogonality. Actually, they must be classified as higher-order singularities. In the light of this, the examples presented in [12], for the three kinds of higher-order singularities in the singularity locus of orthogonal 3R robots, should be revisited.

2 EXAMPLE II: SWALLOWTAILS

Let us consider an orthogonal 3R robot with the following DH parameters

θ_i	d_i	a_i	α_i
θ_1	0	1	$\pi/4$
θ_2	1/2	4/5	$\pi/2$
θ_3	5/3	1	0

The singularities of this robot, plotted in the $(s_{1,7}, s_{2,7})$ plane, appear in Fig. 3(center). On the same drawing we have plotted the curves defined by $\delta_2=0$ (in red) and $\delta_3=0$ (in green). Both curves intersect at four points: the cusps of the singularity locus. To better understand what happens at these four points, we have plotted $\mathscr{A}$ (in blue) and $\mathscr{B}$ (in red), as well as the harmonic conic they define (in green) at each of them. The two close cusps are so close as to almost form a swallowtail. In a swallowtail the four points of intersection between $\mathscr{A}$ and $\mathscr{B}$ coincide in single point. In this case $\mathscr{A}$ and $\mathscr{B}$ are said to have an hyper-osculating contact.

CONCLUSION

In this paper we have explored a bit further the kinematics of generic 3R robots using nested determinant formulations. New results concerning closure polynomials, singularity loci, nodes, and swallowtail higher-order singularities have been presented. However, in order to give a complete characterization of the singularities of 3R robots in terms of nested determinants, it remains to obtain such a kind of characterization for beaks and lips higher-order singularities. Our current efforts are aimed at this.

The characterization of higher-order singularities have been revealed much more important than initially suspected, as we have proved through a simple example. Classifying 3R robots in terms of the number of nodes and cusps in their singularity loci seemed meaningful because nodes and cusps are assumed to be stable features of these loci, in the sense that small variations in the parameters defining the robot lead to small perturbations in their locations, but they still remain there. This was the standard approach when analyzing orthogonal 3R robots. Nevertheless, perfect orthogonality cannot be guaranteed in practice and some of the nodes in the singularity locus behave as higher-order singularities when orthogonality errors exist. As a result, taking into account higher-order singularities seems to be unavoidable when characterizing the singularity loci of 3R robots with some kind of constraint in their geometry.

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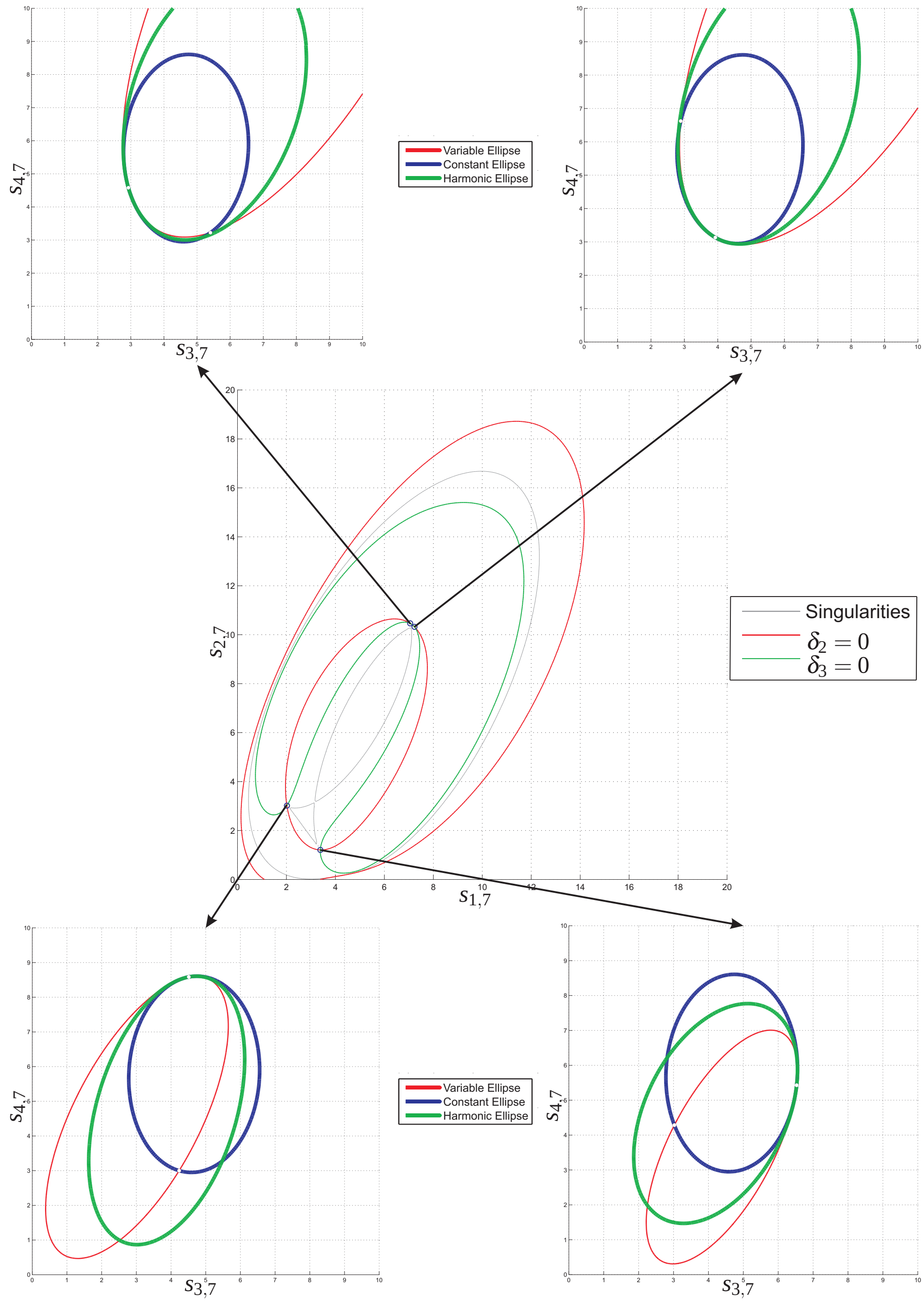

FIGURE 3. ANALYSIS OF THE FOUR CUSPS DETECTED IN THE SINGULARITY LOCUS OF THE 3R ROBOT ANALYZED IN EXAMPLE II. THE CONICS $\mathscr{A}$ (IN BLUE) AND $\mathscr{B}$ (IN RED), AND THE HARMONIC CONIC THEY DEFINE $\mathscr{H}$ (IN GREEN), ARE DEPICTED FOR THE CORRESPONDING VALUES OF $s_{1,7}$ AND $s_{2,7}$ AT WHICH THE FOUR CUSPS OCCUR. OBSERVE THAT TWO CUSPS ARE SO CLOSE AS TO ALMOST FORM A SWALLOWTAIL, IN WHICH CASE THE FOUR POINTS OF INTERSECTION BETWEEN $\mathscr{A}$ AND $\mathscr{B}$ WOULD COINCIDE IN A SINGLE POINT.

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